Rational Segments with Specified Tangents and Curvatures

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Rational Segments with Specified Tangents and Curvatures

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Abstract

We obtain the distribution of inflection points and singularities on a parametric cubic segment with specified tangent directions and curvatures at two data points. Its use enables us to check whether the segment has unwanted inflection points or singularities and gives us an idea how to assign the curvatures at the data points for the shape preserving segment. We also obtain the sufficient conditions for the fair parametric rational segments of the cubic/quadratic and cubic/linear forms.

Key words: cubic segments, inflection points, singularities, curvatures.

1 Introduction

Much attention has been focused on a single- and vector-valued shape preserving interpolation. There is a considerable literature on numerical methods for generating shape preserving interpolation; see for example, [1], [6] and the references therein. Parametric cubic splines of cumulative chord length have been widely used because of their simple computation and good interpolation effects. However, the cubic splines do not always generate "visually pleasing", "shape preserving" (or simply "fair") interpolants which do not contain *unwanted or unplanned* inflection points and singularities to a set of planar data points or have the minimum number of inflection points and singularities compatible with the data. A way to overcome this problem is to consider nonlinear approximation sets, for example, exponential splines, lacunary splines, rational splines or splines with variable additional nodes. For functional data, Delbourgo [3] has successfully treated monotonicity and convexity preserving rational functions of the cubic/quadratic form which contain

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tension parameters for shape adjustments to be made as necessary. However, the functions in tension have also a lack of flexibility in some applications since when the tension parameters become large enough, they approximate the polygon formed by chords joining the data points. To provide more flexibility, the requirements of continuity are relaxed from C^2-continuity to GC^2-continuity where "*the curve is* GC^2*-continuous*" means "*the unit tangent vector and curvature of the curve vary continuously along the curve but each component of the curve is not of* C^2*-continuity at each knot*". Goodman & Unsworth [1] and Su & Lui [6] have obtained the sufficient conditions for a shape preserving cubic interpolation of GC^2-continuity with specified tangent vectors and curvatures at the data points. Note that use of the C^2-continuous spline would require a polynomial of degree five, i.e., a quintic spline which is hard to control as it may have three inflection points.

The object of this paper is to describe the distribution of inflection points and singularities (a loop or a cusp) on a rational cubic segment with specified tangents and curvatures at two end points. Now, consider two data points I_0 and I_1, and suppose we have assigned tangent vectors T_0 and T_1 at these points. Let p (≥ 0) be a rationality parameter. Then, the rational cubic segment $z(t), 0 \leq t \leq 1$ with some $a, b > 0$ is given by

$$(1.1)\qquad (1+ptu)z(t) = u^2\{1+(2+p)t\}I_0 + t^2\{1+(2+p)u\}I_1 + atu^2T_0 - bt^2uT_1,\quad u = 1-t$$

with $z_0'(= z'(0)) = aT_0$ and $z_1'(= z'(1)) = bT_1$. It satisfies the given conditions at the end points $I_i, i = 0, 1$ and is equivalently rewritten as with $\theta(t) = tu^2/(1+ptu)$

$$(1.2)\qquad z(t) = uI_0 + tI_1 + (aT_0 - \Delta I)\theta(t) + (\Delta I - bT_1)\theta(u), \quad \Delta I = I_1 - I_0.$$

where $z \in \text{Span}\{t, u, tu^2/(1+ptu), t^2u/(1+ptu)\}$ or $z \in \text{Span}\{t, u, t^2/(1+qt), u^2/(1+qu)\}$ with $p = q^2/(1+q)$ ([2], [5]). Sections 2-3 describe the distribution of inflection points and singularities on the parametric cubic segment of the form (1.1) with $p = 0$ which is an extension of the sufficient conditions for the convexity preserving interpolation in [1] and [6]. Its use enables us to check whether the segment has unwanted inflection points or singularities when the tangents and curvatures at the data points are approximated by any means and gives us an idea how to assign the curvatures at the data points for the shape preserving segment. Section 4 describes a sufficient condition for the fair parametric rational segment with $p > 0$, that is, a large value of p always gives the fair segment if at I_0 and I_1, it is turning towards the line joining I_0 to I_1. Section 5 considers a sufficient condition for another fair rational segment z of the cubic/linear form, i.e., $z \in \text{Span}\{t, u, t^3/(1+pt), u^3/(1+pu)\}$ with a rationality parameter $-1 < p \leq 0$ as

$$(1.3)\qquad z(t) = uI_0 + tI_1 + (aT_0 - \Delta I)\tau(t) + (\Delta I - bT_1)\tau(u)$$

with

$$(1.4)\qquad \tau(t) = \frac{(1+p)^2\{3+2p+(p+p^2)t\}}{(3+2p)(1+pt)(1+pu)}tu^2 \quad ([4]).$$

Then, for p sufficiently close to $-1+$, the segment (1.3) is fair if at I_0 and I_1, it is turning towards the line joining I_0 to I_1.

2 Inflection points and singularities on the segment (1.1) $(p = 0)$

In Sections 2-3, we consider the case $p = 0$ when the segment of the form (1.1) reduces to the well-known cubic one. We use the similar notations in [1] as $\alpha = T_0 \times \Delta I, \beta = \Delta I \times T_1, \gamma = T_0 \times T_1$ with the usual vector product $'\times'$ and $\| \ \|$ is the Euclidean norm. Assume that $\alpha\beta \neq 0$ as in [1], i.e., neither T_0 nor T_1 are parallel to ΔI. In addition, we assume $\gamma \neq 0$. We require the following simple but easy to use lemma ([4], [5]).

Lemma 2.1 *Assume $z_0' \times z_1' \neq 0$. Then, $\Delta I (= I_1 - I_0)$ can be represented in terms of z_0' and z_1' as $\Delta I = \lambda z_0' + \mu z_1'$ where $(z_0' \times z_1')(\lambda, \mu) = (\Delta I \times z_1', z_0' \times \Delta I)$. The planar cubic segment $z(t), 0 \leq t \leq 1$ has i-inflection points or a loop or a cusp if $(\lambda, \mu) \in N_i, 0 \leq i \leq 2$ or L or C where the boundary of the region L is composed of A (a part of the hyperbola: $\lambda(3\mu - 1) = \mu^2$ limited by the second quadrant), B (a part of the hyperbola: $\mu(3\lambda - 1) = \lambda^2$ limited by the fourth quadrant) and C (a branch of the hyperbola: $(\lambda - 1/3)(\mu - 1/3) = 1/36, \lambda < 1/3, \mu < 1/3$); see Fig. 2.1.*

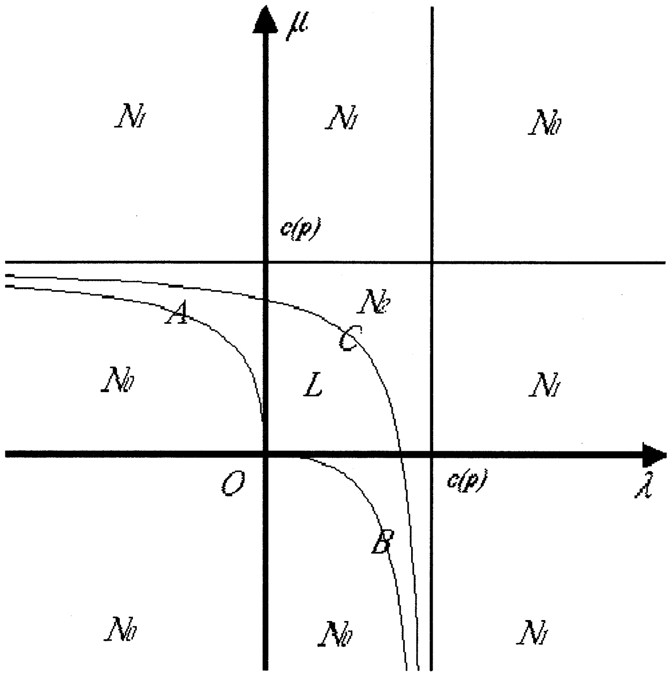

Fig. 2.1. Distribution of inflection points and singularities.

Here, we note that the tangent vector z' vanishes if and only if the segment has a cusp since

$$z'(t) = u\{1 + (6\lambda - 3)t\}z_0' + t\{1 + (6\mu - 3)u\}z_1' \tag{2.1}$$

from which putting $z'(t) = 0, t \in (0,1)$ gives $1 + (6\lambda - 3)t = 0, 1 + (6\mu - 3)u = 0$ and eliminating t gives $(\lambda - 1/3)(\mu - 1/3) = 1/36, \lambda < 1/3, \mu < 1/3$.
Now, a simple calculation gives

$$(z' \times z'')(t) = 2ab\gamma(3\lambda t^2 + 3\mu u^2 + 3tu - 1), \quad 0 < t < 1, u = 1 - t. \tag{2.2}$$

Hence, the curvatures $k_i(= k(i))$ at $I_i, i = 0, 1$ are given by

$$k_0 = \frac{6ab\gamma(\mu - 1/3)}{a^3 \|T_0\|^3}, \quad k_1 = \frac{6ab\gamma(\lambda - 1/3)}{b^3 \|T_1\|^3}. \tag{2.3}$$

Since $\gamma(a\lambda, b\mu) = (\beta, \alpha)$,

$$\lambda^2(1 - \frac{1}{3\mu}) = \left(\frac{\beta}{\gamma}\right)^2 \frac{\|T_0\|^3 k_0}{6\alpha}, \quad \mu^2(1 - \frac{1}{3\lambda}) = \left(\frac{\alpha}{\gamma}\right)^2 \frac{\|T_1\|^3 k_1}{6\beta}. \tag{2.4}$$

Define $D_i, i = 0, 1$ by $\gamma(D_0, D_1) = (\beta\sqrt{\|T_0\|^3 |k_0/(6\alpha)|}, \alpha\sqrt{\|T_1\|^3 |k_1/(6\beta)|})$ to obtain a system of equations in (λ, μ):

$$\lambda\sqrt{\pm(1 - \frac{1}{3\mu})} = D_0, \quad \mu\sqrt{\pm(1 - \frac{1}{3\lambda})} = D_1 \tag{2.5}$$

with the sign in D_0 (or D_1) to be $+$ and $-$ according to αk_0 (or βk_1) > 0 and < 0 since $a, b > 0 \Leftrightarrow \lambda D_0, \mu D_1 > 0$. Refer to Lemma 2.1 to obtain Theorems 2.1-2.3 concerning the distributions of inflection points and singularities on the segment with respect to (D_0, D_1) (being dependent only on the prescribed quantities $I_i, T_i, k_i, i = 0, 1$) where

$f_1(x, y) = (xy)^2\{x^2 + y^2 - 9(xy)^2 - 1/8\} + 1/6912, \quad x, y > 0$
$f_2(x, y) = (xy)^2\{x^2 + y^2 + 9(xy)^2 - 1/8\} - 1/6912, \quad x, y > 0$
$f_3(x, y) = (xy)^2\{-x^2 + y^2 + 9(xy)^2 - 1/8\} - 1/6912, \quad x < 0, y > 0$
$f_4(x, y) = x^2\{y^2 - x^2 + 9(xy)^2 - (1/3 - y^2)(1/3 + 2y^2)\} - (1/3 - y^2)^3/9, \quad x < 0, y > 0.$

For the relative positions of $f_i(x, y) = 0, 1 \leq i \leq 4$, $f_1(x, y) = 0$ is above $f_2(x, y) = 0$ and intersects $x = 1/3$ (or $y = 1/3$) at $y = \sqrt{6}/8$ (or $x = \sqrt{6}/8$). In addition, $f_3(x, y) = 0$ is over $f_4(x, y) = 0$ and $f_i(x, y) = 0, i = 3, 4$ are over $y = 1/3$.

Note that Lemma 2.1 very easily gives the numerically determined distributions of inflection points and singularities since $D_i, i = 0, 1$ are represented in terms of the parameters (λ, μ).

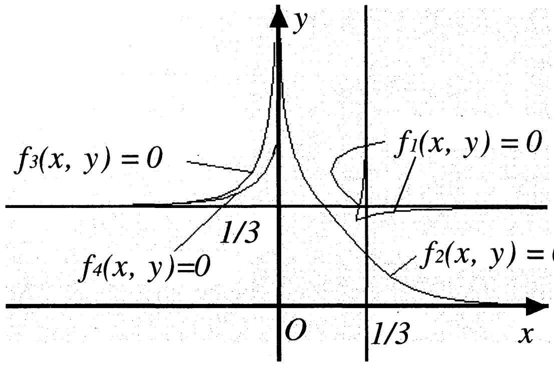

Fig. 2.2. Graphs of curves $f_i(x, y) = 0, 1 \le i \le 4$.

Theorem 2.1 *Assume* $\alpha k_0 \ge 0, \beta k_1 \ge 0$. *Then, the segment has*
(i) no inflection point if $D_0, D_1 < 0$ *or* $0 < D_0, D_1 < 1/3$ *or* (D_0, D_1) *in the first quadrant is limited by* $f_1(D_0, D_1) = 0$*;*
(ii) one inflection point if $D_0 \le 0, D_1 > 1/3$ *or* $D_0 > 1/3, D_1 \le 0$.

Theorem 2.2 *Assume* $\alpha k_0 < 0, \beta k_1 < 0$ *to note that* (D_0, D_1) *is in the first quadrant. Then, the segment has*
(i) two inflection points or a loop if (D_0, D_1) *is in the interior of the region limited by* $f_2(D_0, D_1) = 0$*;*
(ii) a cusp with the pair (D_0, D_1) *which lies on* $f_2(D_0, D_1) = 0$.

Theorem 2.3 *Assume* $\alpha k_0 > 0, \beta k_1 < 0$ *to note that* (D_0, D_1) *is in the first or second quadrants. In the first quadrant, the segment has an inflection point if* $D_1 < 1/3$. *In the second quadrant, it has*
(i) no inflection point if (D_0, D_1) *is in the region limited by* $f_4(D_0, D_1) = 0$ *or on* $f_4(D_0, D_1) = 0$*;*
(ii) two inflection points if (D_0, D_1) *is in the region characterized by* $D_1 = 1/3, f_3(D_0, D_1) = 0$*;*
(iii) a cusp with the pair (D_0, D_1) *on* $f_3(D_0, D_1) = 0$*;*
(iv) a loop if (D_0, D_1) *is in the region limited by* $f_3(D_0, D_1) = 0$ *and* $f_4(D_0, D_1) = 0$.
If the regions in (i)-(iv) have the common part, for example, (D_0, D_1) *is in the region limited by* $f_3(D_0, D_1) = 0$ *and* $f_4(D_0, D_1) = 0$, *the segment has two inflection points or a loop.*

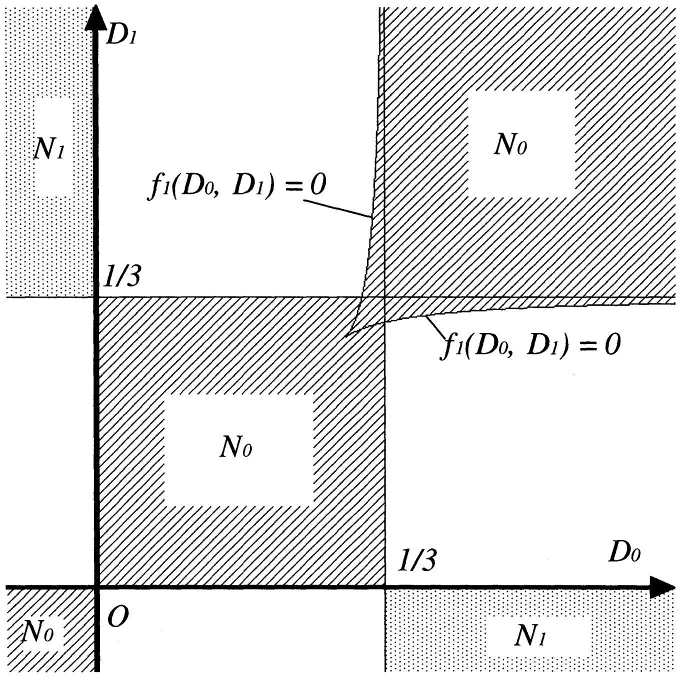

Fig. 2.3. Distribution of inflection points when $\alpha k_0 > 0, \beta k_1 > 0$.

Theorem 2.1-2.3 imply that if possible (we can specify the curvatures k_0, k_1 unconditionally), it is desirable to assign the curvatures as $\alpha k_0 > 0$ and $\beta k_1 > 0$. Fig. 2.3 gives the distribution of inflection points in the case when at I_0 and I_1, the segment is turning towards the line joining I_0 to I_1 or equivalently, $\alpha k_0 > 0$ and $\beta k_1 > 0$. Note that then it has no singularity.

3 Proof of Theorems 2.1-2.3

Four cases depending on the signs of αk_0 and βk_1 will be discussed separately.
Case I $\alpha k_0, \beta k_1 \geq 0$: Since $\lambda(\lambda - 1/3) \geq 0,\ \mu(\mu - 1/3) \geq 0, \lambda\mu \neq 0$, the segment has one or no inflection point and no singularity. Use (2.4) to have

$$\lambda\sqrt{1 - \frac{1}{3\mu}} = D_0, \quad \mu\sqrt{1 - \frac{1}{3\lambda}} = D_1 \tag{3.1}$$

which give

$$\mu = \frac{\lambda^2}{3(\lambda^2 - D_0^2)}, \quad D_1(:= D_1(\lambda)) = \frac{\lambda^2}{3(\lambda^2 - D_0^2)}\sqrt{1 - \frac{1}{3\lambda}}. \tag{3.2}$$

Since (i) $\lambda, \mu < 0 \leftrightarrow D_0, D_1 < 0$ and (ii) $\lambda < 0, \mu \geq 1/3 \leftrightarrow D_0 \leq 0, D_1 > 1/3$ or $\lambda \geq 1/3, \mu < 0 \leftrightarrow D_0 > 1/3, D_1 \leq 0$, Lemma 2.1 shows that the segment has no or one inflection point in the above (i) or (ii), respectively. Now, for $\lambda, \mu \geq 1/3$, first hold $D_0 (\geq 1/3)$ fixed. Then, with $u = 6D_0^2\{1 + \sqrt{1 - 1/(12D_0^2)}\}, D_1$ is monotone decreasing (or increasing) on (D_0, u) (or (u, ∞)) and $D_1(D_0+) = \infty$. Hence, the segment has no inflection point if $D_0 \geq 1/3, D_1 \geq D_1(u)$. $D_1 = D_1(u)$ reduces to $f_1(D_0, D_1) = 0$ as follows. Let $(r, s) = (u^2/\{3(u^2 - D_0^2)\}, u)$ to get (3.1) with $(\lambda, \mu) = (s, r)$. Since $(r - 1/3)(s - 1/3) = 1/36$, by (3.1)

$$r + s = 108(D_0 D_1)^2 + 1/4, \quad rs = 36(D_0 D_1)^2. \tag{3.3}$$

Note $D_0^2 + D_1^2 = r^2 + s^2 - (r^3 + s^3)/(3rs)$ to get $f_1(D_0, D_1) = 0$. Next, hold $D_0 \in (0, 1/3)$ fixed. Since $D_1(1/3+) = 0$ and $D_1(\infty) = 1/3$, the segment has no inflection if $0 < D_0, D_1 < 1/3$. The symmetry of (λ, μ) bringing the symmetry of (D_0, D_1), change of D_0 from 0 to ∞ gives the distribution of the inflection point in Theorem 2.1.

For given curvatures k_0, k_1 satisfying $\alpha k_0, \beta k_1 > 0$, it is not always possible to construct the segment (1.1) with $p = 0$ since the above proof of Theorem 2.1 shows that it exists only for a choice of (k_0, k_1) from the region $N_i, i = 0, 1$ in Fig. 2.3.

It is possible to relate our results in Theorem 2.1 to how to assign the curvatures in [1] and [6]. Theorem 2.1 enables us to choose the curvatures k_0, k_1 so that $D_i = 1/3, i = 0, 1$, or explicitly speaking

$$(3/2)|k_0|\, \|T_0\|^3 = \gamma^2|\alpha|/\beta^2, \quad (3/2)|k_1|\, \|T_1\|^3 = \gamma^2|\beta|/\alpha^2. \tag{3.4}$$

The above assignment of the curvatures is essential the same to the one in ([6], p.85). Next, note that

$$\gamma^2 < 2\{\beta^2 \|T_0\|^2 + \alpha^2 \|T_1\|^2\} / \|\Delta I\|^2 \ (:= 2\delta) \tag{3.5}$$

since $\gamma \Delta I = \beta T_0 + \alpha T_1, \gamma (= T_0 \times T_1) \neq 0$. Hence, substitution of γ^2 by 2δ in the above assignment in [6] gives the one in [1]:

$$(3/2)|k_0|\, \|T_0\|^3 = 2\delta|\alpha|/\beta^2, \quad (3/2)|k_1|\, \|T_1\|^3 = 2\delta|\beta|/\alpha^2 \tag{3.6}$$

which $|D_i| = \sqrt{2\delta}/(3|\gamma|) > 1/3, i = 0, 1$. If $\alpha\gamma, \beta\gamma > 0$, Theorem 2.1 shows that both the assignments of the curvatures ensure the fair segment of the form (1.1) with $p = 0$ since $D_i \geq 1/3, i = 0, 1$. In addition, note that [6] uses smaller values of $D_i, i = 0, 1$ than [1].

Suppose $I_i = (x_i, y_i), 0 \leq i \leq N$ are data points in the plane and we have assigned tangents T_i at I_i. As in [1], define $\alpha_i = T_i \times \Delta I_i, \beta = \Delta I_i \times T_{i+1}, \gamma_i = T_i \times T_{i+1}$. Then, the curvatures k_i at I_i are assigned as

$$(3/2)|k_i|\, \|T_i\|^3 = \mathrm{Max}(\gamma_i^2|\alpha_i|/\beta_i^2, \gamma_{i-1}^2|\beta_{i-1}|/\alpha_{i-1}^2), \quad 1 \leq i \leq N-1 \tag{3.7}$$

where $\alpha_i k_i, \beta_i k_{i+1} > 0$.
Since $\gamma_i \Delta I_i = \beta_i T_i + \alpha_i T_{i+1}$,

$$\gamma_i^2 < 2\{\beta_i^2 \|T_i\|^2 + \alpha_i^2 \|T_{i+1}\|^2\} / \|\Delta I_i\|^2 \ (:= 2\delta_i). \tag{3.8}$$

Hence, from (3.7) and (3.8) we have the assignment of the curvatures in [1].
Case II $\alpha k_0, \beta k_1 < 0$: Since $0 < \lambda, \mu < 1/3$, the segment contains either two inflection points or a singularity. Use (2.4) to get

$$\lambda\sqrt{\frac{1}{3\mu}-1} = D_0, \quad \mu\sqrt{\frac{1}{3\lambda}-1} = D_1. \tag{3.9}$$

Keeping $D_0(>0)$ fixed as $\lambda > 0$, from above

$$\mu = \frac{\lambda^2}{3(\lambda^2+D_0^2)}, \quad D_1(:= D_1(\lambda)) = \frac{\lambda^2}{3(\lambda^2+D_0^2)}\sqrt{\frac{1}{3\lambda}-1}. \tag{3.10}$$

Note that with $v = 6D_0^2\{-1+\sqrt{1+1/(12D_0^2)}\}$, $(\lambda,\mu) \in L$ or C or N_2 if $\lambda \in (0,v)$ or $\lambda = v$ or $\lambda \in (v, 1/3)$, respectively. Easily, D_1 is monotone increasing (or decreasing) on $(0,v)$ (or on $(v,1/3)$) and $D_1(+0)=0, D_1(1/3-)=0$. Hence, the segment has two inflections or a loop if $0 < D_1 < D_1(v)$ where a cusp occurs if $D_1 = D_1(v)$. For a simple form of $D_1 = D_1(v)$, let $(r,s) = (v^2/\{3(v^2+D_0^2)\}, v)$ to obtain (3.5) with $(\lambda,\mu) = (s,r)$. Since (r,s) is on C, by (3.5)

$$r+s = 108(D_0D_1)^2+1/4, \quad rs = 36(D_0D_1)^2. \tag{3.11}$$

Note $D_0^2+D_1^2 = -(r^2+s^2)+(r^3+s^3)/(3rs)$ to get $f_2(D_0,D_1)=0$. Change of D_0 from 0 to 1/3 gives Theorem 2.2.

In Case II, given I_i, T_i and k_i, determine (D_0, D_1). Then, the above analysis shows that the system of equations (3.7) has two solutions (λ,μ) which lead to two values of (a,b) if $D_i, i=0,1$ are in the interior of $f_2(D_0,D_1)=0$ in the first quadrant. The two solutions give the segments of the form (1.1) with two inflection points and a loop, respectively. If $D_i, i=0,1$ are out of $f_2(D_0,D_1)=0$, the segment (1.1) does not exist.
Case III $\alpha k_0 < 0, \beta k_1 > 0$: Since $\lambda(\lambda-1/3) > 0, 0 < \mu < 1/3$, the segment contains zero to two inflection points or a singularity. Use (2.4) to have

$$\lambda\sqrt{\frac{1}{3\mu}-1} = D_0, \quad \mu\sqrt{1-\frac{1}{3\lambda}} = D_1 \tag{3.12}$$

which give

$$\mu = \frac{\lambda^2}{3(\lambda^2+D_0^2)}, \quad D_1(:= D_1(\lambda)) = \frac{\lambda^2}{3(\lambda^2+D_0^2)}\sqrt{1-\frac{1}{3\lambda}}. \tag{3.13}$$

First, since $\lambda > 1/3, 0 < \mu < 1/3 \leftrightarrow D_0 > 0, 0 < D_1 < 1/3$, Lemma 1 shows that the segment has one inflection point if $D_0 > 0, 0 < D_1 < 1/3$. Next, if $\lambda < 0, 0 < \mu < 1/3$, keep $D_0(<0)$ fixed. Then, note that with $w_0 = -6D_0^2\{1+\sqrt{1+1/(12D_0^2)}\}$ and w_1 the root of $t^3+9D_0^2t^2+9D_0^4=0$, $(\lambda,\mu) \in N_2, C, L, N_0$ if $\lambda \in (-\infty, w_0), w_0, (w_0,w_1), [w_1,0)$, respectively. Easily, D_1 is monotone increasing (or decreasing) on $(-\infty, w_0)$ (or $(w_0, 0)$)

and $D_1(-\infty) = 1/3, D_1(0-) = 0$. Hence, the segment has two inflection points or a cusp or a loop or no inflection point if $D_0 < 0$ and in addition if $1/3 < D_1 < D_1(w_0)$ or $D_1 = D_1(w_0)$ or $D_1(w_1) < D_1 < D_1(w_0)$ or $0 < D_1 \leq D_1(w_1)$. For a simple form of $D_1 = D_1(w_0)$, let $(r, s) = (w_0^2/\{3(w_0^2 + D_0^2)\}, w_0)$ to get (3.8) with $(\lambda, \mu) = (s, r)$. Since (r, s) is on C, by (3.8)

$$(3.14) \qquad r + s = -\ 108(D_0 D_1)^2 + 1/4, \quad rs = -\ 36(D_0 D_1)^2.$$

Note $D_0^2 - D_1^2 = -\ (r^2 + s^2) + (r^3 + s^3)/(3rs)$ to get $f_3(D_0, D_1) = 0$. For $D_1 = D_1(w_1)$, let $(r, s) = (w_1^2/\{3(w_1^2 + D_0^2)\}, w_1)$ to get (3.7) with $(\lambda, \mu) = (s, r)$. Since (s, r) is on A, by (3.8)

$$(3.15) \qquad s + r = -\ D_1^2 - 9D_0^2 + 1/3, \quad rs = -\ 3D_0^2.$$

Note $D_0^2 - D_1^2 = -\ (r^2 + s^2) + (r^3 + s^3)/(3rs)$ to get $f_4(D_0, D_1) = 0$. Change of D_0 from $-\infty$ to 0 gives Theorem 2.3.

Case IV $\alpha k_0 > 0, \beta k_1 < 0$: The similar treatment in Case III would give the similar result in Case III.

Here we may say a few words here on the exceptional case when $\gamma = 0$, i.e., $z_1' = mz_0'$. Then, note that if $m > 0$ (or < 0), the segment has no (or one) inflection point and no singularity ([4]). A simple and direct calculation gives

$$(3.16) \quad k_0 = \frac{6\alpha}{a^2 \|T_0\|^3}, \quad k_1 = \frac{6\beta}{b^2 \|T_1\|^3} \quad \rightarrow \quad a = \sqrt{\frac{6\alpha}{k_0 \|T_0\|^3}}, \quad b = \sqrt{\frac{6\beta}{k_1 \|T_1\|^3}}$$

provided that $\alpha k_0, \beta k_1 > 0$.
Hence, the segment has no (or one) inflection point and no singularity if $T_1 = cT_0, c > 0$ (or $c < 0$).

4 A condition for the fair segment (1.1) ($p > 0$)

In this case, refer to [5] to get the distributions of inflection points and singularities with respect to (D_0, D_1). For the sake of simplicity, we obtain the sufficient condition for the fair rational segment (1.1) with $p > 0$.

Lemma 4.1 *([5]). Assume $z_0' \times z_1' \neq 0$, and $\Delta I = \lambda z_0' + \mu z_1'$. Then, the rational cubic segment is fair if $\lambda, \mu \geq 1/(3 + p)$.*

Then, the tangent vector z' does not vanish as follows. Letting $r = tu, 0 \leq r \leq 1/4$,

$$(4.1) \qquad \begin{aligned} (1 + pr)^2 z'(t) &= [\{(6 + 2p)r + (2p + p^2)r^2\}\lambda + u^2 - 2r - pr^2]z_0' \\ &\quad + [\{(6 + 2p)r + (2p + p^2)r^2\}\mu + t^2 - 2r - pr^2]z_1'. \end{aligned}$$

Putting $z'(t)=0$ gives

$$\lambda+\mu=\frac{2pr^2+6r-1}{(2p+p^2)r^2+(6+2p)r}. \tag{4.2}$$

The right hand side of (4.2) is monotone increasing in r, and so

$$\lambda+\mu\le\frac{2(1+p)}{12+6p+p^2}<\frac{2}{(3+p)} \tag{4.3}$$

from which the tangent vector z' does not vanish if $\lambda,\mu\ge 1/(3+p)$. Strictly speaking, as in the case $p=0$, it follows from [5] shows that the tangent vector vanishes if and only if the segment has a cusp as follows. From (4.1),

$$\{(6+2p)r+(2p+p^2)r^2\}(\lambda,\mu)=(pr^2+2r-u^2,pr^2+2r-t^2). \tag{4.4}$$

A simple calculation (or for example, use of *Mathematica*) shows that (λ,μ) satisfies:

$$\begin{aligned}k(\lambda,\mu)&=4\lambda^3\{(3+p)\mu-1\}+4\mu^3\{(3+p)\lambda-1\}-3\lambda^2\mu^2\\&+\{(3+p)\lambda-1\}^2\{(3+p)\mu-1\}^2-6\lambda\mu\{(3+p)\lambda-1\}\{(3+p)\mu-1\}=0.\end{aligned} \tag{4.5}$$

Therefore, by (4.3) (λ,μ) is on the branch $k_1(\lambda,\mu)=0$ of $k(\lambda,\mu)=0$ characterized by $\lambda,\mu<1/(3+p)$. As in [5], with $0\le t\le 1, u=1-t$

$$(1+ptu)^3(z'\times z'')(t)=2ab\gamma\{(3+pt)t^2\lambda+(3+pu)u^2\mu+3tu-1\} \tag{4.6}$$

which gives

$$k_0=\frac{6ab\gamma\{(1+p/3)\mu-1/3\}}{a^3\left\|T_0\right\|^3},\quad k_1=\frac{6ab\gamma\{(1+p/3)\lambda-1/3\}}{b^3\left\|T_1\right\|^3}. \tag{4.7}$$

Hence

$$\lambda^2(1+\frac{p}{3}-\frac{1}{3\mu})=\left(\frac{\beta}{\gamma}\right)^2\frac{\left\|T_0\right\|^3k_0}{6\alpha},\quad \mu^2(1+\frac{p}{3}-\frac{1}{3\lambda})=\left(\frac{\alpha}{\gamma}\right)^2\frac{\left\|T_1\right\|^3k_1}{6\beta}. \tag{4.8}$$

As for $p=0$, let $\gamma(D_0,D_1)=(\,\beta\sqrt{\left\|T_0\right\|^3|k_0/(6\alpha)|},\ \alpha\sqrt{\left\|T_1\right\|^3|k_1/(6\beta)|}\,)$ to obtain a system of equations in (λ,μ):

$$\lambda\sqrt{\pm(1+\frac{p}{3}-\frac{1}{3\mu})}=D_0,\quad \mu\sqrt{\pm(1+\frac{p}{3}-\frac{1}{3\lambda})}=D_1 \tag{4.9}$$

with the sign in D_0 (or D_1) to be $+$ and $-$ according to αk_0 (or βk_1) >0 and <0. Use Lemma 4.1 and the similar argument in Case I of Section 3 to show that $D_0,D_1\ge 1/(3\sqrt{1+p/3})$ give $\lambda,\mu\ge 1/(3+p)$. Thus,

Theorem 4.1 *Assume* $\alpha k_0,\beta k_1\ge 0$. *Then, the segment (1.1) with* $p>0$ *is fair if* $D_0,D_1\ge 1/(3\sqrt{1+p/3})$.

Hence, the segment (1.1) of the cubic/quadratic form is fair for p sufficiently large if at I_0 and I_1, it is turning towards the line joining I_0 to I_1. In practical calculation, it suffices to increase the parameter p, starting at $p=0$, until it is satisfactory.

5 A condition for the fair segment (1.3) ($-1 < p \le 0$)

In this case, we require

Lemma 5.1 *([4]). Assume* $z_0' \times z_1' \neq 0$, *and* $\Delta I = \lambda z_0' + \mu z_1'$. *Then, the rational cubic segment (1.3) with* $-1 < p \le 0$ *is fair if* $\lambda, \mu \ge (1+p)/(3+2p)$.

Note that the tangent vector z' does not vanish as follows. Put $z'(t) = 0$ or let the coefficients of $z_i', i = 0, 1$ of $z'(t)$ equal be zero to get

(5.1) $$\tau'(t) + \{1 - \tau'(t) - \tau'(u)\}\lambda = 0, \tau'(u) + \{1 - \tau'(t) - \tau'(u)\}\mu = 0.$$

Note

(5.2) $$\tau'(t) + \tau'(u) = \frac{(1+p)^2(1-4r)}{(1+p+p^2r)^2}, \quad 0 < r = tu \le \frac{1}{4}$$

which gives

(5.3) $$0 \le \tau'(t) + \tau'(u) < 1, \quad 0 < t < 1, u = 1 - t.$$

By means of (5.1) and (5.3),

(5.4) $$\lambda + \mu = \frac{\tau'(t) + \tau'(u)}{\tau'(t) + \tau'(u) - 1} < \frac{2(1+p)}{3+2p}.$$

Therefore, the tangent vector does not vanish if $\lambda, \mu \ge (1+p)/(3+2p)$. As in the segment of the form (1.1), numerical experiments imply that the tangent vector would vanish if and only if the segment has a cusp, however it is impossible to check it analytically since the exact distribution of a singularity (a loop or a cusp) has never been obtained yet ([4]). Now,

(5.5) $$(z' \times z'')(0) = \frac{2ab\gamma(1+p+p^2/2)}{(1+p)^3}\{(3+2p)\mu - (1+p)\}$$
$$(z' \times z'')(1) = \frac{2ab\gamma(1+p+p^2/2)}{(1+p)^3}\{(3+2p)\lambda - (1+p)\}.$$

Use the same $D_i, i = 0, 1$ in Section 4 to give a system of equation in (λ, μ):

(5.6) $$\lambda\sqrt{\pm(1 + \frac{2p}{3} - \frac{1+p}{3\mu})} = D_0/c(p), \quad \mu\sqrt{\pm(1 + \frac{2p}{3} - \frac{1+p}{3\lambda})} = D_1/c(p)$$

with $c(p) = \sqrt{(1+p+p^2/2)/(1+p)^3}$ where the sign in D_0 (or D_1) are chosen to be + and − according to αk_0 (or βk_1) > 0 and < 0. As in Section 4, Lemma 5.1 gives

Theorem 5.1 *Assume* $\alpha k_0, \beta k_1 \ge 0$. *Then, the segment (1.3) with* $-1 < p \le 0$ *is fair if* $D_0, D_1 \ge (1+p)/(3\sqrt{1+2p/3})$.

Hence, the rational cubic segment of the linear/cubic form is fair for p sufficiently close to − 1 if at I_0 and I_1, it is turning towards the line joining I_0 to I_1.

References

[1] T.N.T. Goodman and K. Unsworth: Shape preserving interpolation by curvature continuous curves, CAGD 5, 1988, 323-340.

[2] J. A. Gregory: Shape preserving spline interpolation, Comput, Aided design 18, 1986, 53-57.

[3] R. Delbourgo: Accurate rational interpolants in tension, SIAM J. Anal. 30, 1993, 595-607.

[4] M. Sakai and R. Usmani: On fair parametric rational cubic curves, BIT 36, 1996, 349-367.

[5] M. Sakai: Inflections and singularity on parametric rational cubic curves, Numer. Math. 76, 1997, 403-417.

[6] B-Q. Su and D-Y. Liu: Computational Geometry-Curve and Surface Modeling, Academic Press, 1989, New York.