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# On Curvatures of Rational Quadratic Bézier segments

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**Abstract**

We give a simple derivation of necessary and sufficient conditions for the rational quadratic Bézier segment to be a spiral or to have local extrema by means of differentiation and Descartes's rule of signs. We also determine (i) how to place control the vertices and (ii) how to give the tangent vectors at the endpoints for the spiral.

**Key words:** inflection points, singularities, rational quadratic Bézier segments, offset curves.

## 1 Introduction

Polynomial curves have been widely used in computer-aided design. However, the curves do not always generate "visually pleasing", "shape preserving" (or simply "fair") interpolants which do not contain *unwanted* interior inflection points and singularities (loop or cusp) to a set of planar data points. There is a considerable literature on numerical methods for generating a shape preserving interpolation; for example, see Ahn & kim [1], Farin [2], Meek & Walton [4], Späth ([5], [6]), and the references therein. A way to overcome this problem is to introduce the quadratic and cubic rational curve segments. In this note, we consider a rational quadratic Bézier segment $\boldsymbol{z}(t)$ with weights $w_i, 0 \leq i \leq 2$ of the form:

$$(1.1) \qquad \boldsymbol{z}(t) = \frac{w_0 u^2 \boldsymbol{b}_0 + 2w_1 tu \boldsymbol{b}_1 + w_2 t^2 \boldsymbol{b}_2}{w_0 u^2 + 2w_1 tu + w_2 t^2}, \quad 0 \leq t \leq 1, u = 1 - t$$

Then the curvature $k(t)$ of the above curve segment $\boldsymbol{z}(t), 0 \leq t \leq 1$ is given by

$$(1.2) \qquad k(t) = (\boldsymbol{z}' \times \boldsymbol{z}'')(t) / \|\boldsymbol{z}'(t)\|^3, \quad 0 \leq t \leq 1$$

where $\times$ means a vector product and $\|\bullet\|$ is the Euclidean norm. The control points $\boldsymbol{b}_i$ belong to $R^2$ and we assume that the weights $w_i$ are all positive. By use of *symmetry* of conics, Ahn & kim[1] obtained necessary and sufficient conditions for the curvature of the quadratic

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rational Bézier curve to be monotone (a spiral), to have a unique local minimum, to have a local maximum, and to have both extrema. Frey & Field[3] found the similar conditions by *differentiation* of the curvature. We assume the quadratic rational Bézier curve to be of the standard form, i.e., $w_0 = w_2 = 1$, $w_1 = \mu(> 0)$ and for simplicity, $\boldsymbol{b}_0 = (0, 0), \boldsymbol{b}_2 = (-1, 0)$. In addition, we assume that the remaining vertex $\boldsymbol{b}_1$ is restricted to be above the $X$-axis and left of the vertical line $u = -1/2$.

In Section 2, we use *differentiation* and Descartes' rule of signs to obtain the same necessary and sufficient conditions for the rational quadratic Bézier spiral segment in terms of (i) the control vertices and (ii) the angles of the tangent vectors at the endpoints. In addition, we shall note that an introduction of the weights does enlarge the region required for the rational quadratic Bézier spiral.

## 2 Main Theorems

The first theorem considers a choice of the control vertex $\boldsymbol{b}_1 = (u, v), u \leq -1/2, v > 0$ for the rational quadratic spiral whose curvature is monotone *increasing*; note that the proof is easier to read and more straightforward than the one given in Ahn & Kim [1]. For later use, we define $D_i, i = 1, 2$ as $D_1 = \{(u, v)|\ 2\mu^2(u^2+v^2)+u \geq 0\}$, $D_2 = \{(u, v)|\ 2\mu^2\{(u+1)^2+v^2\}-(u+1) \leq 0\}$, and $D_1^c(D_2^c)$ is the complimentary set of $D_1(D_2)$. Then for $u < -1/2$, we have

**Theorem 2.1** ([1])

$$\text{(2.1)} \qquad \textit{If } (u, v) \in \quad \text{(i) } D_1 \cap D_2, \quad \text{(ii) } D_1 \cap D_2^c, \quad \text{(iii) } D_1^c \cap D_2, \quad \text{(iv) } D_1^c \cap D_2^c$$

*then the curvature of the rational quadratic Bézier curve segment of the form* (1.1) *is* (i) *monotone increasing,* (ii) *has just one local maximum,* (iii) *has just one local minimum,* (iv) *first just one local minimum and next just one local maximum.*

**Proof** With help of *Mathematica* or not so lengthy calculation by hand,

$$k'(t) = \frac{3v\mu\left[(s+1)(s^2+2\mu s+1)\right]^2 q_4(s)}{2\{r_4(s)\}^{5/2}}, \quad t = 1/(1+s), 0 \leq s < \infty \tag{2.2}$$

where quartic polynomials $q_4(s), r_4(s)$ are given by

$$\begin{aligned} q_4(s) &= \mu\{2\mu^2(u^2+v^2)+u\}s^4 + \{4\mu^2(u^2+v^2)-1\}s^3 - 3\mu(2u+1)s^2 \\ &\quad - \left[4\mu^2\{(u+1)^2+v^2\}-1\right]s - \mu\left[2\mu^2\{(u+1)^2+v^2\}-(u+1)\right] \\ r_4(s) &= \{s+\mu+\mu u(1-s^2)\}^2 + \{\mu v(1-s^2)\}^2. \end{aligned} \tag{2.3}$$

□

Depending on the signs of the coefficient $a_4(= \mu\{2\mu^2(u^2+v^2)+u\})$ of $s^4$ and the constant term $a_0(= -\mu\left[2\mu^2\{(u+1)^2+v^2\}-(u+1)\right])$ in $q_4(s)$, we consider the four cases in which we shall count of the number of the positive roots of $q_4(s) = 0$:

(i) for $a_4 \geq 0, a_0 \geq 0$ ($\Leftrightarrow (u, v) \in D_1 \cap D_2$); then the coefficients of $s^k, k = 1, 3$ are non-negative as follows

$$(2.4) \quad 4\mu^2(u^2 + v^2) - 1 \geq -(2u+1)\ (> 0), -\Big[4\mu^2\{(u+1)^2 + v^2\} - 1\Big] \geq -(2u+1)\ (> 0)$$

In addition, note the positivity of the coefficient of $s^2$ since $-3\mu(2u+1) > 0$. In this case, all the coefficients of $s^k, 0 \leq k \leq 4$ being nonnegative, Descartes' rule of signs shows that the segment is a spiral.

(ii) for $a_4 \geq 0$, $a_0 < 0(\Leftrightarrow (u, v) \in D_1 \cap D_2^c)$; then the coefficient of $s^3$ being nonnegative as (i), the sequence of the signs of the coefficients of $s^k, 0 \leq k \leq 4$ of ascending order is $(-, ?, +, +, +$ or $0)$ from which combining Descartes' rule of signs and theorem of intermediate value shows that the curvature has just one local maximum; note that $t = 0$ and $t = 1$ correspond to $s = \infty$ and $s = 0$, respectively.

(iii) for $a_4 < 0$, $a_0 \geq 0(\Leftrightarrow (u, v) \in D_1^c \cap D_2)$; the coefficient of $s$ is nonnegative as

$$(2.5) \qquad -\Big[4\mu^2\{(u+1)^2 + v^2\} - 1\Big] \geq 1 - 2(u+1) = -(2u+1) > 0$$

Hence, the sequence of the signs of the coefficients of $s^k, 0 \leq k \leq 4$ is $(+, +, +, ?, -)$, and so combine the rule of signs and theorem of intermediate value to show that the curvature has just one local minimum.

(iv) for $a_4 < 0$, $a_0 < 0$ ($\Leftrightarrow (u, v) \in D_1^c \cap D_2^c$); then the sequence of the signs of the coefficients $s^k, 0 \leq k \leq 4$ is $(-, ?, +, ?, -)$ and $q_4(0) < 0, q_4(1)(= -2\mu(\mu+1)^2(2u+1)\ ) > 0, q_4(\infty) < 0$ which imply that the curvature has first just one local minimum and next just one local maximum as the segment starts at $\boldsymbol{b}_0$ and ends at $\boldsymbol{b}_2$.

**Remark 1.** For $u = -1/2$,

$$(2.6) \qquad q_4(s) = \{4\mu^2(v^2 + \frac{1}{4}) - 1\}(s^2 - 1)\{\frac{\mu}{2}(s^2 + 1) + s\}$$

from which the segment (1.1) is a spiral (circular arc) if $4\mu^2(v^2 + 1/4) - 1 = 0$. If otherwise, it has just one local maximum or minimum. Strictly speaking, the segment has a local maximum (minimum) if $v^2 > (<)\ (1/\mu^2 - 1)/4$.

Since

$$(2.7) \qquad \frac{u+1}{(u+1)^2 + v^2} - \{-\frac{u}{u^2 + v^2}\} = \frac{(2u+1)(u^2 + v^2 + u)}{(u^2 + v^2)\{(u+1)^2 + v^2\}},$$

combine Theorem 2.1 and Remark 1 to obtain

**Remark 2.** For a control vertex $\boldsymbol{b}_1 = (u, v), u \leq -1/2, v > 0$, the segment (1.1) whose curvature is monotone increasing is a spiral if

$$(2.8) \qquad u^2 + v^2 + u \leq 0$$

where the weight $\mu$ ($>0$) must satisfy

$$-\frac{u}{u^2+v^2} \le 2\mu^2 \le \frac{u+1}{(u+1)^2+v^2} \tag{2.9}$$

Here we note that the quadratic segment of the form (1.1) with $\mu = 1$ (when (1.1) reduces to the quadratic polynomial) is a spiral if $2(u^2+v^2)+u \le 0$. Therefore, an introduction of the weight $\mu$ enlarges the region for the rational quadratic segment to be a spiral.

Assume that the the tangent vector rotates counterclockwise as one traverses the segment which starts at $\boldsymbol{b}_0$ with tangent vector $\boldsymbol{t}_0$ at angle $\pi-\theta$, and ends at $\boldsymbol{b}_2$ with tangent vector $\boldsymbol{t}_2$ at angle $\pi+\psi$; note $(\theta,\psi) = (\pi - \arg\boldsymbol{t}_0, -\arg\boldsymbol{t}_2), 0 < \theta, \psi < \pi/2$. Then, Remark 2 gives the necessary and sufficient condition on the angles of the tangent vectors $\boldsymbol{t}_0, \boldsymbol{t}_2$ at $\boldsymbol{b}_0, \boldsymbol{b}_2$ for the the rational quadratic spiral segment as follows.

**Theorem 2.2** *If the rational quadratic segment of the form* (1.1) *satisfies the Hermite interpolation conditions: $z'(0) \parallel \boldsymbol{t}_0, z'(1) \parallel \boldsymbol{t}_2$, it is a spiral whose curvature is monotone increasing if*

$$0 < \theta \le \psi < \pi/2, \quad \theta+\psi \le \pi/2 \tag{2.10}$$

*where the weight $\mu$ ($>0$) must be*

$$\frac{\cos\theta\sin(\theta+\psi)}{\sin\psi} \le 2\mu^2 \le \frac{\cos\psi\sin(\theta+\psi)}{\sin\theta} \tag{2.11}$$

**Proof** By a simple calculation,

$$z'(0) = 2\mu(u,v), \quad z'(1) = -2\mu(1+u,v) \tag{2.12}$$

from which we have with $r_i > 0, i = 1,2$

$$2\mu(u,v) = r_1(-\cos\theta, \sin\theta), \quad -2\mu(1+u,v) = r_2(-\cos\psi, -\sin\psi) \tag{2.13}$$

Solve the above equations for $(u,v)$, and $(r_1,r_2)$ to obtain

$$(u,v) = \frac{\sin\psi}{\sin(\theta+\psi)}(-\cos\theta,\sin\theta), \quad (r_1,r_2) = \frac{2\mu}{\sin(\theta+\psi)}(\sin\psi,\sin\theta) \tag{2.14}$$

Substitute the above $(u,v)$ into (2.9) to obtain (2.11) and note

$$u+\frac{1}{2} = \frac{\sin(\theta-\psi)}{2\sin(\theta+\psi)}, \quad u^2+v^2+u = -\frac{\sin\theta\sin\psi\cos(\theta+\psi)}{\sin^2(\theta+\psi)} \tag{2.15}$$

to have (2.10) (which is geometrically trivial from (2.8)). This completes the proof of Theorem 2.2. □

**Remark 3**. The quadratic segment of the form (1.1) with $\mu = 1$ (i.e., the quadratic polynomial segment) is a spiral whose curvature is monotone increasing if

$$2\sin\theta \le \cos\psi\sin(\theta+\psi) \tag{2.16}$$

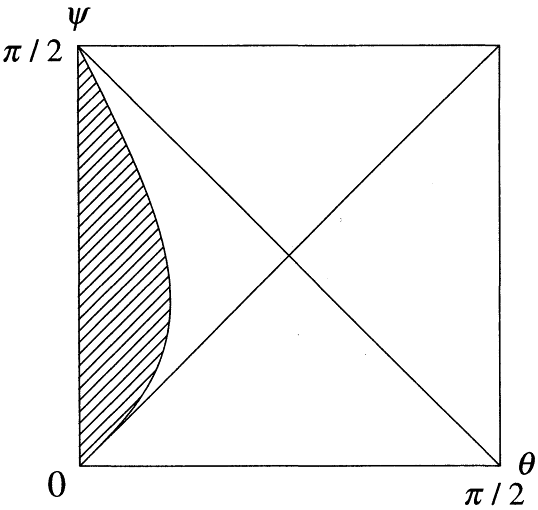


**Figure 1. Angles $(\theta, \psi)$ of tangent vectors at both endpoints for a spiral.**

Figure 1 gives an restriction on the angles $(\theta, \psi)$ of the tangent vectors at the both endpoints $\boldsymbol{b}_0 = (0,0), \boldsymbol{b}_2 = (-1,0)$ for the rational quadratic Bézier segment (1.1) to be a spiral with a monotone increasing curvature where the region $\{(\theta, \psi) \mid 0 < \theta \leq \psi < \pi/2,\ \theta + \psi \leq \pi/2\}$ is divided by the curve: $2\sin\theta = \cos\psi \sin(\theta + \psi)$. Remark 3 means that the dashed region is the one for the quadratic segment ($\mu = 1$) to be a spiral.

By means of Theorem 2.1, we obtain a spiral condition for an offset curve $\boldsymbol{z}_d$ with $\boldsymbol{n}(t)$ the unit normal vector of $\boldsymbol{z}$ at $\boldsymbol{z}(t)$ and its direction outward from the vector $\boldsymbol{z}$

$$\boldsymbol{z}_d(t) = \boldsymbol{z}(t) + d\boldsymbol{n}(t), \quad d \in R \tag{2.17}$$

Note

$$\boldsymbol{n}(t) = (y'(t), -x'(t)) / \|\boldsymbol{z}'(t)\|, \quad \boldsymbol{z}'(t) = (x'(t), y'(t)) \tag{2.18}$$

to obtain

$$\boldsymbol{z}_d'(t) = \{1 + dk(t)\}\boldsymbol{z}'(t), \quad (\boldsymbol{z}_d' \times \boldsymbol{z}_d'')(t) = \{1 + dk(t)\}^2(\boldsymbol{z}' \times \boldsymbol{z}'')(t) \tag{2.19}$$

Hence, we have a condition on radius $d$ for the offset (2.17) to be a spiral.

**Remark 4**. Assume the conditions in Remark 2, i.e., $u^2 + v^2 + u \leq 0,\ u \leq -1/2$. Then the offset curve (2.17) is also a spiral whose curvature is monotone increasing and has the same tangent directions to the one of the segment (1.1) at both endpoints $\boldsymbol{b_0}, \boldsymbol{b_2}$ if and only if

$$d > -1/\max_{0 \leq t \leq 1}\{k(t)\} \Rightarrow d > -\frac{1}{k(0)} \ (= -\frac{2\mu^2(u^2 + v^2)^{3/2}}{v}) \tag{2.20}$$

where $\mu$ satisfies (2.9).

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