SIMULTANEOUS CUBIC HYPER-RESOLUTIONS OF LOCALLY TRIVIAL ANALYTIC FAMILIES OF COMPLEX PROJECTIVE VARIETIES AND COHOMOLOGICAL DESCENT

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Introduction

In [10] the notion of *cubic hyper-resolutions* of algebraic varieties has been introduced, and its cohomological descent property together with several applications has been shown. For example, the mixed Hodge structure on the cohomology of an algebraic variety can be described by use of its cubic hyper-resolution. In this paper we shall consider *simultaneous cubic hyper-resolutions* of *locally trivial* analytic families of complex projective varieties, and prove that they have also cohomological descent property. This might be considerd as a relative analogue of the second author's result in [10, Exposés I, III]. The motivation of this generalization is to describe the variations of mixed Hodge structure arising from *locally trivial* families of complex projective varieties with *ordinary singularites* (for terminology see Definition 1.10 and Definition 2.2 below) by use of simultaneous cubic hyper-resolutions of their fibers. We shall treat the *infinitesimal mixed Torelli problem* for algebraic surfaces with ordinary singularities in a forthcoming paper, using the result of this paper.

Throughout this paper, we shall always work over the complex number field. Our method is basically complex analytic and we shall always regard algebraic manifolds and algebraic varieties over the complex number field as complex manifols and complex analytic varieties. Here we use the term of complex analytic varieties in the sense of reduced complex spaces (possibly not irreducible).

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§1 Simultaneous cubic hyper-resolutions of locally trivial analytic families of complex projective varieties

First, we refer to some terminology and notation from [10]. We denote by $\mathbb{Z}$ the integer ring. For a non-negative integer n, let $\square_n^+$ the *augmented n-cubic category*, i.e., the category whose objects $\mathrm{Ob}(\square_n^+)$ and the set of homomorphisms $\mathrm{Hom}_{\square_n^+}(\alpha,\beta)$ $(\alpha=(\alpha_0,\alpha_1,\cdots,\alpha_n),\ \beta=(\beta_0,\beta_1,\cdots,\beta_n)\in \mathrm{Ob}(\square_n^+))$ are given as follows:

$$\mathrm{Ob}(\square_n^+) := \{\alpha=(\alpha_0,\alpha_1,\cdots,\alpha_n)\in\mathbb{Z}^{n+1} \mid 0\le\alpha_i\le 1 \text{ for } 0\le i\le n\},$$

$$\mathrm{Hom}_{\square_n^+}(\alpha,\beta) := \begin{cases} \alpha\to\beta \text{ (an arrow from } \alpha \text{ to } \beta) & \text{if } \alpha_i\le\beta_i \text{ for } 0\le i\le n \\ \emptyset & \text{otherwise.}\end{cases}$$

For $n=-1$ we define $\square_{-1}^+$ to be the punctual category $\{*\}$, i.e., the category consisting of a single point. For $n\ge 0$ the *n-cubic category*, denoted by $\square_n$, is defined to be the full subcategory of $\square_n^+$ with $\mathrm{Ob}(\square_n)=\mathrm{Ob}(\square_n^+)-\{(0,\cdots,0)\}$. Notice that $\mathrm{Ob}(\square_n^+)$ and $\mathrm{Ob}(\square_n)$ can be considered as finite ordered sets whose order are defined by $\alpha\le\beta \Longleftrightarrow \alpha\to\beta$ for $\alpha,\beta\in\mathrm{Ob}(\square_n^+)$.

1.1 Definition. A $\square_n^+$-object (resp. $\square_n$-object) of a category $\mathcal{C}$ is a contravariant functor $X_\bullet^+$ (resp. $X_\bullet$) from $\square_n^+$ (resp. $\square_n$) to $\mathcal{C}$. It is also called an *augmented n-cubic object of* $\mathcal{C}$ (resp. an *n-cubic object of* $\mathcal{C}$).

1.2 Definition. Let $X_\bullet, Y_\bullet$ be $\square_n^+$-objects (resp. $\square_n$-objects) of a category $\mathcal{C}$. We define a morphism $\Phi_\bullet : X_\bullet\to Y_\bullet$ to be a natural transformation from the functor $X_\bullet$ to the one $Y_\bullet$ over the identity functor $\mathrm{id}:\square_n^+\to\square_n^+$ (resp. $\mathrm{id}:\square_n\to\square_n$).

1.3 Definition. Let $X_\bullet$ be an n-cubic object of $\mathcal{C}$ $(n\ge 0)$, X an object of $\mathcal{C}$. An *augmentation of* $X_\bullet$ *to* X is a natural transformation from the functor

$X_\bullet$ to the one X over the trivial functor $\square_n \to \square^+_{-1}$, where we consider X as a $\square^+_{-1}$-object of $\mathcal{C}$.

1.4 Remark. Notice that we may think of an n-cubic object of $\mathcal{C}$ with an augmentation to X as an augmented n-cubic object of $\mathcal{C}$. Conversely, an augmented n-cubic object $X^+_\bullet : (\square^+_n)^\circ \to \mathcal{C}$ of $\mathcal{C}$ can be identified with an n-cubic object $X_\bullet := X^+_{\bullet|\square_n} : (\square_n)^\circ \to \mathcal{C}$ of $\mathcal{C}$ with an augmentation to $X^+_{(0,\cdots,0)}$, where $\circ$ denotes the dual category.

In what follows we shall interchangeably use an augmented n-cubic object of $\mathcal{C}$ and an n-cubic object of $\mathcal{C}$ with an augmentation.

1.5 Definition. A $\square^+_n$-*complex projective variety* (resp. $\square^+_n$-*complex analytic variety*) is defined to be a $\square^+_n$-object of the category of complex projective varieties (Proj/$\mathbb{C}$) (resp. complex anlytic varieties (An/$\mathbb{C}$)) . It is also called an *augmented n-cubic complex projective variety* (resp. *augmented n-cubic complex analytic variety*).

1.6 Example. Let X be a complex projective variety and $\{X_r\}_{0\le r\le n}$ all of irreducible components of X. For each $\alpha = (\alpha_0, \cdots, \alpha_n) \in \square_n$ we define

$$X_\alpha := \bigcap \{X_i \mid \alpha_i = 1\}.$$

If $\alpha \le \beta$ in $\square_n$, then there is the natural inclusion map $X_\beta \subseteq X_\alpha$. Hence the correspondence $\alpha \in \square_n \to X_\alpha \in$ (Proj/$\mathbb{C}$) defines an n-cubic complex projective variety $X_\bullet : (\square_n)^\circ \to$ (Proj/$\mathbb{C}$). We consider X as a $\square^+_{-1}$-complex projective variety. Then there exists naturally an augmentation $X_\bullet \to X$, which can be considered as an augmented n-cubic complex projective variety (cf. Remark 1.4)

1.7 Definition. For a $\square^+_n$-complex projective variety $X_\bullet$, a contravariant functor $Y_\bullet$ from $\square^+_1$ to the category of $\square^+_n$-complex projective varieties is called a *2-resolution of* $X_\bullet$ if $Y_\bullet$ is defined by a cartesian square of morphisms of $\square^+_n$-complex projective varieties

$$\begin{array}{ccc} Y_{11\bullet} & \longrightarrow & Y_{01\bullet} \\ \downarrow & & \downarrow f \\ Y_{10\bullet} & \longrightarrow & Y_{00\bullet}, \end{array} \tag{1.1}$$

which satisfies the following conditions:

(i) $Y_{00\bullet} = X_\bullet$,
(ii) $Y_{01\bullet}$ is a smooth $\square^+_n$-complex projective variety, i.e., a contravariant functor from $\square^+_n$ to the category of smooth complex projective varieties,
(iii) the horizontal arrows are closed immersions of $\square^+_n$-complex projective varieties,
(iv) f is a proper morphism between $\square^+_n$-complex projective varieties, and
(v) f induces an isomorphism from $Y_{01\beta} - Y_{11\beta}$ to $Y_{00\beta} - Y_{10\beta}$ for any $\beta \in \mathrm{Ob}(\square^+_n)$.

We think of the cartesian square in (1.1) as a morphism from the $\square^+_{n+1}$-complex projective variety $Y_{1\bullet\bullet}$ to the one $Y_{0\bullet\bullet}$ and write it as $Y_{1\bullet\bullet} \to Y_{0\bullet\bullet}$. For a 2-resolution $Z_\bullet$ of $Y_{1\bullet\bullet}$, we define the $\square^+_{n+3}$-complex projective variety $rd(Y_\bullet, Z_\bullet)$ by

$$rd(Y_\bullet, Z_\bullet) := \begin{array}{ccc} Z_{11\bullet} & \longrightarrow & Z_{01\bullet} \\ \downarrow & & \downarrow \\ Z_{10\bullet} & \longrightarrow & Y_{0\bullet\bullet}. \end{array}$$

and call it the *reduction* of $\{Y_\bullet, Z_\bullet\}$.

1.8 Definition. Let X be a complex projective variety and let $\{X^1_\bullet, X^2_\bullet, \cdots, X^n_\bullet\}$ be a sequence of $\square^+_r$-complex projective varieties $X^r_\bullet$ $(1 \leq r \leq n)$ such that

(i) $X^1_\bullet$ is a 2-resolution of X,

(ii) $X^{r+1}_\bullet$ is a 2-resolution of $X^r_{1\bullet}$ for every r with $1 \leq r \leq n-1$.

Then, by induction on n, we define

$$Z_\bullet := rd(X^1_\bullet, X^2_\bullet, \cdots, X^n_\bullet) := rd(rd(X^1_\bullet, X^2_\bullet, \cdots, X^{n-1}_\bullet), X^n_\bullet).$$

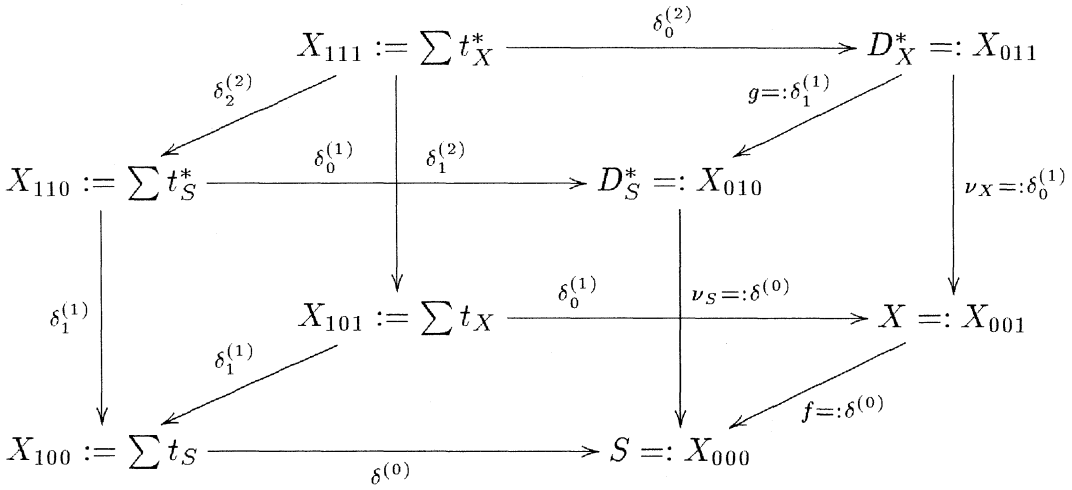
With this notation, if Z_α are smooth for all $\alpha \in \mathrm{Ob}(\square_n)$, we call $Z_\bullet$ an *augmented n-cubic hyper-resolution* of X.

1.9 Example. A 2-dimensional complex projective varity is said to be *with ordinary singularities* if it is locally isomorphic to one of the following germs of hypersurfaces of the complex 3-space $\mathbb{C}^3$:

$$\begin{cases} (i)\ z = 0 \ (\text{simple point}), & (ii)\ yz = 0 \ (\text{ordinary double point}), \\ (iii)\ xyz = 0 \ (\text{ordinary triple point}), & (iv)\ xy^2 - z^2 = 0 \ (\text{cuspidal point}), \end{cases}$$

where (x, y, z) is the coordinate on $\mathbb{C}^3$. We fix notation as follows:

Let S be a complex projective surface with ordinary singularities. We denote by D_S the singular locus of S, and call it the *double curve* of S. D_S is a singular curve with triple points. We denote by Σt_S the triple point locus of S, and by $\Sigma \mathfrak{c}_S$ the cuspidal point locus of S. Let $f : X \to S$ be the normalization. Note that X is non-singular. We put $D_X := f^{-1}(D_S)$ and $\Sigma t_X := f^{-1}(\Sigma t_S)$. D_X is a singular curve with nodes and Σt_X coincides with the set of nodes of D_X. We denote by $n_S : D^*_S \to D_S$ and $n_X : D^*_X \to D_X$ the normalizations of D_S and D_X, respectively. We denote by $g : D^*_X \to D^*_S$ the lifting of the map $f_{|D_X} : D_X \to D_S$. We put $\Sigma t^*_S := n_S^{-1}(\Sigma t_S)$, $\Sigma \mathfrak{c}^*_S := n_S^{-1}(\Sigma \mathfrak{c}_S)$ and $\Sigma t^*_X := n_X^{-1}(\Sigma t_X)$. With this natation, we have a *2-cubic hyper-resolution* of S as follows:



where ν_S and ν_X are the composites of the normalizations $n_S : D_S^* \to D_S$ and $n_X : D_X^* \to D_X$ and the inclusion maps $D_S \hookrightarrow S$ and $D_X \hookrightarrow X$, respectively, and the square on the left-hand side is the one induced from the square on the right-hand side.

The important property of a cubic hyper-resolution is *cohomological descent*. There are two sorts of *cohomological descent*; one is that of R-module sheaves (R:a commutative ring with identity element 1, especially R=$\mathbb{Z}, \mathbb{Q}$ and $\mathbb{C}$)([10, p.41, Théorème 6.9]) and the second is that of de Rham complexes ([10, p.61, Théorème 1.3]).

Now we are going to give the definitions of *locally trivial* analytic families of complex projecitive varieties (resp. complex analytic varieties) and their *simultaneous cubic hyper-resolutions*.

1.10 Definition. By an analytic family of complex projective varieties (resp. complex analytic varieties), parametrized by a complex space M, we mean a triple $(\mathfrak{X}, \pi, M)$ satisfying the following conditions:

(i) π: $\mathfrak{X} \to M$ is a flat surjective holomorphic map of complex spaces,and

(ii) $X_t := \pi^{-1}(t)$ is a complex projective variety (resp. complex analytic variety) for any $t \in M$.

Let $(\mathfrak{X}, \pi, M)$ and $(\mathfrak{X}', \pi', M)$ be analytic families of complex projective varieties (resp. complex analytic varieties) parametrized by the same complex space M.

1.11 Definition. By a morphism (resp. an isomorphism) for $(\mathfrak{X}, \pi, M)$ to $(\mathfrak{X}', \pi', M)$ we mean a holomorphic (resp. biholomorphic) map H: $\mathfrak{X} \to \mathfrak{X}'$ such

that the diagram

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{H} & \mathfrak{X}' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow{\mathrm{id}_M} & M \end{array}$$

commutes, where id_M is the identity map on M.

1.12 Definition. An analytic family of complex projective varieties (resp. complex analytic varieties) $(\mathfrak{X}, \pi, M)$ is said to be *locally trivial* if it satisfies the following condition: for every point $p \in \mathfrak{X}$, there exist open neighborhoods $\mathcal{U}$ of p in $\mathfrak{X}$, V of $\pi(p)$ in M with $\pi(\mathcal{U}) = V$, and a biholomorphic map $\phi : \mathcal{U} \longrightarrow U \times V$, where we define $U := \mathcal{U} \cap X_{\pi(p)}$, such that;

(a) the diagram

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\phi} & U \times V \\ & {\scriptstyle\pi|_{\mathcal{U}}}\searrow \quad \swarrow{\scriptstyle Pr_V} & \\ & V & \end{array}$$

commuts,

(b) $\phi_{|U} := id_U$.

We denote by $\mathcal{F}_M(\mathrm{Proj}/\mathbb{C})$ (resp. $\mathcal{F}_M(\mathrm{An}/\mathbb{C})$) the category of analytic families of complex projective (resp. analytic) varieties, parametrized by a complex space M.

1.13 Definition. We call a $\square_n^+$-object (resp. $\square_n$-object) of $\mathcal{F}_M(\mathrm{Proj}/\mathbb{C})$, or of $\mathcal{F}_M(\mathrm{An}/\mathbb{C})$, an *analytic family of augmented n-cubic (resp. n-cubic) complex projective varieties, or complex analytic varieties, parametrized by a complex space* M.

Let $b_\bullet : X_\bullet \longrightarrow X$ be an augmented n-cubic complex projective (resp. analytic) variety and M a complex space. Then $X_\alpha \times M (\alpha \in \square_n)$, $X \times M$, $a_\alpha := b_\alpha \times \mathrm{id}_M : X_\alpha \times M \longrightarrow X \times M$ and $\pi := \mathrm{Pr}_M : X \times M \longrightarrow M$, the projection to M consititute an analytic family of augmented n-cubic complex projective (resp. analytic) varieties , parametrized by a complex space M, which we denote by

$$X_\bullet \times M \xrightarrow{a_\bullet := b_\bullet \times \mathrm{id}_M} X \times M \xrightarrow{\pi := \mathrm{Pr}_M} M$$

and call the *product family of augmented n-cubic complex projective (resp. analytic) varieties, parametrized by a complex space* M. Let $\mathfrak{X}_\bullet^+ = \{a_\bullet : \mathfrak{X}_\bullet \longrightarrow \mathfrak{X}\}$ be an analytic family of augmented n-cubic complex projective (resp. analytic) varieties (for notation see Remark 1.4 above), parametrized by a complex space M. Whenever we wish to express its parameter space M explicitly, we write

$$\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M. \tag{1.2}$$

For $t \in M$, $X_{\alpha t} := (\pi \circ a_\alpha)^{-1}(t)$ $(\alpha \in \square_n)$, $X_t := \pi^{-1}(t)$ and $a_{\alpha t} := a_{\alpha|X_{\alpha t}} : X_{\alpha t} \to X_t$ constitute an augmented n-cubic complex projective (resp. analytic) variety, which we denote by $X_{\bullet t} \xrightarrow{a_{\bullet t}} X_t$ and call the fiber at $t \in M$ of an analytic family of augmented n-cubic complex projective (resp. analytic) varieties in (1.2). For an open subset U of $\mathfrak{X}$, we form an analytic family

$$a_\bullet^{-1}(\mathcal{U}) \xrightarrow{a_{\bullet|a_\bullet^{-1}(\mathcal{U})}} \mathcal{U} \xrightarrow{\pi} \pi(\mathcal{U})$$

of augmented n-cubic *analytic varieties*, parametized by a complex space $\pi(\mathcal{U})$. With these notions, we define a *simultaneous cubic hyper-resolution* of a locally trivial analytic family of complex projective varieties, parametrized by a complex space as follows:

1.14 Definition. Let $\pi : \mathfrak{X} \longrightarrow M$ be a *locally trivial* analytic family of complex projective varieties, parametrized by a complex space M. A *simultaneous* (n-) *cubic hyper-resolution* of the family $\pi : \mathfrak{X} \to M$ is defined to be an analytic family $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$ of augmented n-cubic complex projective varieties with a certain non-negative integer n, parametrized by the complex space M, which satisfies the following conditions:

(i) for any pont $t \in M$, $a_{\bullet t} : X_{\bullet t} \to X_t$ is an augmented n-cubic hyper-resolution of X_t,

(ii)(analytical "local triviality") for any point $p \in \mathfrak{X}$, there exists an open neighborhood $\mathcal{U}$ of p in $\mathfrak{X}$ such that $a_\bullet^{-1}(\mathcal{U}) \xrightarrow{a_{\bullet|a_\bullet^{-1}(\mathcal{U})}} \mathcal{U} \xrightarrow{\pi} \pi(\mathcal{U})$ is analytically isomorphic to

$$(a_\bullet^{-1}(\mathcal{U}) \bigcap X_{\bullet\pi(p)}) \times \pi(\mathcal{U}) \to (\mathcal{U} \bigcap X_{\pi(p)}) \times \pi(\mathcal{U}) \xrightarrow{\mathrm{Pr}_{\pi(\mathcal{U})}} \pi(\mathcal{U})$$

over the identy map $\mathrm{id}_{\pi(\mathcal{U})} : \pi(\mathcal{U}) \to \pi(\mathcal{U})$

If the parameter space M of a locally trivial analytic family $\pi : \mathfrak{X} \longrightarrow M$ of complex projective varieties is smooth, we have the following theorems.

1.15 Theorem. *Let $\pi : \mathfrak{X} \longrightarrow M$ be a locally trivial analytic family of complex projective varieties, parametrized by a complex manifold M, and $a_\bullet : \mathfrak{X}_\bullet \longrightarrow \mathfrak{X}$ the canonical cubic hyper-resolution of $\mathfrak{X}$. Here "canonical" means in the sense of Bierstone-Millman ([2]). Then $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$ is a simultaneous cubic hyper-resolution of $\pi : \mathfrak{X} \longrightarrow M$.*

Proof. The construction of the canonical hyper-resolution of $\mathfrak{X}$ is obtained, using the canonical process of desingularisation in the proof of the existence of the resolution of a diagram of complex projective varieties (or compact complex analytic varieties) (cf. [10, Théorème 2.6]). Then, because of the hypothesis of locally triviality of $\pi : \mathfrak{X} \longrightarrow M$, the fibre $X_{\bullet t} \longrightarrow X_t$ for each $t \in M$ is also the canonical hyper-resolution. Hence $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$ is a simultaneous cubic hyper-resolution of the family $\pi : \mathfrak{X} \longrightarrow M$ in the sense of Definition 1.14.

Q.E.D.

1.16 Theorem. *Let $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$ be a simultaneous n-cubic hyper-resolution of locally trivial analytic family $\pi : \mathfrak{X} \to M$ of complex projective varieties, parametrized by a complex manifold M. Then the $\square_n$-object $\pi_\bullet : \mathfrak{X}_\bullet \to M (\pi_\bullet := \pi \circ a_\bullet)$ of smooth families of complex manifolds, parametrized by M is C^∞ trivial at any point of M; that is, for any point $t_0 \in M$, there exist an open neighborhood N of t_0 in M and a diffeomorphism $\Phi_\bullet : (\pi_\bullet^{-1})(N) \to X_{\bullet t_0} \times N$ of $\square_n$-objects of complex manifolds over the identy map $\mathrm{id}_N : N \to N$. Furthermore, $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$ is topologically trivial at any point of M.*

Proof. Let N_1 be a coordinate neighborhood of t_0 in M with a holomorphic local coordinate system $(t_1, \cdots, t_m)$, and N a relatively compact open subset of N_1 with $\overline{N} \subset N_1$. Let $t_i = x_i + \sqrt{-1}x_{m+i} (1 \leq i \leq m)$ be the expression of t_i in real local coordinate functions x_i, y_i. To prove the theorem it suffices to show that for every $\partial/\partial x_i$ $(1 \leq i \leq 2m)$ and every $\alpha \in \square_n$ there exists its liftings v_i^α to $\pi_\alpha^{-1}(N)$, i.e., a C^∞ vector field on $\pi_\alpha^{-1}(N)$ with the property

$$(d\pi_\alpha)(v_i^\alpha) = \pi_\alpha^*(\frac{\partial}{\partial x_i}),$$

subject to the requirement

$$dE_{\alpha\beta}(v_i^\beta) = E_{\alpha\beta}^*(v_i^\alpha) \tag{1.3}$$

in $E_{\alpha\beta}^*\Theta_{\mathfrak{X}_\alpha}$ for every pair (α, β) of elements of $\mathrm{Ob}(\square_n)$ with $\alpha \leq \beta$ in the category $\square_n$, where $E_{\alpha\beta} : \mathfrak{X}_\beta \to \mathfrak{X}_\alpha$ denotes a holomorphic map corresponding to an arrow $\alpha \to \beta$ in $\square_n$ and $\Theta_{\mathfrak{X}_\alpha}$ the sheaf of germs of holomorphic vector fields on $\mathfrak{X}_\alpha$. In fact, if such liftings $\{v_i^\alpha\}_{\alpha \in \square_n}$ exist, integrating v_i^α, we have a C^∞-trivialization of the family $\pi_\alpha : \mathfrak{X}_\alpha \to N$ along the x_i-axis in N for all $\alpha \in \square_n$ such that those trivializations commute with the maps $E_{\alpha\beta} : \mathfrak{X}_\beta \to \mathfrak{X}_\alpha$ for every pair (α, β) of elements of $\mathrm{Ob}(\square_n)$ with $\alpha \to \beta$ in the category $\square_n$ due to the requiement (1.3). Arguing inductively on the dimension of M, we finally get the trivialization asserted in the proposition (cf. for more precise argument we refer to Theorem 3.3 in [8]). Now we are going to prove the existence of the liftings v_i^α to $\pi_\alpha^{-1}(N)$ of $\partial/\partial x_i$ subject to the requirement (1.3).

We take open coverings $\mathcal{V} = \{\mathcal{V}_\lambda\}_{\lambda \in \Lambda_0}$ and $\mathcal{V}' = \{\mathcal{V}'_\lambda\}_{\lambda \in \Lambda_0}$ of $\pi^{-1}(\overline{N})$ in $\mathfrak{X}$ that satisfy the following conditions:

for every $\lambda \in \Lambda_0$,

(i) $\overline{\mathcal{V}_\lambda}$ is a compact subset of $\mathcal{V}'_\lambda$,

(ii) there exists an embedding $\varphi_\lambda : \mathcal{V}'_\lambda \to \mathbb{C}^{n_\lambda}$, and

(iii) $a_\bullet^{-1}(\mathcal{V}'_\lambda) \xrightarrow{a_\bullet} \mathcal{V}'_\lambda \xrightarrow{\pi} \pi(\mathcal{V}'_\lambda)$ is analytically trivial.

We are allowed to put the condition (iii) due to the analytical "local triviality" of the family $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$ (cf. Definition 1.14 (ii)). By this condition there exist liftings $v^\alpha_{\lambda i}$ of $\partial/\partial x_i$ to $a_\alpha^{-1}(\mathcal{V}'_\lambda)$ for every $\alpha \in \square_n$ and every $\lambda \in \Lambda_0$, subject to the requirement (1.3). We take a C^∞ partition of unity $\{\rho_\lambda\}_{\lambda\in\Lambda_0}$ on $\mathfrak{X}' := \bigcup_{\lambda\in\Lambda_0} \mathcal{V}'_\lambda$ subordinate to the covering $\mathcal{V} = \{\mathcal{V}_\lambda\}_{\lambda\in\Lambda_0}$, i.e., ρ_λ's are "C^∞ functions" on $\mathfrak{X}' := \bigcup_{\lambda\in\Lambda_0} \mathcal{V}'_\lambda$ satisfying the following conditions:

(i) $0 \leq \rho_\lambda \leq 1$ for $\lambda \in \Lambda_0$,

(ii) $Supp\, \rho_\lambda \subset \mathcal{V}_\lambda$ for $\lambda \in \Lambda_0$,

(iii) $\sum_{\lambda\in\Lambda_0} \rho_\lambda \equiv 1$ on $\mathfrak{X}'$.

Notice that $\mathfrak{X}'$ is a singular space. We use here the term "C^∞ functions" in the sense of that they are locally pull-backs of C^∞ functions on $\mathbb{C}^{n_\lambda}$ via embeddings $\varphi_\lambda : \mathcal{V}'_\lambda \to \mathbb{C}^{n_\lambda}$. The existence of C^∞-partition of unity $\{\rho_\lambda\}_{\lambda\in\Lambda_0}$ as above is guaranteed by the fact that the proof of the existence of C^∞-partition of unity subordinate to a countably indexed open covering of a C^∞-manifold is also applicable in our case (cf.[8, Chapter I, Theorem 4.6]). We define

$$v^\alpha_i := \sum_{\lambda\in\Lambda_0} a^*_\alpha(\rho_\lambda) v^\alpha_{\lambda i}$$

for $\alpha \in \square_n$. Then we can easily check that

$$(d\pi_\alpha)(v^\alpha_i) = \pi^*_\alpha(\frac{\partial}{\partial x_i}) \quad \text{and}$$

$$(dE_{\alpha\beta})(v^\beta_i) = E^*_{\alpha\beta}(v^\alpha_i)$$

for every pair (α, β) of elements of $Ob(\square_n)$ with $\alpha \leq \beta$ in the category $\square_n$.

Finally, we shall show that the C^∞ triviality of the family $\pi_\bullet : \mathfrak{X}_\bullet \to M$ implies the topological triviality of the family $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$. For a fiber $X_{\bullet t}$ $(t \in M)$ of the family $\pi_\bullet : \mathfrak{X}_\bullet \to M$, we define an equivalence relation on the topological space $\coprod_{\alpha\in\square_n} X_{\alpha t}$ (disjoint sum) by

$$p \sim q \text{ iff } p \in X_{\alpha t}, q \in X_{\beta t} \text{ such that } \begin{cases} \alpha \leq \beta & \text{and } e_{\alpha\beta}(q) = p \\ \text{or} \quad \alpha > \beta & \text{and } e_{\beta\alpha}(p) = q, \end{cases}$$

where $e_{\alpha\beta} : X_{\beta t} \to X_{\alpha t}$ (resp. $e_{\beta\alpha} : X_{\alpha t} \to X_{\beta t}$) is the holomorphic map corresponding to an arrow $\alpha \to \beta$ (resp. $\beta \to \alpha$) in $\square_n$. Then the natural map from $(\coprod_{\alpha\in\square_n} X_{\alpha t}/\sim)$ (the quotient topological space of $\coprod_{\alpha\in\square_n} X_{\alpha t}$ by the equivalence relation $\sim$ defined above) to X_t gives rise to a homeomorphism between these spaces, because $X_{\bullet t}$ is a cubic hyper-resolution of X_t. Therefore a diffeomorphism between different fibers $X_{\bullet t}$ and $X_{\bullet t'}(t, t' \in M)$ gives rise to

a homeomorphism between different fibers $X_{\bullet t} \to X_t$ and $X_{\bullet t'} \to X_{t'}$ of the family $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$.

Q.E.D.

§2 Examples

In this section we shall show that we can obtain a simultaneous cubic hyperresolution of a locally trivial family of complex projective varieties with *ordinary singularities* of dimension ≤ 3 as well as of a locally trivial family of complex projective varieties with normal crossing of any dimension by taking normalizations of their fibers successively. Though, using the local equations of ordinary singularities obtained in [15], we can prove that the same statement holds for locally trivial families of complex projective varieties with ordinary singularities of dimension 4 and 5, we omit its proof (for the case of dimension 4 see [19, Example 4.2.10]).

By definition a 1-dimensional complex projective varietiy with ordinary singularities is no more than a curve with nodes (possibly reducible). The definition of 2-dimensional complex projective varieties with ordinary singularities has been given in Example 1.7.

2.1 Definition. A 3-dimensional complex projective variety is said to be *with ordinary singularities* if it is locally isomorphic to one of the germs of hypersurfaces of the complex 4-space $\mathbb{C}^4$ as follows:

(2.1)
$$\begin{cases} (i)\ w = 0\ (\text{simple point}), & (ii)\ zw = 0\ (\text{ordinry double point}), \\ (iii)\ yzw = 0\ (\text{ordinary triple point}), & (iv)\ xyzw = 0\ (\text{ordinary quadruple point}), \\ (v)\ xy^2 - z^2 = 0\ (\text{cuspidal point}), & (vi)\ w(xy^2 - z^2) = 0\ (\text{stationary point}), \end{cases}$$

where (x, y, z, w) is the coordinate on $\mathbb{C}^4$.

2.2 Definition. By a *locally trivial* analytic family of complex projective varieties with *ordinary singularities*, parametrized by a complex space M, we mean a *locally trivial* analytic family $\pi : \mathfrak{X} \to M$ of complex projective varieties all of whose fibers $X_t := \pi^{-1}(t)$ are complex projective varieties with *ordinary singularities*.

Now we are going to show that we can obtain a simultaneous cubic hyperresolution of a locally trivial analytic family of complex projective varieties with *ordinary singularities* of dimension ≤ 3 by taking normalizations of their fibers successively. Our arguments in the subsequence are rather "set-theoretical" (not scheme-theoretic) and all complex analytic varieties and subvarieties are assumed to be reduced. First, we introduce a general notion and mention a fundamental fact on it, which will be needed later. Let I be a finite ordered set. Remember

that $\mathrm{Ob}(\square_n^+)$ and $\mathrm{Ob}(\square_n)$ can be considered as finite ordered sets. We think of I as a category. Let $X_\bullet : I^\circ \to (\mathrm{An}/\mathbb{C})$ be an I-object of complex analytic varieties, that is, a contravariant functor from the category I to the category $(\mathrm{An}/\mathbb{C})$ of complex analytic varieties. We shortly call an I-object of complex analytic varieties an *I-complex analytic variety*.

2.3 Definition. A morphism of I-complex analytic varieties $f_\bullet : X_\bullet \to Y_\bullet$ is defined to be a *normalization* of $Y_\bullet$ if $f_i : X_i \to Y_i$ is the normalization for every $i \in I$.

For an I-complex analytic variety $X_\bullet$, we denote by $e_{ij} : X_j \to X_i$ the corresponding holomorphic map to $i, j \in I$ with $i \leq j$, and by $N(X_i)$ the non-normal locus of X_i for each $i \in I$.

2.4 Lemma. *With the same notation as above, for an I-complex analytic varity $X_\bullet$ we assume that $e_{ij}^{-1}(N(X_i))$ is analytically rare in X_j, i.e., for any open subset U of X_j the restriction map $\mathcal{O}_{X_j}(U) \to \mathcal{O}_{X_j}(U \setminus e_{ij}^{-1}(N(X_i)))$ is injective, for every $i, j \in I$ with $i \leq j$. Then there exists a normalization $\nu_\bullet : X_\bullet^\nu \to X_\bullet$ of $X_\bullet$ and it is unique up to isomorphisms in the category of I-complex analytic varieties over the identity map $\mathrm{id}_{X_\bullet} : X_\bullet \to X_\bullet$.*

Proof. For any $i \in I$ we take the normalization $\nu_i : X_i^\nu \to X_i$. By the assumption, every $e_{ij} : X_j \to X_i$ for $i, j \in I$ with $i < j$ can be uniquely lifted to $e_{ij}^\nu : X_j^\nu \to X_i^\nu$ ([6, p.121, Proposition 2.28]). Then $\{X_i^\nu, e_{ij}^\nu\}$ constitutes an I-complex analytic variety due to the uniqueness of the liftings e_{ij}^ν, and by definition, this is a normalization of $X_\bullet$. The uniqueness of $X_\bullet^\nu$ up to isomorphisms over the identity map $\mathrm{id}_{X_\bullet} : X_\bullet \to X_\bullet$ results from the uniqueness of each X_i^ν up to isomorphisms over the identity map $\mathrm{id}_{X_i} : X_i \to X_i$ for every $i \in I$.

Q.E.D.

2.5 Definition. For a morphism of I-complex analytic varieties $f_\bullet : X_\bullet \to Y_\bullet$, the *discriminant* of $f_\bullet$ is defined to be the smallest, closed I-complex analytic subvariety $D_{f_\bullet}$ of $Y_\bullet$ such that $f_\bullet$ induces an isomorphism $f_i : X_i - f_i^{-1}(D_{f_i}) \to Y_i - D_{f_i}$ for every $i \in I$.

2.6 Remark. Let $f_\bullet : X_\bullet \to Y_\bullet$ be a proper morphism of I-complex analytic varieties, i.e., $f_i : X_i \to Y_i$ is proper for every $i \in I$. Then one has

$$D_{f_i} = \cup_{i \to j} Im(T_j \to Y_i) \quad (i \in I),$$

where T_j denotes the discriminant of $f_j : X_j \to Y_j$ (cf. [10, p.9, Proposition 2.3]).

The notion of *a cubic hyper-resolution of a complex analytic variety being obtained by successive normalizations* is defined as follows:

Let X be a complex analytic variety. First, we define a $\square_1^+$-complex analytic variety $X_\bullet^1$ to be

$$(2.2)_1 \qquad \begin{array}{ccc} X^1_{11} := D^*_{\nu_1} & \xrightarrow{j_1} & X^\nu =: X^1_{01} \\ {\scriptstyle \mu_1}\downarrow & & \downarrow{\scriptstyle \nu_1} \\ X^1_{10} := D_{\nu_1} & \underset{i_1}{\longrightarrow} & X =: X^1_{00}, \end{array}$$

where $\nu_1 : X^\nu \to X$ is the normalization of X, D_{ν_1} the discriminant of ν_1, $D^*_{\nu_1} := \nu_1^{-1}(D_{\nu_1})$, and $\mu_1 := \nu_{1|D^*_{\nu_1}} : D^*_{\nu_1} \to D_{\nu_1}$ the restriction of ν_1 to $D^*_{\nu_1}$. Inductively, for an integer $r \geq 2$ we define $\square^+_r$-complex analytic variety $X^r_\bullet$ to be

$$(2.2)_r \qquad \begin{array}{ccc} X^r_{11\bullet} := D^*_{\nu_r} & \xrightarrow{j_r} & (X^{r-1}_{1\bullet})^\nu =: X^r_{01\bullet} \\ {\scriptstyle \mu_r}\downarrow & & \downarrow{\scriptstyle \nu_r} \\ X^r_{10\bullet} := D_{\nu_r} & \underset{i_r}{\longrightarrow} & X^{r-1}_{1\bullet} =: X^r_{00\bullet}, \end{array}$$

where $X^{r-1}_{1\bullet}$ is the $\square^+_{r-2}$-complex analytic variety, $\mu_{r-1} := \nu_{r-1|D^*_{\nu_{r-1}}} : X^{r-1}_{11\bullet} := D^*_{\nu_{r-1}} \to X^{r-1}_{10\bullet} := D_{\nu_{r-1}}$ in $(2.2)_{r-1}$, $\nu_r : (X^{r-1}_{1\bullet})^\nu \to X^{r-1}_{1\bullet}$ is the normalization of $X^{r-1}_{1\bullet}$, D_{ν_r} is the discriminant of ν_r, $D^*_{\nu_r} := \nu_r^{-1}(D_{\nu_r})$, and $\mu_r := \nu_{r|D^*_{\nu_r}}$ is the restriction of ν_r to $D^*_{\nu_r}$.

2.7 Definition. In the above procedure we assume that the normalization $(X^{r-1}_{1\bullet})^\nu$ is always non-singular for every $r \geq 1$, where we understand $X^0_{1\bullet} = X$. Then, after finite steps, say n-th step, the reduction

$$Z_\bullet := rd(X^1_\bullet, X^2_\bullet, \cdots, X^n_\bullet)$$

of the sequence $\{X^1_\bullet, X^2_\bullet, \cdots, X^n_\bullet\}$ of $\square^+_r$-complex analytic varieties $X^r_\bullet$ $(1 \leq r \leq n)$ gives an augmented n-cubic hyper-resolution of X. If this is the case, we say that a *cubic hyper-resolution of X is obtained by successive normalizations.*

2.8 Definition. We say a complex analytic varity Z is *with normal crossing* if, at each point of Z, it is locally isomorphic to the germ of a subvariety $\{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} | z_0 \cdots z_r = 0\}$ at the origin of $\mathbb{C}^{n+1}$ for some r $(0 \leq r \leq n)$.

2.9 Proposition. *For a complex analytic variety with normal crossing its cubic hyper-resolution is obtained by successive normalizations.*

Proof. Since the problem is local, it suffices to show that, for the subvariety Z in $\mathbb{C}^{n+1}$ defined by $z_0 \cdots z_r = 0$ $(0 \leq r \leq n)$ its cubic hyper-resolution is obtained by successive normalizations. Furthermore, we may assume that $r = n$, because the subvariety $\{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} | z_0 \cdots z_r = 0\}$ is isomorphic to the product $\{(z_0, \cdots, z_r) \in \mathbb{C}^{r+1} | z_0 \cdots z_r = 0\} \times \mathbb{C}^{n-r}$. In fact, we shall prove the following by double induction on n and k.

Claim. *For the analytic subvariety*

$$Z = \{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} | z_0 \cdots z_n = 0\},$$

we define

$$Z_{(z_0 \cdots \check{z}_{i_0} \cdots \check{z}_{i_k} \cdots z_n)} := \{(z) \in \mathbb{C}^{n+1} | z_{i_0} = \cdots = z_{i_k} = 0\} \ (0 \leq i_0 < \cdots < i_k \leq n)$$

and

$$Z_k^{(n)} := \cup_{0 \leq i_0 < \cdots < i_k \leq n} Z_{(z_0 \cdots \check{z}_{i_0} \cdots \check{z}_{i_k} \cdots z_n)} \ (0 \leq k \leq n) \ (a\, subvariety\ of\ \mathbb{C}^{n+1})$$

Then a cubic hyper-resolution of $Z_k^{(n)}$ $(0 \leq n, 0 \leq k \leq n)$ *is obtained by successive normalizations.*

Proof of the claim.

(I) In the case of $n = k = 0$: $Z_0^{(0)}$ is non-singular (a single point), so there is nothing to be proved.

(II) In the case of $n \geq 1$: we assume that the claim is true for $Z_\ell^{(m)}$ with $0 \leq m \leq n-1$ and $0 \leq \ell \leq m$. $Z_n^{(n)}$ is non-singular (a single point), so there is nothing to be proved. Next we shall show that if the claim is true for $Z_\ell^{(n)}$ with $0 \leq k < \ell \leq n$, then it is also true for $Z_k^{(n)}$. We consider the 2-resolution

$$\begin{array}{ccc} D_{\nu_1}^* & \xrightarrow{j_1} & (Z_k^{(n)})^\nu \\ {\scriptstyle \mu_1}\downarrow & & \downarrow{\scriptstyle \nu_1} \\ D_{\nu_1} & \underset{i_1}{\longrightarrow} & Z_k^{(n)} \end{array}$$

in $(2.2)_1$ for $Z_k^{(n)}$. Then

$$(Z_k^{(n)})^\nu = \coprod_{0 \leq i_0 < \cdots < i_k \leq n} Z_{(z_0 \cdots \check{z}_{i_0} \cdots \check{z}_{i_k} \cdots z_n)} \ \text{(disjoint sum)}$$

$$D_{\nu_1} = Z_{k+1}^{(n)}, \ \text{and}$$

$$D_{\nu_1}^* = \coprod_{0 \leq i_0 < \cdots < i_k \leq n} \cup_{\substack{i \neq i_0, \cdots, i_k \\ 0 \leq i \leq n}} Z_{(z_0 \cdots \check{z}_{i_0} \cdots \check{z}_{i_k} \cdots \check{z}_i \cdots z_n)}.$$

Here we consider

$$(2.3) \qquad \cup_{\substack{i \neq i_0, \cdots, i_k \\ 0 \leq i \leq n}} Z_{(z_0 \cdots \check{z}_{i_0} \cdots \check{z}_{i_k} \cdots \check{z}_i \cdots z_n)}$$

as a subvariety of $Z_{(z_0 \cdots \check{z}_{i_0} \cdots \check{z}_{i_k} \cdots z_n)}$. By the induction hypothesis, a cubic hyper-resolution of $D_{\nu_1} = Z_{k+1}^{(n)}$ is obtained by successive normalizations, which we denote by $\nu_{1\bullet} : D_{\nu_1 \bullet} \to D_{\nu_1}$ (an augmented $\square_{n-k-1}^+$-object of complex analytic

varieties). Since the complex analytic variety in (2.3) is isomorphic to $Z_0^{(n-k-1)}$ for every $(i_0, \cdots, i_k)$ with $0 \leq i_0 < \cdots < i_k \leq n$, by the induction hypothesis, a cubic hyper-resolution of $D^*_{\nu_1}$ is also obtained by successive normalizations, which we denote by $\nu_{1\bullet}^* : D^*_{\nu_1 \bullet} \to D^*_{\nu_1}$ (an augmented $\square^+_{n-k-1}$-object of complex analytic varieties). Obviously, there naturally exists a homomorphism $\mu_{1\bullet} : D^*_{\nu_1 \bullet} \to D_{\nu_1 \bullet}$ of $\square_{n-k-1}$-objects of complex analytic subvarieties such that the following diagram commutes:

$$\begin{array}{ccc} D^*_{\nu_1 \bullet} & \xrightarrow{\nu_{1\bullet}^*} & D^*_{\nu_1} \\ {\scriptstyle \mu_{1\bullet}}\downarrow & & \downarrow{\scriptstyle \mu_1} \\ D_{\nu_1 \bullet} & \xrightarrow[\nu_{1\bullet}]{} & D_{\nu_1}, \end{array}$$

of which we think as a $\square_{n-k+1}$-object of complex analytic varieties. This is nothing but the cubic hyper-resoluion of the $\square_0^+$-complex analytic variety $\mu_1 : D^*_{\nu_1} \to D_{\nu_1}$ by successive normalizations. Therefore,

$$\begin{array}{ccc} D^*_{\nu_1 \bullet} & \xrightarrow{j_1 \circ \nu_{1\bullet}^*} & (Z_k^{(n)})^\nu \\ {\scriptstyle \mu_1}\downarrow & & \downarrow{\scriptstyle \nu_1} \\ D_{\nu_1 \bullet} & \xrightarrow[i_1 \circ \nu_{1\bullet}]{} & Z_k^{(n)}, \end{array}$$

is the cubic hyper-resolution of $Z_k^{(n)}$ by successive normalizations. This completes the proof of the claim.

Since $Z = Z_0^{(n)}$, the proposition follows from this claim.

Q.E.D.

2.10 Proposition. *A cubic hyper-resolution of a complex analytic variety with ordinary singularities of dimension* ≤ 3 *is obtained by successive normalizations.*

Proof. The proof is straightforward caluculation in terms of local coordinates. We shall show only in the case of dimension 3. First we fix notation as follows:

T : a threefold with ordinary singularities,
S : the singular locus of T,
Δ : the singular locus of S,
Σq : the set of ordinary quadruple points of T,
Σs : the set of stationary points of T.

Notice that Δ is non-singular outside Σq and that, at each point of Σq, Δ is isomorphic to the union of four coordinate axes of $\mathbb{C}^4$ at the origin. It suffices to prove the proposition for each hypersuface in $\mathbb{C}^4$ in (2.1). The proofs for the hypersurfaces (ii), (iii), (iv) in (2.1) are included in Proposition 2.9.

(v) In the case of $xy^2 - z^2 = 0$ (cuspidal point):

Let us take the 2-resolution of T by normalization in $(2.2)_1$:

$$\begin{array}{ccc} D^*_{\nu_1} & \xrightarrow{j_1} & T^\nu \\ {\scriptstyle \mu_1}\downarrow & & \downarrow{\scriptstyle \nu_1} \\ D_{\nu_1} & \xrightarrow[i_1]{} & T. \end{array} \tag{2.4}$$

Then $T^\nu \simeq \mathbb{C}^3$ and the normalization $\nu_1 : T^\nu \to T \subset \mathbb{C}^4$ is given by $(r, s, t) \to (r^2, s, rs, t) = (x, y, z, w)$, where (r, s, t) is the coordinate on $\mathbb{C}^3$ and (x, y, z, w) is that on $\mathbb{C}^4$. Hence $D_{\nu_1} = S : y = z = 0$ and $D^*_{\nu_1} : s = 0$, which are non-singular. Therefore the 2-resolution of T by normalization in (2.4) gives a cubic hyper-resolution of T.

(vi) In the case of $w(xy^2 - z^2) = 0$ (stationary point):

T and S have the following irreducible decompositions:

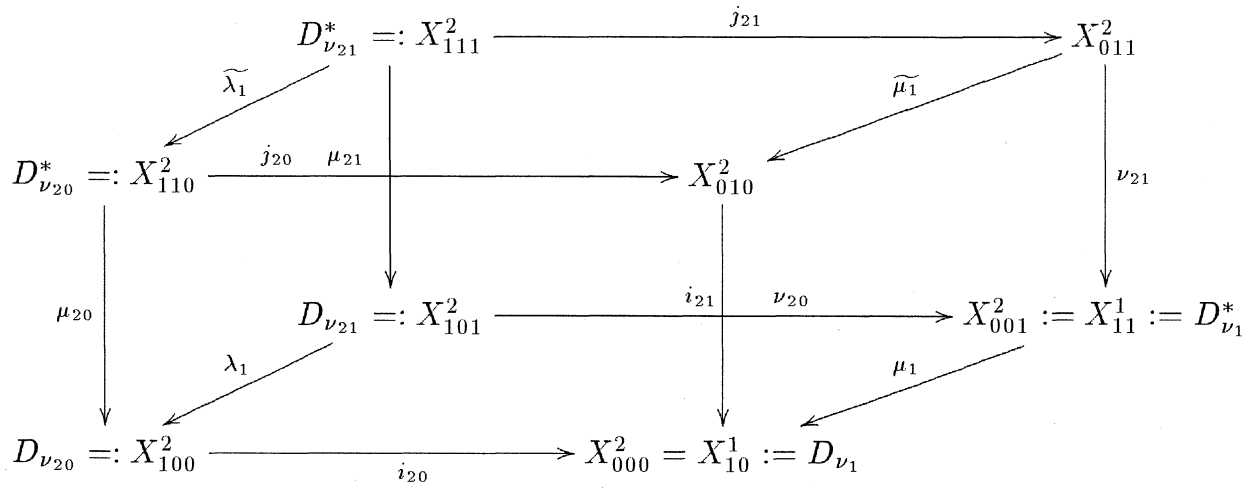
$$\begin{aligned} &T = T_0 + T_c, \ \ T_0 : \ w = 0, \ \ T_c : \ xy^2 - z^2 = 0, \\ &S = S_d + S_c, \ \ S_d : \ y = z = 0, \ \ S_c : \ w = xy^2 - z^2 = 0. \end{aligned}$$

Notice that S_d = the singular locus of T_c, $S_c = T_0 \cap T_c$ and $\Delta = S_d \cap S_c = S_d \cap T_0 : y = z = w = 0$. The reduced ideal of S is $(xy^2 - z^2, wy, wz)$. The 2-resolution of T by normalization in $(2.2)_1$ is explicitly described as follows:

$$\begin{array}{ccc} X^1_{11} := D^*_{\nu_1} := S^*_{0c} \coprod (S^*_{1c} + S^*_d) & \xrightarrow{j_1} & T^\nu_0 \coprod T^\nu_c =: X^1_{01} \\ {\scriptstyle \mu_1}\downarrow & & \downarrow{\scriptstyle \nu_1} \\ X^1_{10} := D_{\nu_1} := S = S_d + S_c & \xrightarrow[i_1]{} & T = T_0 + T_c =: X^1_{00}, \end{array}$$

$$\begin{aligned} &\nu_{1|T^\nu_0} : T^\nu_0 \simeq \mathbb{C}^3 \to T_0 \subset \mathbb{C}^4; (r, s, t) \to (r, s, t, 0) = (x, y, z, w), \\ &\nu_{1|T^\nu_c} : T^\nu_c \simeq \mathbb{C}^3 \to T_c \subset \mathbb{C}^4; (r', s', t') \to (r'^2, s', r's', t') = (x, y, z, w). \\ &S^*_{0c} := \{rs^2 - t^2 = 0\} \subset T^\nu_0, \\ &S^*_{1c} := \{t' = 0\} \subset T^\nu_c, \ \ S^*_d := \{s' = 0\} \subset T^\nu_c. \end{aligned}$$

The 2-resolution of a $\square^+_0$-complex analytic variety $\mu_1 : D^*_{\nu_1} \to D_{\nu_1}$ by normalization in $(2.2)_2$



is explicitly described as follows:

$$\text{(I)} \qquad X^2_{010} := S_d \coprod S^\nu_{\mathfrak{c}} \xleftarrow{\tilde{\mu}_1} (S^*_{0\mathfrak{c}})^\nu \coprod (S^*_{1\mathfrak{c}} \coprod S^*_d) =: X^2_{011}$$

$$\nu_{20} \downarrow \qquad\qquad \downarrow \nu_{21}$$

$$X^2_{000} := S = S_d + S_{\mathfrak{c}} \xleftarrow[\mu_1]{} S^*_{0\mathfrak{c}} \coprod (S^*_{1\mathfrak{c}} + S^*_d) =: X^2_{001};$$

$(S^*_{0\mathfrak{c}})^\nu \simeq S^\nu_{\mathfrak{c}} \simeq S^*_{1\mathfrak{c}} \simeq \mathbb{C}^3,\ S_d \simeq S^*_d \simeq \mathbb{C}^3;$

$\nu_{20|S_d} : S_d \to S_d \subset S$: identity map,

$\nu_{20|S^\nu_{\mathfrak{c}}} : S^\nu_{\mathfrak{c}} \to S_{\mathfrak{c}} \subset S$: normalization map,

$\nu_{21|(S^*_{0\mathfrak{c}})^\nu} : (S^*_{0\mathfrak{c}})^\nu \to S^*_{0\mathfrak{c}} \subset X^2_{001}$: normalization map,

$\nu_{21|S^*_{1\mathfrak{c}}} : S^*_{1\mathfrak{c}} \to S^*_{1\mathfrak{c}} \subset X^2_{001}$: identity map,

$\nu_{21|S^*_d} : S^*_d \to S^*_d \subset X^2_{001}$: identity map;

$$\begin{array}{ccc} T^\nu_0 & \simeq & T_0 \subset \mathbb{C}^4 \\ \cup & & \cup \end{array}$$

$\mu_{1|S^*_{0\mathfrak{c}}} : S^*_{0\mathfrak{c}} \xrightarrow{\simeq} S_{\mathfrak{c}}$: identity map,

$$\begin{array}{ccc} T^\nu_{\mathfrak{c}} & \to & T_{\mathfrak{c}} \subset \mathbb{C}^4 \\ \cup & & \cup \end{array}$$

$\mu_{1|S^*_{1\mathfrak{c}}} : S^*_{1\mathfrak{c}} \to S_{\mathfrak{c}},\ (r', s', 0) \to (r'^2, s', r's', 0) = (x, y, z, w)$:
normalization map,

$$\begin{array}{ccc} T_{\mathfrak{c}}^{\nu} & \to & T_{\mathfrak{c}} \subset \mathbb{C}^4 \\ \cup & & \cup \end{array}$$

$\mu_{1|S_d^*} : S_d^* \to S_d,\ (r', 0, t') \to (r'^2, 0, 0, t') = (x, y, z, w)$:

double covering

$\tilde{\mu}_{1|(S_{0\mathfrak{c}}^*)^{\mu}} : (S_{0\mathfrak{c}}^*)^{\nu} \xrightarrow{\simeq} S_{\mathfrak{c}}^{\nu}$: natural isomorphism,

$\tilde{\mu}_{1|S_{1\mathfrak{c}}^*} : S_{1\mathfrak{c}}^* \xrightarrow{\simeq} S_{\mathfrak{c}}^{\nu}$: natural isomorphism,

$\tilde{\mu}_{1|S_d^*} : S_d^* \to S_d$: the same double covering as $\mu_{1|S_d^*} : S_d^* \to S_d$;

(II)
$$\begin{array}{ccccccc} & & S_d & S_{\mathfrak{c}}^{\nu} & (S_{0\mathfrak{c}}^*)^{\nu} & S_{1\mathfrak{c}}^* & S_d^* \\ & & \cup & \cup & \cup & \cup & \cup \\ X_{110}^2 & := & \triangle \coprod \triangle^* & \xleftarrow{\tilde{\lambda}_1} & (\triangle^* \coprod (\triangle^* \coprod \triangle^*) & =: & X_{111}^2 \\ & & \mu_{20}\downarrow & & \downarrow \mu_{21} & & \\ X_{100}^2 & := & \triangle & \xleftarrow[\lambda_1]{} & \triangle \coprod \triangle^* & =: & X_{101}^2; \\ & & \cap & & \cap\ \cap & & \\ & & S = S_{\mathfrak{c}} + S_d & & S_{0\mathfrak{c}}^*\ S_{1\mathfrak{c}}^* + S_d^* & & \end{array}$$

Here $\triangle^*$ are the inverse images of Δ by the normalization maps $\nu_{20|S_{\mathfrak{c}}^{\nu}} : S_{\mathfrak{c}}^{\nu} \to S_{\mathfrak{c}}$, $\nu_{21|(S_{0\mathfrak{c}}^*)^{\nu}} : (S_{0\mathfrak{c}}^*)^{\nu} \to S_{0\mathfrak{c}}^* \simeq S_{\mathfrak{c}}$, $\mu_{1|S_{1\mathfrak{c}}^*} : S_{1\mathfrak{c}}^* \to S_{\mathfrak{c}}$, respectively, which are non-singular. This shows that a cubic hyper-resolution is obtained by successive normalizations for a stationary point.

Q.E.D.

By Proposition 2.9 and Proposition 2.10 we obtain the following theorem.

2.11 Theorem. *Taking successive normalizations fiberwise, we obtain a simultaneous cubic hyper-resolution of a locally trivial family of the following kinds of complex analytic varieties :*

(i) complex analytic varieties with ordinary singularities of dimension ≤ 3,

(ii) complex analyitc varieties with normal crossing of any dimension.

Proof. Let $\pi : \mathfrak{X} \to M$ be a locally trivial family of above kinds of complex analytic varieties, parametrized by a complex space M. Taking relative normalization $\nu_1 : \mathfrak{X}^{\nu} \to \mathfrak{X}$ of $\mathfrak{X}$ over M (cf. [16, Theorem 3.6]), we obtain the "relative 2-resolution" of the family $\pi : \mathfrak{X} \to M$, which we denote as follows:

$$
\begin{array}{ccc}
\mathfrak{X}^1_{11} := \mathfrak{D}^*_{\nu_1/M} & \xrightarrow{j_1} & \mathfrak{X}^\nu =: \mathfrak{X}^1_{01} \\
{\scriptstyle \mu_1}\downarrow & & \downarrow{\scriptstyle \nu_1} \\
\mathfrak{X}^1_{10} := \mathfrak{D}_{\nu_1/M} & \xrightarrow[i_1]{} & \mathfrak{X} =: \mathfrak{X}^1_{00},
\end{array}
\tag{2.5}
$$

where $\mathfrak{D}_{\nu_1/M}$ denotes the "relative discriminant" of the map $\nu_1 : \mathfrak{X}^\nu \to \mathfrak{X}$ over M and $\mathfrak{D}^*_{\nu_1/M} := \nu_1^{-1}(\mathfrak{D}_{\nu_1/M})$. All maps in the diagram (2.5) are over M. Notice that $\mathfrak{D}_{\nu_1/M}$ and $\mathfrak{D}^*_{\nu_1/M}$ are locally trivial families of complex analytic varieties over M, since $\pi : \mathfrak{X} \to M$ is locally trivial. Next, we take the "relative normalizations" of the families $\mathfrak{D}_{\nu_1/M}$ and $\mathfrak{D}^*_{\nu_1/M}$, respectively, which we denote as follows:

$$
\begin{array}{ccc}
 & \overset{\tilde{\mu}_1}{\Longleftarrow} & \\
(\mathfrak{D}_{\nu_1/M})^\nu & & (\mathfrak{D}^*_{\nu_1/M})^\nu \\
{\scriptstyle \nu_{20}}\downarrow & & \downarrow{\scriptstyle \nu_{21}} \\
\mathfrak{D}_{\nu_1/M} & \xleftarrow[\mu_1]{} & \mathfrak{D}^*_{\nu_1/M},
\end{array}
\tag{2.6}
$$

where $\tilde{\mu}_1$ stands for the "fiberwise" lifting of the map μ_1. Here the "fiberwise" lifting means that, for every $t \in M$, $\tilde{\mu}_{1t} : (D^*_{\nu_1/M,t})^\nu \to (D_{\nu_1/M,t})^\nu$ is the lifting of the map $\mu_{1t} : D^*_{\nu_1/M,t} \to D_{\nu_1/M,t}$ between fibers of the families $(\mathfrak{D}^*_{\nu_1/M})^\nu$ and $\mathfrak{D}^*_{\nu_1/M}$ over M. This is possible due to the fact that $(\mathfrak{D}_{\nu_1/M})^\nu$ and $(\mathfrak{D}^*_{\nu_1/M})^\nu$ are the "relative normalizations of $\mathfrak{D}_{\nu_1/M}$ and $\mathfrak{D}^*_{\nu_1/M}$, respectively. In fact, $\tilde{\mu}_1 := \{\tilde{\mu}_{1t}\}_{t\in M}$ is a holomorphic map from $(\mathfrak{D}^*_{\nu_1/M})^\nu$ to $(\mathfrak{D}_{\nu_1/M})^\nu$, since the family $\mu_1 : \mathfrak{D}^*_{\nu_1/M} \to \mathfrak{D}_{\nu_1/M}$ of holomorphic maps over M is locally trivial. Therefore we conclude that the diagram (2.6) gives a "relative normalization" of the $\square^+_0$-object $\mathfrak{X}^1_{1\bullet} := \{\mu_1 : \mathfrak{D}^*_{\nu_1/M} \to \mathfrak{D}_{\nu_1/M}\}$ of locally trivial families of complex analytic varieties over M in (2.5). Using this "relative normalization", we obtain the "relative 2-resolution" of the $\square^+_0$-object $\mathfrak{X}^1_{1\bullet} := \{\mu_1 : \mathfrak{D}^*_{\nu_1/M} \to \mathfrak{D}_{\nu_1/M}\}$, which we denote as follows:

$$
\begin{array}{ccc}
\mathfrak{X}^2_{11\bullet} := \mathfrak{D}^*_{\nu_{2\bullet}/M} & \xrightarrow{j_2} & (\mathfrak{X}^1_{1\bullet})^\nu =: \mathfrak{X}^2_{01\bullet} \\
{\scriptstyle \mu_{2\bullet}}\downarrow & & \downarrow{\scriptstyle \nu_{2\bullet}} \\
\mathfrak{X}^2_{10\bullet} := \mathfrak{D}_{\nu_{2\bullet}/M} & \xrightarrow[i_2]{} & \mathfrak{X}^1_{1\bullet} =: X^2_{00\bullet},
\end{array}
$$

where $\nu_{2\bullet} : (\mathfrak{X}^1_{1\bullet})^\nu \to \mathfrak{X}^1_{1\bullet}$ is the relative normalization of $\mathfrak{X}^1_{1\bullet}$ in (2.6), $\mathfrak{D}_{\nu_{2\bullet}/M}$ is the "relative discriminant" of the map $\nu_{2\bullet} : (\mathfrak{X}^1_{1\bullet})^\nu \to \mathfrak{X}^1_{1\bullet}$, $\mathfrak{D}^*_{\nu_{2\bullet}/M} := \nu_{2\bullet}^{-1}(\mathfrak{D}_{\nu_{2\bullet}/M})$, and $\mu_{2\bullet}$ is the restriction of $\nu_{2\bullet}$ to $\mathfrak{D}^*_{\nu_{2\bullet}/M}$. The procedure of taking this "relative normalization" can be continued similarly like the absolute

case and obtain a sequence $\mathfrak{X}_\bullet^0 = \mathfrak{X}, \mathfrak{X}_\bullet^1, \mathfrak{X}_\bullet^2, \cdots, \mathfrak{X}_\bullet^r, \cdots$ of $\square_r^+$-objects $\mathfrak{X}_\bullet^r$ of locally trivial analytic families of complex analytic varieties, parametrized by M, such that $\mathfrak{X}_\bullet^r$ is the 2-resolution of $\mathfrak{X}_{1\bullet}^{r-1}$ by "relative normalization" for every $r \geq 0$. Then, after finite steps, say n-th step, the reduction

$$\mathfrak{X}_\bullet^+ := rd(\mathfrak{X}_\bullet^1, \mathfrak{X}_\bullet^2, \cdots, \mathfrak{X}_\bullet^n),$$

which can be defined in the same manner as in the absolute case, gives a "relative" cubic hyper-resolution of $\mathfrak{X}$, i.e., if we write $\mathfrak{X}_\bullet^+$ as

$$\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M, \tag{2.7}$$

where $\mathfrak{X}_\bullet$ is the "$\square_n$"-part of $\mathfrak{X}_\bullet^+$, then the fiber $a_{\bullet t} : \mathfrak{X}_{\bullet t} \to \mathfrak{X}_t$ is a cubic hyper-resolution of $\mathfrak{X}_t$ for every $t \in M$. The analytical "local triviality" of the family in (2.7) is obvious, because the original family $\pi : \mathfrak{X} \to M$ is so. That is, by defintiton, the family in (2.7) is a simultaneous cubic hyper-resolution of the family $\pi : \mathfrak{X} \to M$.

Q.E.D.

§3 Cohomological descent

The relative version of "cohomological descent" holds for a simultaneous cubic hyper-resolution of a locally trivial analytic family of complex projective varieties. In order to state this fact we refer to some notation and terminology from [10]. Let $\Phi_\bullet : X_\bullet \to X$ be an n-cubic topological space with an augmentation to a topological space X, i.e., $X_\bullet$ is a contravariant functor from the n-cubic category $\square_n$ to the category of topological space (Top) and $\Phi_\bullet$ is a natural transformation from the functor $X_\bullet$ to the one X over the trivial functor $\square_n \to \square_{-1}^+$, where X is considered as a $\square_{-1}^+$-object of the category (Top) (cf. Definition 1.1, Definition 1.3 and Remark 1.4).

3.1 Definition. For a commutative ring R with identity elemeny 1, an *R-module preshef $F^\bullet$* on an n-cubic topological space $X_\bullet : \square_n \to$ (Top) is defined to be a contravariant functor from the total category $\mathrm{tot}(X_\bullet)$ to the category of R-modules, where we identify a topological space with the category of open subsets of it. We say an R-module presheaf $F^\bullet$ on an n-cubic topological space $X_\bullet$ is an *R-module sheaf* if the presheaves F^α on X_α, defined by $F^\bullet$, are sheaves for all $\alpha \in \square_n$. For R−module (pre)sheaves $F^\bullet$ and $G^\bullet$ on $X_\bullet$, a *morphism from $F^\bullet$ to $G^\bullet$* is defined to be a natural transformation from $F^\bullet$ to $G^\bullet$.

We denote by $\mathcal{M}(X_\bullet, R)$ and $\mathcal{M}(X, R)$ the categories of R-module sheaves on $X_\bullet$ and X, respectively, where R is a commutative ring with identity element 1. For an R-module sheaf $\mathcal{F}$ on X we define its inverse image $\Phi_\bullet^*\mathcal{F} \in \mathcal{M}(X_\bullet, R)$ in a natural way. The functor $\Phi_\bullet^* : \mathcal{M}(X, R) \to \mathcal{M}(X_\bullet, R)$ has a right adjoint $\Phi_{\bullet *} : \mathcal{M}(X_\bullet, R) \to \mathcal{M}(X, R)$. Since the functor $\Phi_\bullet^*$ is exact, it defines a functor

$$\Phi_\bullet^* : D^+(X, R) \to D^+(X_\bullet, R), \tag{3.1}$$

where $D^+(X,R)$ and $D^+(X_\bullet,R)$ denote the derived categories of lower bounded complexes of R-module sheaves on X and $X_\bullet$, respectively. The functor in (3.1) has a right adjoint

$$\mathbb{R}\Phi_{\bullet *} : D^+(X_\bullet,R) \to D^+(X,R).$$

Let $F^\bullet$ be a lower bounded complex of R-module sheaves on an n-cubic topological space $X_\bullet$. We take the factorization

$$X_\bullet \xrightarrow{\Phi_{1\bullet}} X \times \square_n \xrightarrow{\Phi_{2\bullet}} X \tag{3.2}$$

of $\Phi_\bullet : X_\bullet \to X$, where $X \times \square_n$ is the n-cubic object of (Top) defined by $(X \times \square_n)(\alpha) := X$ for $\alpha \in \square_n$, $\Phi_{1\bullet}$ is the natural transformation defined by $\Phi_{1\alpha} := \Phi_\alpha$ for $\alpha \in \square_n$, and $\Phi_{2\bullet}$ the one defined by $\Phi_{2\alpha} := id_X$ for $\alpha \in \square_n$. By definition $\Phi_{1\bullet *}F^\bullet = \{\Phi_{1\alpha *}F^\alpha\}_{\alpha \in \mathrm{Ob}(\square_n)}$, to which we associate a simple complex $s(\Phi_{1\bullet *}F^\bullet)$ of R-module sheaves on X. To explain this we give the definition of an *n-ple complex of an abelian category*. Let A be an abelian category. We denote by $C^+(A)$ the category of lower bounded complexes of A. Let n be an integer≥ 1. We denote by e_i the i-th vector of the canonical basis of $\mathbb{Z}^n$, i.e., $e_i = (0,\cdots,1,\cdots,0)$ (1 is at the i-th place) for $1 \leq i \leq n$.

3.2 Definition. With the notation above, an n-ple complex of A consisits of the following entities:

(i) a $\mathbb{Z}^n$-graded object $\{K^\alpha\}_{\alpha \in \mathbb{Z}^n}$ of A, and

(ii) a family $\{d_i\}_{1 \leq i \leq n}$ of differentials of $K^\bullet$ such that d_i is of defree e_i and they commute each other.

We denote by n-$C^+(A)$ the category of n-ple complexes of an abelian category A.

3.3 Definition. For $K \in n$-$C^+(A)$ its *associated simple complex* $s(K) \in C^+(A)$ is defined to be as follows:

$$s(K)^p := \sum_{\sum p_i = p} K^{p_1 \cdots p_n}, \quad p \in \mathbb{Z} \text{ and}$$

the differential d of $s(K)$ is defined by

$$d = \sum_{j=1}^{n} (-1)^{\varepsilon_j} d_j \text{ on } K^{p_1 \cdots p_n},$$

where $\varepsilon_j = \sum_{i<j} p_i$.

Let $\mathcal{A}$ be a $(\square_n^+)^\circ$-object of lower bounded complexes of R-module sheaves on a topological space, say Y, i.e., a functor $\mathcal{A} : (\square_n^+)^\circ \to C^+(Y,R)$, where $C^+(Y,R)$ is the category of lower bounded complexes of R-module sheaves on

Y. We denote $\mathcal{A}(\alpha) \in C^+(Y,R)$ by $\mathcal{A}^{\alpha\bullet}$ for each $\alpha \in \mathrm{Ob}(\square_n^+)$. We associate to such $\mathcal{A}$ an object $K(\mathcal{A})$ of (n+2)-$C^+(Y,R)$, i.e., an $(n+2)$-ple lower bounded complex of $\mathcal{M}(Y,R)$ as follows:

$$K(\mathcal{A})^{\alpha_0\cdots\alpha_n q} := \begin{cases} \mathcal{A}^{\alpha q} & \text{if } \alpha \in \mathrm{Ob}(\square_n^+) \\ 0 & \text{if } \alpha \in \mathbb{Z}^{n+1} - \mathrm{Ob}(\square_n^+); \end{cases}$$

the $(i+1)$-th differential is the one induced by the morphism $\alpha \to \alpha + e_i$ in $\square_n^+$ for $0 \leq i \leq n$, and $(n+2)$-th differential is the one of the complex $\mathcal{A}^{\alpha\bullet}$. For the sake of simplicity we denote $s(K(\mathcal{A}))$ by $s(\mathcal{A})$.

We think of $\Phi_{1\bullet *}F^\bullet = \{a_{1\alpha *}F^\alpha\}_{\alpha\in\mathrm{Ob}(\square_n)}$ as a $(\square_n^+)^\circ$-object of lower bounded complexes of R-module sheaves on X by defining $F^{(0,\cdots,0)} = \{0\}$ for $(0,\cdots,0) \in \square_n^+$, and form $s(\Phi_{1\bullet *}F^\bullet)$. Then we have

$$\mathbb{R}\Phi_{2\bullet *}(\Phi_{1\bullet *}F^\bullet) \cong s(\Phi_{1\bullet *}F^\bullet)[1]$$

in $D^+(X,R)$, where [1] stands for the shift of the degree of complexes to the left by 1, i.e., $s(\Phi_{1\bullet *}F^\bullet)[1]^p = s(\Phi_{1\bullet *}F^\bullet)^{p+1}$. Then we have

$$\mathbb{R}\Phi_{\bullet *}F^\bullet \cong s(\Phi_{1\bullet *}F^\bullet)[1] \tag{3.3}$$

in $D^+(X,R)$. This description of $\mathbb{R}\Phi_{\bullet *}F^\bullet$ is necessary for our arguments in the following. For more details we refer to [10, Exposé I].

The following is the relative version of the cohomological descent for R-module sheaves.

3.4 Theorem. *Let $\mathfrak{X}_\bullet \xrightarrow{a_\bullet} \mathfrak{X} \xrightarrow{\pi} M$ be a simultaneous n-cubic $(n \geq 1)$ hyper-resolution of a locally trivial analytic family of complex projective varieties,parametrized by a complex space M. Then, for an R-module sheaf $\mathcal{A}$ on $\mathfrak{X}$, the adjunction map*

$$\mathcal{A} \longrightarrow \mathbb{R}a_{\bullet *}a_\bullet^*\mathcal{A}$$

is an isomorphism in $D^+(\mathfrak{X},R)$.

Proof. In order to prove the theorem, it suffices to show that for any point $x \in \mathfrak{X}$, the homomorphism

$$\mathcal{A}_x \longrightarrow (\mathbb{R}a_{\bullet *}a_\bullet^*\mathcal{A})_x \tag{3.4}$$

is a quasi-isomorphism of complexes of R-modules. We put $t := \pi(x)$, $X_t := \pi^{-1}(t)$, $X_{\bullet t} := \pi_\bullet^{-1}(t)$ and

$$b_{\bullet t} := a_{\bullet|X_{\bullet t}} : X_{\bullet t} \to X_t \quad .$$

Since $b_{\bullet t} : X_{\bullet t} \to X_t$ is a a cubic hyper-resolution by the assumption, it follows from its cohomological descent property that the homomorphism

$$(\mathcal{A}_{|X_t})_x \longrightarrow (\mathbb{R}b_{\bullet t *}b_{\bullet t}^*\mathcal{A}_{|X_t})_x \tag{3.5}$$

is a quasi-isomorphism. Therefore, since $\mathcal{A}_x = (\mathcal{A}_{|X_t})_x$, it suffices to show that the canonical map

$$(\mathbb{R}a_{\bullet *}a_\bullet^*\mathcal{A})_x \longrightarrow (\mathbb{R}b_{t\bullet *}b_{t\bullet}^*\mathcal{A}_{|X_t})_x, \tag{3.6}$$

is a quasi-isomorphism in order to prove that the homomorphism in (3.5) is a quasi-isomorphism. We use the following lemma which is a consequence of the proper base change formula of Godement ([7, II.4.11]), and of [10, Exposé I,5.13]:

3.5 Lemma. *Let $T_\bullet$ be a cubic paracompact topological space, S a paracompact space, and $f_\bullet : T_\bullet \longrightarrow S$ a proper augmentation. For all complexe of sheaves $F^\bullet$ on $T_\bullet$ and all $s \in S$, the fibre at s of the complex of sheaves $\mathbb{R}f_{\bullet *}F^\bullet$ is quasi-isomorphic to the hypercohomology $\mathbb{H}(T_{\bullet s}, F^\bullet_{|T_{\bullet s}})$ of the fiber $T_{\bullet s} := f_\bullet^{-1}(s)$.*

Then one obtains the following quasi-isomorphisms,

$$(\mathbb{R}a_{\bullet *}a_\bullet^*\mathcal{A})_x \cong \mathbb{H}(a_\bullet^{-1}(x), a_\bullet^*\mathcal{A}_{|a_\bullet^{-1}(x)})$$
$$(\mathbb{R}b_{\bullet t*}b_{\bullet t}^*\mathcal{A}_{|X_t})_x, \cong \mathbb{H}(b_{\bullet t}^{-1}(x), b_{\bullet t}^*(\mathcal{A}_{|X_t})_{|b_{\bullet t}^{-1}(x)})$$

and the obvious identity $a_\bullet^{-1}(x) = b_{\bullet t}^{-1}(x)$, from which one deduces that (3.6) is a quasi-isomorphism as required.

Q.E.D.

We are now going to define the *cohomological relative de Rham complex* $DR_{\mathfrak{X}/M} \in D^+(\mathfrak{X}, \mathbb{C})$ for an analyitc family $\pi : \mathfrak{X} \to M$ of complex analyitc varieties, parametrized by a complex space M. For this end we take a system of relative local embeddings $\mathcal{U} := \{(\mathcal{U}_i', \mathcal{U}_i), \varphi_i, (\mathcal{Y}_i', \mathcal{Y}_i, \pi_i)\}$ of $\pi : \mathfrak{X} \to M$ which consists of the following entities:

(i) $\{\mathcal{U}_i'\}, \{\mathcal{U}_i\}$ are open coverings of $\mathfrak{X}$ with $\mathcal{U}_i$ being a relatively compact open subset of $\mathcal{U}_i'$ for every i,

(ii) $\mathcal{Y}_i' = D_i \times \pi(\mathcal{U}_i')$, where D_i are polycylinders in complex number spaces $\mathbb{C}^{n_i}$,

(iii) $\mathcal{Y}_i \xrightarrow{\pi_i} \pi(\mathcal{U}_i)$ are smooth families of complex manifolds, parametrized by $\pi(\mathcal{U}_i)$ such that

(a) $\mathcal{Y}_i$ are relatively compact open subsets of $\mathcal{Y}_i'$, and

(b) the following diagrams commute:

$$\begin{array}{ccc} \mathcal{Y}_i & \longrightarrow & \mathcal{Y}_i' \\ {\scriptstyle \pi_i}\downarrow & & \downarrow{\scriptstyle \mathrm{Pr}_{\pi(\mathcal{U}_i')}} \\ \pi(\mathcal{U}_i) & \longrightarrow & \pi(\mathcal{U}_i'), \end{array}$$

(iv) $\varphi_i : \mathcal{U}_i' \to \mathcal{Y}_i'$ are closed embeddings over $\pi(\mathcal{U}_i')$ such that $\varphi_i(\mathcal{U}_i) = \mathcal{Y}_i$.

For each $(p+1)$-tuple $(i) = \{i_0 < i_1 < \cdots < i_p\}$ we consider an open set $\mathcal{U}'_{(i)} = \mathcal{U}'_{i_0} \cap \cdots \cap \mathcal{U}'_{i_p}$ and a relative closed embedding

$$\begin{aligned}\mathcal{U}'_{(i)} &\to \mathcal{Y}'_{(i)} \\ &:= (D_{i_0} \times \pi(\mathcal{U}'_{(i)})) \times_{\pi(\mathcal{U}'_{(i)})} (D_{i_1} \times \pi(\mathcal{U}'_{(i)})) \times_{\pi(\mathcal{U}'_{(i)})} \cdots \\ &\qquad \times_{\pi(\mathcal{U}'_{(i)})} (D_{i_P} \times \pi(\mathcal{U}'_{(i)}))\end{aligned}$$

over $\pi(\mathcal{U}'_{(i)})$, where $\times_{\pi(\mathcal{U}'_{(i)})}$ denotes the fiber product over $\pi(\mathcal{U}'_{(i)})$; and define

$$\Omega^\bullet_{\mathcal{Y}'_{(i)}/\pi(\mathcal{U}'_{(i)})}\widehat{|}\mathcal{U}'_{(i)} := \varprojlim_k \Omega^\bullet_{\mathcal{Y}'_{(i)}/\pi(\mathcal{U}'_{(i)})}/\mathcal{I}^k_{\mathcal{U}'_{(i)}} \Omega^\bullet_{\mathcal{Y}'_{(i)}/\pi(\mathcal{U}'_{(i)})}$$

where $\Omega^\bullet_{\mathcal{Y}'_{(i)}/\pi(\mathcal{U}'_{(i)})}$ is the relative de Rham complex of the smooth family $\mathrm{Pr}_{\pi(\mathcal{U}'_{(i)})} : \mathcal{Y}'_{(i)} \to \pi(\mathcal{U}'_{(i)})$ of comlex manifolds and $\mathcal{I}_{\mathcal{U}'_{(i)}}$ is the ideal sheaf of $\mathcal{U}'_{(i)}$ in the structure sheaf $\mathcal{O}_{\mathcal{Y}'_{(i)}}$ of $\mathcal{Y}'_{(i)}$. We call $\Omega^\bullet_{\mathcal{Y}'_{(i)}/\pi(\mathcal{U}'_{(i)})}\widehat{|}\mathcal{U}'_{(i)}$ the completion of $\Omega^\bullet_{\mathcal{Y}'_{(i)}/\pi(\mathcal{U}'_{(i)})}$ along $\mathcal{U}'_{(i)}$. Then we consider a complex of sheaves of $\mathbb{C}$-vector spaces on $\mathfrak{X}$

$$C^\bullet_{(i)} := j_*(\Omega^\bullet_{\mathcal{Y}'_{(i)}/\pi(\mathcal{U}'_{(i)})}\widehat{|}\mathcal{U}'_{(i)})|_{\mathcal{U}_{(i)}},$$

where j is the inclusion of $\mathcal{U}'_{(i)}$ into $\mathfrak{X}$ and $\mathcal{U}_{(i)} = \mathcal{U}_{i_0} \cap \cdots \cap \mathcal{U}_{i_p}$. Here, putting 0 outside $\mathcal{U}_{(i)}$, we consider $C^\bullet_{(i)}$ as a complex of sheaves of $\mathbb{C}$-vector spaces on $\mathfrak{X}$. Now for any $0 \leq j \leq p$, let $(i') = \{i_0, \cdots, \check{i}_j, \cdots, i_p\}$ (omit i_j). Then we have a natural inclusion $\mathcal{U}'_{(i)} \to \mathcal{U}'_{(i')}$, which maps $\mathcal{U}_{(i)}$ into $\mathcal{U}_{(i')}$; and a natural inclusion $\mathcal{Y}'_{(i)} \to \mathcal{Y}'_{(i')}$ over $\pi(\mathcal{U}'_{(i)}) \to \pi(\mathcal{U}'_{(i')})$, which maps $\mathcal{Y}_{(i)}$ into $\mathcal{Y}_{(i')}$ over $\pi(\mathcal{U}_{(i)}) \to \pi(\mathcal{U}_{(i')})$. Hence there is a natural map

$$\Omega^\bullet_{\mathcal{Y}'_{(i')}/\pi(\mathcal{U}'_{(i')})}\widehat{|}\mathcal{U}'_{(i')|\mathcal{U}'_{(i)}} \to \Omega^\bullet_{\mathcal{Y}'_{(i)}/\pi(\mathcal{U}'_{(i)})}\widehat{|}\mathcal{U}'_{(i)},$$

and a morphism of complexes on $\mathfrak{X}$

$$\delta_{j,(i)} : C^\bullet_{(i')} \to C^\bullet_{(i)}$$

Notice that, by the construction, for two integers $0 \leq j < k \leq p$, the corresponding four δ maps are compatible with each other. Hence we can define a double complex $\mathcal{C}(\mathcal{U})$ by

$$\mathcal{C}(\mathcal{U})^p = \prod_{|i|=p} C^\bullet_{(i)}$$

where $|i|$ is defined to be p for $i = (i_0, \cdots, i_p)$, and

$$\delta^{p-1} := \prod_{|i1=p} \sum_{j=0}^{p} (-1)^j \delta_{j,(i)} : \mathcal{C}(\mathcal{U})^{p-1} \to \mathcal{C}(\mathcal{U})^p$$

We denote by $\widehat{\Omega}^{\bullet}_{\mathfrak{X}/M}(\mathcal{U})$ the associated single complex of $\mathcal{C}(\mathcal{U})$. If $\mathcal{V} = \{(\mathcal{V}'_j, \mathcal{V}_j), \psi_j, (\mathcal{Z}'_j, \mathcal{Z}_j, \pi_j)\}$ is a refinement of a system of relative local embeddings $\mathcal{U}$, then there is a natural map of double complexes $\varphi : \mathcal{C}(\mathcal{U}) \to \mathcal{C}(\mathcal{V})$ and, as in the absolute case, we can see that the map $\widehat{\Omega}^{\bullet}_{\mathfrak{X}/M}(\mathcal{U}) \to \widehat{\Omega}^{\bullet}_{\mathfrak{X}/M}(\mathcal{V})$ of simple complexes associated to φ is a quasi-isomorphism (cf. [11, p.29]). Therefore we conclude that $\widehat{\Omega}^{\bullet}_{\mathfrak{X}/M}(\mathcal{U})$ defines an element of $D^+(\mathfrak{X}, \mathbb{C})$, which is independent of the choice of $\mathcal{U}$.

3.6 Definition. We call such an element of $D^+(\mathfrak{X}, \mathbb{C})$ determined by the $\widehat{\Omega}^{\bullet}_{\mathfrak{X}/M}(\mathcal{U})$ the *cohomological relative de Rham complex of* the family $\pi : \mathfrak{X} \to M$ and denote by $DR^{\bullet}_{\mathfrak{X}/M}$.

Let $\mathfrak{X}_{\bullet} \xrightarrow{a_{\bullet}} \mathfrak{X} \xrightarrow{\pi} M$ be a simultaneous n-cubic hyper-resolution of a locally trivial analytic family of comlex projective varieties, parametrized by a complex space M. For each $\alpha \in \square_n$ we denote by $\Omega^{\bullet}_{\mathfrak{X}_\alpha/M}$ the relative de Rham complex of a smooth family $\pi \circ a_\alpha : \mathfrak{X}_\alpha \to M$ of complex manifolds. Then $\Omega^{\bullet}_{\mathfrak{X}_\bullet/M} := \{\Omega^{\bullet}_{\mathfrak{X}_\alpha/M}\}_{\alpha \in \square_n}$ is obviously a complex of sheaves of $\mathbb{C}$-vector spaces on a $\square_n$-complex manifold $\mathfrak{X}_\bullet$. The rest of this section will be devoted to proving the following theorems and a corollary.

3.7 Theorem. (*Cohomological descent of relative de Rham complexes*) *Under the same setting as above, there naturally exists an isomorphism*

$$DR^{\bullet}_{\mathfrak{X}/M} \simeq \mathbb{R}a_{\bullet *}\Omega^{\bullet}_{\mathfrak{X}_\bullet/M}$$

in $D^+(\mathfrak{X}, \mathbb{C})$.

3.8 Theorem. (*Relative formal analytic Poincaré lemma*) *Under the same setting as above,* $\widehat{\Omega}^{\bullet}_{\mathfrak{X}/M}(\mathcal{U})$ *yields a resolution of the sheaf* $\pi^{\cdot}(\mathcal{O}_M)$ *for a system of relative local embeddings* $\mathcal{U} = \{(\mathcal{U}'_i, \mathcal{U}_i), \varphi_i, (\mathcal{Y}'_i, \mathcal{Y}_i, \pi_i)\}$ *of* $\pi : \mathfrak{X} \to M$, *where* $\pi^{\cdot}(\mathcal{O}_M)$ *denotes the topological inverse of the structure sheaf of* M *by the map* $\pi : \mathfrak{X} \to M$.

3.9 Corollary. *There exist isomorphisms*

$$H^i(\mathfrak{X}, \pi^{\cdot}(\mathcal{O}_M)) \simeq H^i(\mathbb{R}\Gamma(\mathfrak{X}, s(a_{1\bullet *}\Omega^{\bullet}_{\mathfrak{X}_\bullet/M})[1]))$$

$$\simeq H^i(\mathbb{R}\Gamma(\mathfrak{X}_\bullet, \Omega^{\bullet}_{\mathfrak{X}_\bullet/M})[1]) \quad (1 \leq i \leq 2\dim_{\mathbb{C}}\mathfrak{X}).$$

(*for the notation* $a_{1\bullet}$ *see* (3.2))

To prove these theorems the following two theorems are essential.

3.10 Theorem. *(Mayer-Vietories sequence for relative de Rham complexes) Let $\pi : \mathfrak{Y} \to M$ be a flat family of analyitc varieties, parametrized by a complex space M. Suppose that $\pi : \mathfrak{Y} \to M$ is relatively embedded in a smooth family $\pi' : \mathfrak{X} \to M$ of complex manifolds, parametrized by the same complex space M, and further suppose that $\mathfrak{Y}$ is a union of two closed subvarieties $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$ of $\mathfrak{X}$. Then there is an exact sequence of relative de Rham complexes*

$$0 \to \Omega^{\bullet}_{\mathfrak{X}/M}\widehat{|\mathfrak{Y}} \to \Omega^{\bullet}_{\mathfrak{X}/M}\widehat{|\mathfrak{Y}}_1 \oplus \Omega_{\mathfrak{X}/M}\widehat{|\mathfrak{Y}}_2 \to \Omega^{\bullet}_{\mathfrak{X}/M}\widehat{|\mathfrak{Y}}_1 \cap \mathfrak{Y}_2 \to 0,$$

where $\Omega^{\bullet}_{\mathfrak{X}/M}\widehat{|\mathfrak{Y}}$ is the completion of the relative de Rham complex $\Omega^{\bullet}_{\mathfrak{X}/M}$ along $\mathfrak{Y}$ and so on.

3.11 Theorem. *Let $f : X' \to X$ be a poper morphism of analytic varieties. Let Y be a closed analyitc subvariety of X, and let $Y' := f^{-1}(Y)$. Assume that f maps $X' - Y'$ isomorphically onto $X - Y$. Suppose we are given coherent sheaves $\mathcal{F}$ on X and $\mathcal{F}'$ on X', and an injective map $\mathcal{F} \to f_*\mathcal{F}'$, whose restriction to $X - Y$ is an isomorphism. Then the single complex associated to the $\square_1^+$-object of lower bounded complexes of sheaves of $\mathbb{C}$-vector spaces on X*

$$\begin{array}{ccc} \mathbb{R}(\widehat{\imath} \circ \widehat{f})_*\widehat{\mathcal{F}}' & \longleftarrow & \mathbb{R}f_*\mathcal{F}' \\ \uparrow & & \uparrow \\ \widehat{\imath}_*\widehat{\mathcal{F}} & \longleftarrow & \mathcal{F} \end{array}$$

is acyclic in $D^+(X, \mathbb{C})$, where $\imath$ is the closed immersion $Y \to X$ and $\widehat{\ }$ denotes the completion along Y, or Y', respectively.

The proof of Theorem 3.10 for the absolute case, i.e., M is a single point, can be found in [11, p.89, Proposition(1.4)]. Since $\Omega^p_{\mathfrak{X}/M}$ are locally free sheaves over $\mathcal{O}_{\mathfrak{X}}$, and since all of $\Omega^p_{\mathfrak{X}/M}\widehat{|\mathfrak{Y}}$, $\Omega^p_{\mathfrak{X}/M}\widehat{|\mathfrak{Y}}_i (i = 1, 2)$ and $\Omega^p_{\mathfrak{X}/M}\widehat{|\mathfrak{Y}}_1 \cap \mathfrak{Y}_2$ are completions with respect to some ideal sheaves of $\mathcal{O}_{\mathfrak{X}}$, the same arguments as in the absolute case also go well for the relative case. Hence we obtain Theorem 3.10. Theorem 3.11 is an analytic analogue of Proposition(4.3) in [11]. The key point of the proof of Proposition(4.3) in [11] is "fundamental theorem of a proper morphism" ([9, 4.1.5]), which tells us that, with the same notation as in Theorem 3.11, though all things should be replaced by algebraic ones,

$$R^i\widehat{f}_*\widehat{\mathcal{F}}' \simeq (R^i f_*\mathcal{F}')\widehat{\ } \quad (i \geq 0),$$

where $(R^i f_*\mathcal{F}')\widehat{\ }$ is the completion of $R^i f_*\mathcal{F}'$ along Y, and $R^i\widehat{f}_*\widehat{\mathcal{F}}'$ the i-th higher direct image sheaf of $\widehat{\mathcal{F}}'$ by the morphism of formal scheames $\widehat{f} : \widehat{X}' \to \widehat{X}$, induced by f, from the completion $\widehat{X}'$ of X' along Y' to that of X along Y. Fortunately, we have an analytic analogue of the "fundamental theorem of a

proper morphism" due to C. Bănică and O. Stănăşilă ([1, p.225, VI, Cor.4.5). Using this theorem, we can carry out the same arguments as in the proof of Proposition(4.3) in [11]. Hence we obtain Theorem 3.11.

To prove Theorem 3.7 we shall use the following theorem, which is an analytic analogue of Theorem(4.4) in [11, p.44].

3.12 Theorem. *Let $\pi' : \mathfrak{X}' \to M$ and $\pi : \mathfrak{X} \to M$ be two flat families of analytic varieties, parametrized by the same complex space M. Let $f : \mathfrak{X}' \to \mathfrak{X}$ be a proper morphism of analytic varieties over M, $\mathfrak{Y}$ a closed subvariety of $\mathfrak{X}, \mathfrak{Y}' := f^{-1}(\mathfrak{Y})$, and $h := f_{|\mathfrak{Y}'} : \mathfrak{Y}' \to \mathfrak{Y}$ the restriction of f to $\mathfrak{Y}'$. We assume the following:*

(i) f maps $\mathfrak{X}' - \mathfrak{Y}'$ isomorphically onto $\mathfrak{X} - \mathfrak{Y}$,

(ii) there exist

(a) smooth families of complex manifolds $\pi' : \mathfrak{Z}' \to M$ and $\pi : \mathfrak{Z} \to M$, parametrized by the complex space M,

(b) closed immersions $\mathfrak{X}' \to \mathfrak{Z}'$ and $\mathfrak{X} \to \mathfrak{Z}$ over M, and

(c) a proper morphism $g : \mathfrak{Z}' \to \mathfrak{Z}$ over M

such that $g_{|\mathfrak{X}'} = f$ and g maps $\mathfrak{Z}' - g^{-1}(\mathfrak{Y})$ isomorphically onto $\mathfrak{Z} - \mathfrak{Y}$.

Then the single complex associated to the following $\square_1^+$-object of lower bounded complexes of sheaves of $\mathbb{C}$-vector spaces on $\mathfrak{X}$

$$\begin{array}{ccc} \mathbb{R}(\widehat{\iota} \circ \widehat{h})_* \Omega^{\bullet}_{\mathfrak{Z}'/M} \widehat{|}\mathfrak{Y}' & \longleftarrow & \mathbb{R}f_* \Omega^{\bullet}_{\mathfrak{Z}'/M} \widehat{|}\mathfrak{X}' \\ \uparrow & & \uparrow \\ \widehat{\iota}_* \Omega^{\bullet}_{\mathfrak{Z}/M} \widehat{|}\mathfrak{Y} & \longleftarrow & \Omega^{\bullet}_{\mathfrak{Z}/M} \widehat{|}\mathfrak{X} \end{array}$$

is acyclic in $D^+(\mathfrak{X}, \mathbb{C})$, where $\iota : \mathfrak{Y} \to \mathfrak{X}$ is the inclusion map.

Since the proof of Theorem 3.12 is almost identical with that in the algebraic case ([11, p.44, Chapter II, Theorem(4.4)), we omit it, just mentioning that we essentially use Theorem 3.10 and Theorem 3.11 to prove it.

3.13 Proposition. *Let $\pi : \mathfrak{Y} \to M$ be a flat family of analytic varieties, parametrized by a complex space M, which is relatively embedded in a smooth family $\pi' : \mathfrak{X} \to M$ of complex manifolds, parametrized by the same complex space M. Suppose $\mathfrak{Y}$ is a union of finite closed subvarieties $\mathfrak{Y}_1, \cdots, \mathfrak{Y}_n (n \geq 2)$. Let $\iota : \mathfrak{Y}_\bullet \to \mathfrak{Y}$ be the n-cubic object of analytic varieties, augmented to $\mathfrak{Y}$, effected by the finite closed cover $\{\mathfrak{Y}_r\}_{1 \leq r \leq n}$ of $\mathfrak{Y}$ (cf. Example 1.6). Then we have a quasi-isomorphism*

$$\Omega^{\bullet}_{\mathfrak{X}/M} \widehat{|}\mathfrak{Y} \to \iota_* \Omega^{\bullet}_{\mathfrak{X}/M} \widehat{|}\mathfrak{Y}_\bullet,$$

where

$$\Omega^{\bullet}_{\mathfrak{X}/M}\widehat{|}\mathfrak{Y}_{\bullet} := \{\Omega^{\bullet}_{\mathfrak{X}/M}\widehat{|}\mathfrak{Y}_{\alpha}\}_{\alpha\in\square_n}$$

is a complex of sheaves of $\mathbb{C}$*-vector spaces on* $\mathfrak{Y}_{\bullet}$ *obtained by the completion of* $\Omega^{\bullet}_{\mathfrak{X}/M}$ *along* $\mathfrak{Y}_{\alpha}$ *for every* $\alpha \in \square_n$.

Proof. We use induction on n. The case n=2 is nothing but Theorem 3.10. In the case $n \geq 3$, the argument is almost identical with that of Proposition 1.4 in [10, p.61] for the absolute and algebraic case. Hence we omit it.

Q.E.D.

3.14 Proposition. *Let* X *be a complex projective variety embedded in a smooth complex projective variety* Y*, and let* $a_{\bullet} : X_{\bullet} \longrightarrow X$ *be an* n*-cubic hyper-resolution of* X *in the category of complex projective varieties. We denote by* X_h *and* Y_h *the corresponding complex analytic varieties, and by* $a_{h\bullet} : X_{h\bullet} \longrightarrow X_h$ *the corresponding* n*-cubic hyper-reosolution of* X_h *in the category of complex analytic varieties. Let* p *be a point of* X_h*. We take an open neighborhood* V *of* p *in* Y_h *and define* $U := V \cap X_h$ *and* $U_{\alpha} := a_{\alpha}^{-1}(U)$ *for each* $\alpha \in \square_n$*. We consider an* n*-cubic object of the product families of complex analytic varieties*

$$a_{\bullet} \times \mathrm{id}_M : U_{\bullet} \times M \longrightarrow U \times M$$

where M *is a complex space and* id_M *is the identity map on* M*. Then we have a quasi-isomorphism*

$$\Omega^{\bullet}_{V\times M/M}\widehat{|}U \times M \longrightarrow \mathbb{R}(a_{\bullet} \times \mathrm{id}_M)_*\Omega^{\bullet}_{U_{\bullet}\times M/M}. \tag{3.7}$$

Proof. By the same argument used in the proof for the absolute case of Theorem 3.1 (cf. [10, p.41, Théorème 6.9]), we can reduce the proof to the case of n=2. Hence it suffices to prove (3.7) for the following $\square_1^+$-object of complex analytic varieties:

$$\begin{array}{ccc} U_{11} \times M & \longrightarrow & U_{01} \times M \\ \downarrow & & \downarrow{\scriptstyle a_{01}\times id_M} \\ U_{10} \times M & \longrightarrow & U_{00} \times M \\ & & \| \\ & & U \times M \subset V \times M, \end{array}$$

which is a cartesian square, where U_{01} is a smooth analytic variety, $a_{01} : U_{01} \longrightarrow U_{00}$ a proper morphism (hence so is $a_{01} \times \mathrm{id}_M : U_{01} \times M \longrightarrow U_{00} \times M$), $U_{11} \longrightarrow U_{01}$ and $U_{10} \longrightarrow U_{00}$ are closed immersions, such that $a_{01} \times \mathrm{id}_M : (U_{10} \times M) \setminus (U_{11} \times M) \longrightarrow (U_{00} \times M) \setminus (U_{10} \times M)$ is an isomorphism. Furthermore, using

Proposition 3.13, we can reduce the proof to that for the case where U_{01} and U_{00} are irreducible (for the details of this procedure we refer to the proof of Théorème 1.5 in [10, p.62]). Now we shall check the proof for this case.

We write X, X', Y, Y', Z and f instead of U_{00}, U_{01}, U_{10}, U_{11}, V and a_{01}, respectively. Since X, X' are open subsets of complex projective varieties, by the result of Hironaka (Elimination of points of indeterminancy of a rational mapping, [12]), there exists a commutative diagram

(3.8)
$$\begin{array}{ccccc} \overline{X'} & \xrightarrow{f_4} & \overline{X} & & \\ & \searrow^{f_3} & \downarrow{\scriptstyle f_2} & \searrow^{f_1} & \\ & & X' & \xrightarrow[f]{} & X \hookrightarrow Z \end{array}$$

such that (i)f_1, f_3 are the composits of blowing-ups along non-singular centers, (ii) $\overline{X}, \overline{X'}$ are non-singular, and (iii) f_2, f_4 are proper morphisms. Blowing up Z along the same centers as those of $f_1 : \overline{X} \to X$, we have the following diagram

(3.9)
$$\begin{array}{ccccc} \overline{Y} & \xrightarrow{\bar{\iota}} & \overline{X} & \longrightarrow & \overline{Z} \\ \downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle g_1} \\ Y & \xrightarrow{\iota} & X & \longrightarrow & Z \end{array}$$

where $\overline{Y} := f_1^{-1}(Y)_{red}$. Forming direct product of each term in the diagram (3.9) with M, we come to the same setting as in Theorem 3.12. Hence, by that theorem, we conclude that the simple complex associated to the following $\square_1^+$-object of lower bounded complex of sheaves of $\mathbb{C}$-vector spaces on $X \times M$

$$\begin{array}{ccc} \mathbb{R}(h_1 \times \mathrm{id}_M)_* \Omega^{\bullet}_{\overline{Z} \times M} \widehat{|\overline{Y} \times M} & \longleftarrow & \mathbb{R}(f_1 \times \mathrm{id}_M)_* \Omega^{\bullet}_{\overline{Z} \times M} \widehat{|\overline{X} \times M} \\ \uparrow & & \uparrow \\ (\iota \times \mathrm{id}_M)_* \Omega^{\bullet}_{Z \times M} \widehat{|Y \times M} & \longleftarrow & \Omega^{\bullet}_{Z \times M} \widehat{|X \times M}, \end{array}$$

where $h_1 := f_1 \circ \bar{\iota}$, is acyclic in $D^+(X \times M, \mathbb{C})$. If we define $s(X \times M / Y \times M)$, $s(\overline{X} \times M / \overline{Y} \times M)$ to be the single complexes associated to the morphisms of complexes

$$\Omega^{\bullet}_{Z \times M} \widehat{|X \times M} \to (\iota \times \mathrm{id}_M)_* \Omega^{\bullet}_{Z \times M} \widehat{|Y \times M} \text{ and}$$

$$\Omega^{\bullet}_{\overline{Z} \times M} \widehat{|\overline{X} \times M} \to (\bar{\iota} \times \mathrm{id}_M)_* \Omega^{\bullet}_{\overline{Z} \times M} \widehat{|\overline{Y} \times M},$$

respectively, then the above statetement is equivalent to that the morphism $(f_1 \times id_M)^* : s(X \times M/Y \times M) \to s(\overline{X} \times M/\overline{Y} \times M)$ induced by $f_1 \times id_M$ is a quasi- isomorphism. Here we should notice that, since X', $\overline{X'}$ are non-singular, $s(X' \times M/Y' \times M)$ and $s(\overline{X'} \times M/\overline{Y'} \times M)$ are defined as the single complexes assciated to the morphisms of complexes

$$\Omega^{\bullet}_{X' \times M/M} \to (\iota' \times \mathrm{id}_M)_* \Omega^{\bullet}_{X' \times M/M} \widehat{|} Y' \times M \quad \text{and}$$

$$\Omega^{\bullet}_{\overline{X'} \times M/M} \to (\overline{\iota'} \times \mathrm{id}_M)_* \Omega^{\bullet}_{\overline{X'} \times M/M} \widehat{|} \overline{Y'} \times M,$$

respectively, where $\overline{Y'} := f_4^{-1}(\overline{Y})_{red} = f_3^{-1}(Y')_{red}$ and $\iota' : Y' \to X'$, $\overline{\iota'} : \overline{Y'} \to \overline{X'}$ are natural inclusions. We consider the following diagram derived from (3.8)

(3.10)

$$\begin{array}{ccc} s(\overline{X'} \times M/\overline{Y'} \times M) & \xleftarrow{(f_4 \times id_M)^*} & s(\overline{X} \times M/\overline{Y} \times M) \\ & \nwarrow^{(f_3 \times id_M)^*} \quad \uparrow_{(f_2 \times id_M)^*} \quad \nwarrow_{(f_1 \times id_M)^*} & \\ & s(X' \times M/Y' \times M) \xleftarrow[f^*]{} s(X' \times M/Y' \times M) & \end{array}$$

By the same reasoning as for $(f_1 \times id_M)^*$, we conclude that $(f_3 \times id_M)^*$, $(f_4 \times id_M)^*$ are quasi-isomprphisms on X, X' , respectively. Hence by the commutativity of the diagram in (3.10), we conclude that $(f_2 \times id_M)^*$ is a quasi-isomorphism on X' and so is $(f \times id_M)^*$. This completes the proof of the proposition.

Q.E.D.

We are now in a position to prove Theorem 3.7 and Theorem 3.8.

Proof of Theorem 3.7: By the assumption, we can take a system $\mathcal{U} = \{(\mathcal{U}'_i, \mathcal{U}_i), \varphi_i, (\mathcal{Y}'_i, \mathcal{Y}_i, \pi_i)\}$ of relative local embeddings of $\mathfrak{X}$ which satisfies the following conditions:

(3.11)

For each i there exists a point $p_i \in \mathcal{U}_i$ and an embedding $e_i : X_{\pi(p_i)} \to Y_{p_i}$ of $X_{\pi(p_i)}$ (the fiber of $\mathfrak{X}$ over $\pi(p_i)$) into a smooth complex projective variety Y_{p_i} such that

(i) $a_{\bullet}^{-1}(\mathcal{U}'_i) \xrightarrow{a_{\bullet}|a_{\bullet}^{-1}(\mathcal{U}'_i)} \mathcal{U}'_i \xrightarrow{\pi|\mathcal{U}'_i} \pi(\mathcal{U}'_i)$ is isomorphic to

$$(a_{\bullet}^{-1}(\mathcal{U}'_i) \cap X_{\pi(p_i)}) \times \pi(\mathcal{U}'_i) \xrightarrow{a_{\bullet} \times \mathrm{id}_{\pi(\mathcal{U}'_i)}} (\mathcal{U}'_i \cap X_{\pi(p_i)}) \times \pi(\mathcal{U}'_i) \xrightarrow{\mathrm{Pr}_{\pi(\mathcal{U}'_i)}} \pi(\mathcal{U}'_i)$$

(for the notation see Definition 1.12)

(ii) $\mathcal{Y}'_i = D'_i \times \pi(\mathcal{U}'_i)$ and $\mathcal{Y}_i = D_i \times \pi(\mathcal{U}_i)$, where D'_i, D_i are open neighborhoods of the point $e_i(p_i)$ in Y_{p_i} with $D_i \subset D'_i$, and

(iii) $\varphi_i(\mathcal{U}'_i) = (e_i(X_{\pi(p_i)}) \cap D'_i) \times \pi(\mathcal{U}'_i)$. and $\varphi_i(\mathcal{U}_i) = (e_i(X_{\pi(p_i)}) \cap D_i) \times \pi(\mathcal{U}_i)$.

Then by Proposition 3.14 the natural map

$$\widehat{\Omega}^{\bullet}_{\mathcal{U}'_i/\pi(\mathcal{U}'_i)} \to \mathbb{R}a_{\bullet|a_\bullet^{-1}(\mathcal{U}'_i)} {}_*\Omega^{\bullet}_{a_\bullet^{-1}(\mathcal{U}'_i)/\pi(\mathcal{U}'_i)} = \mathbb{R}a_{\bullet *}\Omega^{\bullet}_{\mathfrak{X}_\bullet/M|\mathcal{U}'_i}$$

is a quasi-isomorphism on $\mathcal{U}'_i$, hence

$$j_*(\widehat{\Omega}^{\bullet}_{\mathcal{U}'_i/\pi(\mathcal{U}'_i)})|_{\mathcal{U}_i} \to j_*(\mathbb{R}a_{\bullet *}\Omega^{\bullet}_{\mathfrak{X}_\bullet/M|\mathcal{U}'_i})|_{\mathcal{U}_i}$$

is a quasi-isomorphism on $\mathfrak{X}$ for every i, where $j : \mathcal{U}'_i \to \mathfrak{X}$ is the inclusion map. From this it follows that for any $(i) = \{i_0 < i_1 < \cdots < i_p\}$

$$C_{(i)} := j_*(\widehat{\Omega}^{\bullet}_{\mathcal{U}'_{(i)}/\pi(\mathcal{U}'_{(i)})})|_{\mathcal{U}_{(i)}} \to D_{(i)} := j_*(\mathbb{R}a_{\bullet *}\Omega^{\bullet}_{\mathfrak{X}_\bullet/M|\mathcal{U}'_{(i)}})|_{\mathcal{U}_{(i)}}$$

is a quasi-isomorphism. Similarly as for $C(\mathcal{U})$, we define a double complex $D(\mathcal{U})$, using $\{D_{(i)}\}$, which is nothing but $\mathbb{R}a_{\bullet *}\Omega^{\bullet}_{\mathfrak{X}_\bullet/M}$. Therefore we conclude that the natural map

$$\widehat{\Omega}^{\bullet}_{\mathfrak{X}/M}(\mathcal{U}) \to \mathbb{R}a_{\bullet *}\Omega^{\bullet}_{\mathfrak{X}_\bullet/M}$$

is a quasi-isomorphism. Since any system of relative local embeddings of $\mathfrak{X}$ has its refinement satisfying the conditions (i),(ii),(iii) in (3.11) we obtain the theorem.

Proof of Theorem 3.8: Since the poblem is local, we may assume that $\pi : \mathfrak{X} \to M$ is a product family, namely $\pi := \mathrm{Pr}_M : \mathfrak{X} = X \times M \to M$, where X is a complex projective variety, M a complex space, and $\pi := \mathrm{Pr}_M$ the projection to M. Furthermore we may assume that X is embedded in a smooth complex projective variety Z. We define $\mathfrak{Z} := Z \times M$ and $\pi_1 := \mathrm{Pr}_M : \mathfrak{Z} = Z \times M \to M$ the projection to M. Under this setting we shall prove that

$$\pi^{\cdot}\mathcal{O}_M \to \Omega^{\bullet}_{\mathfrak{Z}/M}|\mathfrak{X} \tag{3.12}$$

is a quasi-isomorphism on $\mathfrak{X}$. In the following we shall confuse complex algebraic objects and their associated analytic objects, and write them by the same letters. To prove (3.12) we proceed by induction on $\dim_{\mathbb{C}} X$. If $\dim_{\mathbb{C}} X = 0$, there is nothing to be proved. We assume that (3.12) holds for any X with $0 \leq \dim_{\mathbb{C}} X < n$. By the Hironaka resolution theorem ([12]) there is the following commutative diagam:

$$\begin{array}{ccccc} Y' & \xrightarrow{\iota'} & X' & \hookrightarrow & Z' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\iota} & X & \hookrightarrow & Z \end{array} \tag{3.13}$$

with the property $g_{|X'-Y'} : X' - Y' \to X - Y$ is an isomorphosm, where X' is a smooth complex projective variety, $f : X' \to X$ a proper morphism, Y a proper closed subvariety of X, $Y' := f^{-1}(Y)_{red}$, and ι, ι' closed immersions. Taking direct product of each term in (3.13) with M, we obtain the commutative diagram

$$\begin{array}{ccccc} \mathfrak{Y}' & \xrightarrow{I'} & \mathfrak{X}' & \hookrightarrow & \mathfrak{Z}' \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle F} \\ \mathfrak{Y} & \xrightarrow{I} & \mathfrak{X} & \hookrightarrow & \mathfrak{Z}, \end{array} \tag{3.14}$$

where $\mathfrak{X} := X \times M$, $\mathfrak{X}' := X' \times M$, $F := f \times \mathrm{id}_M$, etc.. Then, by Theorem 3.12 it follows that

$$\begin{array}{ccc} \mathbb{R}(I \circ H)_* \Omega^{\bullet}_{\mathfrak{X}'/M} \widehat{|}\mathfrak{Y}' & \longleftarrow & \mathbb{R}G_* \Omega^{\bullet}_{\mathfrak{X}'/M} \\ \uparrow & & \uparrow \\ \mathbb{R}I_* \Omega^{\bullet}_{\mathfrak{Z}/M} \widehat{|}\mathfrak{Y} & \longleftarrow & \Omega^{\bullet}_{\mathfrak{Z}/M} \widehat{|}\mathfrak{X} \end{array}$$

is acyclic in $D^+(\mathfrak{X}, \mathbb{C})$. Therefore, for any relatively compact open subset $\mathfrak{X}_0$ of $\mathfrak{X}$, we have the following long exact sequence of cohomology

$$\begin{aligned} &\to H^i(\mathfrak{X}_0, \Omega^{\bullet}_{\mathfrak{Z}/M} \widehat{|}\mathfrak{X}) \to H^i(\mathfrak{X}_0, \mathbb{R}I_* \Omega^{\bullet}_{\mathfrak{Z}/M} \widehat{|}\mathfrak{Y}) \oplus H^i(\mathfrak{X}_0, \mathbb{R}G_* \Omega^{\bullet}_{\mathfrak{X}'/M}) \\ &\to H^i(\mathfrak{X}_0, \mathbb{R}(I \circ H)_* \Omega^{\bullet}_{\mathfrak{X}'/M} \widehat{|}\mathfrak{Y}') \to H^{i+1}(\mathfrak{X}_0, \Omega^{\bullet}_{\mathfrak{Z}/M} \widehat{|}\mathfrak{X}) \to \\ &\cdots \end{aligned} \tag{3.15}$$

On the other hand, applying Theorem 3.4 for $\mathcal{A} = \pi'^{\cdot} \mathcal{O}_M$, we derive from (3.14) that

$$\begin{array}{ccc} (I \circ H)_* \pi'^{\cdot}_{|\mathfrak{Y}'} \mathcal{O}_M & \longleftarrow & G_* \pi'^{\cdot} \mathcal{O}_M \\ \uparrow & & \uparrow \\ I_* \pi^{\cdot}_{|\mathfrak{Y}} \mathcal{O}_M & \longleftarrow & \pi^{\cdot} \mathcal{O}_M \end{array}$$

is acyclic in $D^+(\mathfrak{X}, \mathbb{C})$, where $\pi' := \mathrm{Pr}_M : \mathfrak{X}' = X' \times M \to M$, the projection to M. Therefore we have the following long exact sequence of cohomology

$$\to H^i(\mathfrak{X}_0, \pi^{\cdot}\mathcal{O}_M) \to H^i(\mathfrak{X}_0, I_*\pi^{\cdot}_{|\mathfrak{Y}}\mathcal{O}_M) \oplus H^i(\mathfrak{X}_0, G_*\pi'^{\cdot}\mathcal{O}_M)$$

(3.16)

$$\to H^i(\mathfrak{X}_0, (I \circ H)_*\pi'^{\cdot}_{|\mathfrak{Y}'}\mathcal{O}_M) \to H^{i+1}(\mathfrak{X}_0, \pi^{\cdot}\mathcal{O}_M) \to \cdots$$

There naturally exist homomorphisms from (3.16) to (3.15). Among these homomorphisms,

$$H^i(\mathfrak{X}_0, I_*\pi^{\cdot}_{|\mathfrak{Y}}\mathcal{O}_M) \to H^i(\mathfrak{X}_0, \mathbb{R}I_*\Omega^{\bullet}_{\mathfrak{Z}/M}\widehat{|}\mathfrak{Y}),$$

$$H^i(\mathfrak{X}_0, (I \circ H)_*\pi'^{\cdot}_{|\mathfrak{Y}'}\mathcal{O}_M) \to H^i(\mathfrak{X}_0, \mathbb{R}(I \circ H)_*\Omega^{\bullet}_{\mathfrak{X}'/M}\widehat{|}\mathfrak{Y}')$$

are isomprphisms on $\mathfrak{X}_0$ by the induction hypothesis, and

$$H^i(\mathfrak{X}_0, G_*\pi'^{\cdot}\mathcal{O}_M) \to H^i(\mathfrak{X}_0, \mathbb{R}G_*\Omega^{\bullet}_{\mathfrak{X}'/M})$$

is also, because $\pi' : \mathfrak{X}' \to M$ is a smooth family ([3, p.15, 2.23.2]). Hence we conclude that

$$H^i(\mathfrak{X}_0, \pi^{\cdot}\mathcal{O}_M) \to H^i(\mathfrak{X}_0, \Omega^{\bullet}_{\mathfrak{Z}/M}\widehat{|}\mathfrak{X})$$

is an isomorphism on $\mathfrak{X}_0$, which means $\pi^{\cdot}\mathcal{O}_M \to \Omega^{\bullet}_{\mathfrak{Z}/M}\widehat{|}\mathfrak{X}$ is a quasi-isomorphism on $\mathfrak{X}_0$ as required. This completes the proof of Theorem 3.8.

Corollary 3.9 follows from Theorem 3.7 and Theorem 3.8.

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