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On Set Representations and Intersection Numbers of Some Graphs

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Abstract

Let G be any simple and connected graph, $i(G)$ be the intersection number of G in the sense of Mckee et al. [6], and $\theta_1(G)$ be the minimum number of cliques $\{Q_j\}$ of G by which the edge set $E(G)$ is covered. In this paper using the fact that $i(G) = \theta_1(G)$ we shall determine the intersection number for the graphs as follows: split graphs, the complete r-partite graphs, the $(n-2)$-regular graphs of order n, and the complementary graphs of the cycle C_n of order n respectively.

Key words: intersection number, edge clique cover, split graph, the complete r-partite graph, regular graph.

1 Introduction and Preliminary

Throughout this paper any graphs are assumed always to be finite, simple and connected. The terminology and notion concerning graphs follow Chartrand et al. [5] unless otherwise stated. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$, and put $q(G) = |E(G)|$. We begin with the definition of set representations and edge clique covers of G. A *set representation* of G is a mapping ϕ of $V(G)$ to the set of non-empty finite sets of positive integers with the following property:

(1.1) *For any $u, v \in V(G)$, u and v are adjacent if and only if* $\phi(u) \cap \phi(v) \neq \emptyset$.

For any set representation ϕ of $G, S(\phi) = \cup\{\phi(u); u \in V(G)\}$ is called the range of ϕ, and $|\phi| = |S(\phi)|$ is the rank of ϕ. We note that any graphs have set representations. For example let $E(G) = \{e_1, e_2, e_3, \cdots, e_q\}$, where $q = q(G)$, and for any $u \in V(G)$ we put $\phi(u) = \{j; e_j$ is incident with $u\}$. Then this ϕ is a representation of G with $|\phi| = q(G)$. The *intersection number* $i(G)$ of G is the minimum number of $|\phi|$ for any set representations ϕ of G. Any representation ϕ of G is said to be *minimal* if $|\phi| = i(G)$.

By the above example of a set representation of G, we have $i(G) \leq q(G)$. A subset W of $V(G)$ is called an *independent set* of G if any distinct two vertices in W are not adjacent. For any independent set W and set representations ϕ of G, the sets $\phi(w), w \in W$, are mutually disjoint by (1.1). So we have an estimation of $i(G)$ as follows:

(1.2) $\beta_0(G) \leq i(G) \leq q(G)$,

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where the $\beta_0(G)$ is the vertex independent number of G, *i.e.* the maximum cardinal number of independent sets of G.

Any complete subgraph of G is called a *clique*, and especially is *maxclique* if it is not properly contained in another cliques. A family of cliques $\mathbf{F} = \{Q_j; j = 1, 2, \cdots, m\}$ of G is called an *edge* [resp. *vertex*] *clique cover* of G if $E(G) \subseteq \cup_{j=1}^{m} E(Q_j)$ [resp. $V(G) \subseteq \cup_{j=1}^{m} V(Q_j)$]. For example the edge set $E(G)$ and the set $MC(G)$ of all maxcliques of G are edge clique covers of G.

Here we introduce the following three kinds of numbers for G defined by:

(1.3) $\theta_1(G) =$ the minimum cardinal number of $|\mathbf{F}|$ for any edge clique covers $\mathbf{F}$ of G,

(1.4) $\theta_0(G) =$ the minimum cardinal number of $|\mathbf{F}|$ for any vertex clique covers $\mathbf{F}$ of G,

(1.5) $\theta_m(G) = |MC(G)|$.

We say that any edge [resp. vertex] clique cover $\mathbf{F}$ of G is *minimal* if $|\mathbf{F}| = \theta_1(G)$ [resp.$\theta_0(G)$]. By the definitions above we have

(1.6) $\beta_0(G) \leq \theta_0(G) \leq \theta_1(G) \leq \theta_m(G)$.

The aim of this paper is to determine the intersection numbers for some graphs. In section 2 we state the relationship between set representations and edge clique covers of G following [6]. By the relationship we have $i(G) = \theta_1(G)$. We note also the conditions for G to be $i(G) = q(G), i(G) = \theta_m(G)$ and $i(G) = \beta_0(G)$ respectively.

In section 3, the relations of $i(G)$ and $\beta_0(G)$ are established for any split graphs G, and the intersection number of the complete r-partite graphs is given. The intersection numbers of the $(n-2)$-regular graph of order n and the complementary graph of the cycle C_n are determined respectively in the final section.

For any graph G we use the following notation:

(1.7) $N[u] = N(u) \cup \{u\}$ for $u \in V(G)$, where $N(u)$ is the neighborhood of u.

(1.8) $E(G, u)$ is the set of all edges of G incident with $u \in V(G)$.

(1.9) $< W >=$ the induced subgraph of G from $W \subset V(G)$.

(1.10) $< W_1, W_2, \cdots, W_k >=< \cup_{j=1}^{k} W_j >$.

Throughout this paper any symbols of variables, say i, j, k, m, n and so on, denote any positive integers unless otherwise stated. For any m, n with $m \leq n$, $[m, n]$ is the set of the consecutive numbers from m to n, and especially $[n] = [1, n]$.

2 Key Lemma and its Consequences

First following [6] we state the relationship between set representations and edge clique covers of any graphs G. Let ϕ be any set representation of G. For any $k \in S(\phi)$ we define a subgraph Q_k by:

(2.1) $Q_k =< \{u \in V(G); k \in \phi(u)\} >$.

Then by (1.1) it is seen easily that Q_k is a clique and $\mathbf{F}_\phi = \{Q_k; k \in S(\phi)\}$ is an edge clique cover of G.

Conversely for some m let $\mathbf{F} = \{Q_j; j \in [m]\}$ be any edge clique cover of G. For any $u \in V(G)$ we define a non-empty subset $\phi(u)$ of $[m]$ by:

(2.2) $\phi(u) = \{j \in [m]; v \in V(Q_j)\}$.

Then ϕ is a set representation of G, which is denoted by $\phi_{\mathbf{F}}$. We note that

(2.3) $|\phi| = |\mathbf{F}_\phi|$ and $|F| = |\phi_{\mathbf{F}}|$.

Summarizing the above discussion we have our Key lemma.

Key Lemma. *Let G be any graph. Then*

(1) *For any set representation ϕ of G there corresponds an edge clique cover $\mathbf{F}_\phi$ of G defined by* (2.1) *with $|\phi| = |\mathbf{F}_\phi|$,*

(2) *For any edge clique cover $\mathbf{F}$ of G there corresponds a set representation $\phi_{\mathbf{F}}$ of G defined by* (2.2) *with $|\mathbf{F}| = |\phi_{\mathbf{F}}|$.* □

Key lemma teaches us that the study on set representations of G is equivalent to the study on edge clique covers of G. The next is an immediate consequence of Key lemma (cf. [6], Slater [8]).

Theorem 2.1. *$i(G) = \theta_1(G)$ for any graphs G.* □

From (1.2), (1.6) and Theorem 2.1 we have

(2.4) $\beta_0(G) \leq i(G) = \theta_1(G) \leq \theta_m(G) \leq q(G)$.

Here we collect some theorems concerning $i(G)$, which are immediate consequences of Key Lemma. If G is triangle-free, then $MC(G) = E(G)$ and $E(G)$ is only one edge clique cover of G. Hence we have

Theorem 2.2. *For any graph $G, i(G) = q(E)$ if and only if G is triangle-free.* □

An edge e of G is said to be *proper* if there is a unique maxclique Q such that $e \in E(Q)$. A maxclique is also said to be *proper* if it has at least one proper edge, and otherwise to be *non-proper.* If any maxcliques of G are proper, then $MC(G)$ is the only one edge maxclique cover of G. In this case we have $i(G) = \theta_m(G)$. Conversely If G has a non-proper maxclique Q, then $MC(G) \setminus \{Q\}$ is an edge maxclique cover of G. Therefore we have

Theorem 2.3. *For any graph $G, i(G) = \theta_m(G)$ if and only if every maxclique of G is proper.* □

On graphs G with $i(G) = \theta_m(G)$ it is investigated in Wallis et al.[10], in which such graph is called a *maximal clique irreducible graph.* In the author's paper [7] some non-proper maxcliques in regular graphs are considered.

We recall that a subset S of $V(G)$ is a *dominating set* of G if $V(G) = \cup\{N[s]; s \in S\}$.

Theorem 2.4. *For any graph G let $W = \{v_1, v_2, ..., v_m\}$ be an independent set of G, where $m = \beta_0(G)$, and put $Z = V(G) \setminus W$. Then the following conditions for G are mutually equivalent.*

(1) $i(G) = \beta_0(G)$,

(2) *W is a dominating set of G, and for any $u, v \in Z, u$ and v are adjacent if and only if $N(u) \cap N(v) \cap W \neq \emptyset$,*

(3) *For every $v_j \in W, Q_j := < N[v_j] >$ is a maxclique and the family $\{Q_j; j \in [m]\}$ is an edge maxclique cover of G.*

Proof. It suffices to see (1) $\Rightarrow$ (2). Let ϕ be a set representation of G with $|\phi| = \beta_0(G)$. Then we may assume that $S(\phi) = [m]$ and $\phi(v_j) = \{j\}$ for $j \in [m]$. Hence for any $u \in Z$

we have $\phi(u) = \{k \in [m]; v_k \in N(u)\}$ and for any $u, v \in Z, \phi(u) \cap \phi(v) \neq \emptyset$ if and only if $N(u) \cap N(v) \cap W \neq \emptyset$ by (1.1). So (2) follows from (1). (3) $\Rightarrow$ (1) is obvious from (2.4). □

In Brigham et al.[3] any graphs G with $i(G) = \theta_0(G)$ are characterized in terms of vertex clique covers of G as follows.

Theorem 2.5. *The following conditions for any graph G are mutually equivalent*:

(1) $i(G) = \theta_0(G)$,

(2) *Every minimal vertex clique cover of G is a minimal edge clique cover*,

(3) $\beta_0(G) = \beta_1(G)$,

where $\beta_1(G)$ is the maximum number of edges of G having the property that no two are in the same clique. □

Combining Theorems 2.3 and 2.4 we have

Theorem 2.6. *Under the same notation as in Theorem 2.4, the following conditions for G are mutually equivalent*:

(1) $i(G) = \beta_0(G) = \theta_m(G)$,

(2) *For every $v_j \in W, Q_j :=< N[v_j] >$ is a maxclique and $MC(G) = \{Q_j; j \in [m]\}$.* □

Examples 2.7. The following are examples of graphs as in the above theorems:

(1) $i(G) = q(G)$ for $G =$ any trees or bipartite graphs.

(2) $i(G) = \theta_m(G)$ for $G =$ the wheel graphs $W_n, n > 3$, the interval graphs (cf. [10]), or 3-regular graphs (cf.[7], [9]).

(3) $i(G) = \beta_0(G) < \theta_m(G)$ for $G =$ the 3-sun (Hajos-graph) S_3 (cf. [2]).

(4) $i(G) = \beta_0(G) = \theta_m(G)$ for $G =$the complete graphs, or the intersection graphs $\Omega(P([m]))$ for any $m \geq 2$. Where $P([m])$ is the family of all non-empty subsets of $[m]$, $\Omega(P([m])$ is the graph with the vertex set $P([m])$ such that any distinct $S_j, S_k \in P([m])$ are adjacent if and only if $S_j \cap S_k \neq \emptyset$.

In [7] any r-regular graph G with $i(G) < \theta_m(G)$ is characterized for $r = 4$ and 5. For example any 4-regular graph G, $i(G) < \theta_m(G)$ if and only if G contains the 3-sun graphs as induced subgraphs.

Theorem 2.8. *For any given $n > 1$ the set of the intersection numbers $i(G)$ of graphs G of order n is the integer interval $[\lfloor \frac{n^2}{4} \rfloor]$.*

Proof. In Erdös et al.[4] it is proved that $i(G) \leq \lfloor \frac{n^2}{4} \rfloor$ for any graph G of order n, and that $i(G_0) = \lfloor \frac{n^2}{4} \rfloor$ for the complete bipartite graph $G_0 = K(k, k)$ or $K(k+1, k)$ according as even $n = 2k$ or odd $n = 2k+1$. Note that any bipartite graph is triangle-free. So removing repeatedly a cycle edge from G_0 until its spanning tree, for every $k \in [n-1, \lfloor \frac{n^2}{4} \rfloor]$ we get a bipartite graph G with $i(G) = k$.

On the other hand, let $p \in [2, n-1]$, and define a graph G_p of order n as follows: $V(G_p) = \{v_1, v_2, \cdots, v_{p-1}, v_p, v_{p+1}, \cdots, v_n\}$ such that $Q :=< v_p, v_{p+1}, \cdots, v_n >$ is the complete subgraph K_{n-p+1} and $E(G_p) = E(Q) \cup \{v_j v_n; j \in [p-1]\}$. Then $i(G_p) = p$ by Theorem 2.4. Obviously $i(K_n) = 1$. This completes the proof. □

3 Intersection numbers of split graphs and the complete r-partite graphs

Let G be any connected split graph, *i.e.*, $V(G)$ is partitioned into two sets D and S such that $< D >$ is a clique and S is an independent set. Let $k = |D|$ and $s = |S|$. Note that $< N[s] >$ is a clique for every $s \in S$ and $\beta_0(G)$ is equal to s or $s+1$. Dividing the following three cases (a)-(c) we determine $i(G)$.

Case (a): There is a $v \in D$ with $deg(v) = k-1$.
Then $N[v] = D$ and $W = S \cup \{v\}$ is the maximal independent set. Moreover the family $\{< N[w] >; w \in W\}$ of cliques is an edge clique cover of G. Hence $i(G) = \beta_0(G) = s+1$ by Theorem 2.4.

Case (b): For any distinct $u, v \in D$ there exists an $s \in S$ for which $u, v \in N(s)$.
Then S is the maximal independent subset of $V(G)$ and the family $\{< N[s] >; s \in S\}$ is an edge clique cover of G. Hence $i(G) = \beta_0(G) = s$. In this case note that $deg(u) \geq k$ for all $u \in D$.

Case (c): $deg(u) \geq k$ for all $u \in D$ and there is at least one edge of $< D >$ which is not covered by any cliques $< N[s] >, s \in S$.
Then S is the maximal independent subset of $V(G)$ and $\{< D >\} \cup \{< N[s] >; s \in S\}$ is the minimal edge clique cover of G. Hence $i(G) = s+1$ and $\beta_0(G) = s$.

Summarizing we have

Theorem 3.1. *Let G be a connected split graph and let* $\mathbf{F} = \cup_{s \in S} E(< N[s] >)$ *under the above notation. Then the following holds:*

(1) $i(G) = \beta_0(G) = |S| + 1$ *if and only if S is not a dominating set of G.*

(2) $i(G) = \beta_0(G) = |S|$ *if and only if* $E(< D >) \subset \mathbf{F}$.

(3) $i(G) = |S|+1, \beta_0(G) = |S|$ *if and only if S is a dominating set of G and* $E(< D >) \backslash \mathbf{F} \neq \emptyset$.

□

Next we consider the intersection number of the complete r-partite graphs with $r \geq 2$. For the complete r-partite graph $G = K(m_1, m_2, \cdots, m_r)$ assume $m_1 \geq m_2 \geq m_3 \geq \cdots \geq m_r \geq 1$ and denote by $V_j(G)$ the j-th partite set with $|V_j(G)| = m_j$ for $j \in [r]$.

Lemma 3.2.

(1) $i(K(m_1, m_2)) = m_1 m_2$.

(2) $i(K(m_1, m_2, \cdots, m_{r-1}, 1)) = i(K(m_1, m_2, \cdots, m_{r-1}))$.

(3) $i(K(m_1, m_2, \cdots, m_{r-1}, m_r + 1)) = i(K(m_1, m_2, \cdots, m_{r-1}, m_r)) + m_1$.

Proof. (1) follows from Theorem 2.2. To see (2), we put $G = K(m_1, m_2, \cdots, m_{r-1}), G' = K(m_1, m_2, \cdots, m_{r-1}, 1)$ with $V_r(G') = \{v\}$, and let $\{Q_j; j \in [s]\}$ be a minimal edge clique cover of G, where $s = i(G)$. Then for every $j \in [s], R_j := < v, Q_j >$ is a maxclique of G' and the family $\{R_j; j \in [s]\}$ is an edge maxclique cover of G'. Hence we have $i(G') = i(G)$, because $N(v) = \cup_{j=1}^{r-1} V_j(G)$ and $i(G) \leq i(G')$. To see (3) put $G = K(m_1, m_2, \cdots, m_r)$, $G' = K(m_1, m_2, \cdots, m_r + 1)$ with $V_r(G') = V_r(G) \cup \{v\}$, and let $\{Q_j; j \in [s]\}$ be a minimal edge clique cover of G, where $s = i(G)$. We regard G as a subgraph of G'. Since $V_r(G')$ is an independent set of G', any clique covering the edges in $E(G', v)$ does not cover any edges in $E(G, w)$ for any $w \in V_r(G)$. Further since $V_1(G')$ is an independent set of G', there are at least m_1 maxcliques of G' which covers $E(G', v)$. Actually we can find a family $\{Q_j; j \in [s+1, s+m_1]\}$ of maxcliques of G' which covers $E(G', v)$. Therefore $\{Q_j; j \in [s+m_1]\}$ is a minimal maxclique cover of G' and $i(G') = i(G) + m_1$. □

The next result is derived inductively from the above lemma.

Theorem 3.3. $i(K(m_1, m_2, \cdots, m_r)) = m_1(\sum_{j=2}^{r} m_j - r + 2)$. □

4 Intersection numbers of some regular graphs

First we consider a minimal set representation of the $(n-2)$-regular graph G_n of order even $n = 2m$. Since G_{2m} is the complementary graph of the 1-regular graph mK_2, we label $V(G_{2m}) = \{u_j, v_j; j \in [m]\}$ such that for every $j \in [m], u_j$ [resp. v_j] is adjacent to all another vertices except v_j [resp. u_j]. As $G_4 = C_4$, we have $i(G_4) = 4$ from Theorem 2.2. In what follows let $m > 2$. Since there are many maxcliques in G_{2m}, precisely $\theta_m(G_{2m}) = 2^m$, in order to determine $i(G_{2m})$ we consider not edge clique covers but set representations of G_{2m}. For some t, the construction of any set representation ϕ of G_{2m} with $|\phi| = t$ is reduce to give a family $\mathbf{F}_m = \{S_j; j \in [m]\}$ of mutually distinct subsets in $[t]$ with the following the properties:

(4.1) $S_j \cap S_k \neq \emptyset$, neither $S_j \subset S_k$ nor $S_k \subset S_j$ for any distinct $j, k \in [m]$,

(4.2) $|S_j| \leq \lfloor \frac{t}{2} \rfloor$ for any $j \in [m]$.

Indeed for such ϕ, we may assume without loss of generality that for every $j \in [m], S_j := \phi(u_j)$ is a non-empty subset of $[t]$ and the family $\{S_j; j \in [m]\}$ satisfies (4.1) and (4.2). Conversely for any $\mathbf{F}_m = \{S_j; j \in [m]\}$ in the above, we put $\phi(u_j) = S_j$ and $\phi(v_j) = [t] \setminus S_j$ for $j \in [m]$. Then for any distinct $j, k \in [m]$, $\phi(v_j) \cap \phi(u_k)$ and $\phi(v_j) \cap \phi(v_k)$ are non-empty by (4.1) and (4.2) respectively. Hence ϕ is a set representation of G_{2m} with $|\phi| = t$. Here we denote $\phi(\mathbf{F}_m)$ by the set representation ϕ defined from $\mathbf{F}_m$. Any family $\mathbf{F} = \{S_j\}$ of subsets of $[t]$ is called an *intersecting Sperner family* if it satisfies the condition (4.1). The determination of $i(G_{2m})$ is to find the smallest positive integer t such that in $[t]$ there exists an intersecting Sperner family $\mathbf{F}, |\mathbf{F}| = m$, with (4.2). We note that from any non-empty subfamily $\mathbf{F}'$ with $|\mathbf{F}'| = m'$ of any intersecting Sperner family $\mathbf{F}, |\mathbf{F}| = m$, with (4.2) there corresponds a set representation $\phi(\mathbf{F}')$ of $G_{2m'}$. So our problem is reduced to a combinatorial problem to find the largest cardinal number of any intersecting Sperner family with (4.2) in $[t]$ for any given t. The answer for this problem is derived from some results stated in Bollobás [1].

For any given $t > 1$ let $X_t = [t]$ and use the notation concerning family of subsets of X_t as follows:

$$X_t(\leq r) = \{S; S \subset X_t, |S| \leq r\} \text{ for any } r \text{ with } 1 \leq r < t,$$

$$X_t(r) = \{S; S \subset X_t, |S| = r\} \text{ for any } r \text{ with } 1 \leq r < t,$$

$$X_t(\{j\}) = \{S; j \in S \subset X_t\} \text{ for any fixed } j \in X_t,$$

$$X_t(r, \{j\}) = X_t(r) \cap X_t(\{j\}).$$

Under these notation we have the next lemma, which is due to [1, Theorem13.2].

Lemma 4.1. *Let* $\mathbf{F}$ *be any intersecting Sperner family in* $X_t(\leq \lfloor \frac{t}{2} \rfloor)$. *Then*

(4.3) $\sum_{A \in \mathbf{F}} \binom{t-1}{|A|-1}^{-1} \leq 1.$

Especially the equality holds in (4.3) *if and only if* $\mathbf{F} = X_t(r, \{j\})$ *for some* $r < \frac{t}{2}, j \in X_t$ *or* t *is even and* $\mathbf{F}$ *contains precisely one of each pair* $\{A, X_t \setminus A\}$ *in any* $A \in X_t(\frac{t}{2})$. □

From Lemma 4.1 it follows that the largest cardinal number $M(t)$ of intersecting Sperner families in $X_t(\leq \frac{t}{2})$ is given as follows:

(4.4) $M(t) = \binom{t-1}{p-1}$, where $p = \lfloor \frac{t}{2} \rfloor$,

and $\mathbf{F}_t := X_t(p, \{1\})$ is a maximal intersecting Sperner family in X_t with $|\mathbf{F}_t| = M(t)$. Therefore for any m with $2 \leq m \leq M(t), G_{2m}$ has a set representation ϕ with $|\phi| = t$, and for any m with $m > M(t), i(G_{2m}) > t$. Consequently we have

Theorem 4.2. *Under the notation* (4.4) *for any* $m > 2, i(G_{2m}) = t$, *where* t *is determined by the inequalities*: $M(t-1) < m \leq M(t)$. □

For examples $i(G_6) = 4, i(G_8) = 5$, and $i(G_{2m}) = 6$ for $4 < m \leq 10$.

Here we consider the intersection number of the complementary graph G_n of the cycle C_n with $n \geq 5$, which is a $(n-3)$-regular graph of order n. The vertex set $\{u_1, u_2, u_3, \cdots, v_n\}$ of G_n and C_n are labeled as $u_j u_{j+1} \in E(C_n)$ for any $j \in [n]$. Now for convenience we use the label j of u_j in the sense of modulo n, say u_{n+1} should be understood as u_1. So $N(u_j) = \{u_{j+k}; k \in [2, n-2]\}$ for any u_j. We note that for any subset W of $V(G_n)$, $< W >$ is a clique in G_n if and only if W is an independent set in C_n. So enumerating the maximal independent sets in C_n for the case $n =$ odd, we have

Lemma 4.3. *For the case* $n = 2m-1, m \geq 3, MC(G_n)$ *is the family* $\{Q_j; j \in [n]\}$ *given as follows*: $Q_j =< W_j >$, *where*

$W_j = \{u_j, u_{j+2}, u_{j+4}, \cdots, u_{j+2m-4}\}$ *for any* $j \in [n]$.

In this case for every $j \in [n]$ *the edge* $u_j u_{j+2m-4}$ *is proper.* □

Let $n = 2m, m \geq 3$. Then we note that $G_{n-1} = G_n - \{v_{2m}\} - \{u_1 u_{2m-1}\}$. So using this fact we can get $MC(G_n)$ from $MC(G_{n-1})$ by a slight modification. Under the notation in Lemma 4.3, we add $\{v_{2m-1}\}$ to W_1, add $\{v_{2m}\}$ to W_2, and for W_3 add $\{v_{2m}\}$ and delete $\{v_{2m-1}\}$. Therefore we have

Lemma 4.4. *For the case* $n = 2m, m \geq 3, MC(G_n)$ *is the family* $\{Q_j; j \in [n-1]\}$ *given as follows*: $Q_j =< W_j >$ *for any* $j \in [n]$, *where*

$$W_1 = \{u_1, u_3, u_5, \cdots, u_{2m-3}, u_{2m-1}\}$$
$$W_2 = \{u_2, u_4, u_6, \cdots, u_{2m-2}, u_{2m}\}$$
$$W_3 = \{u_3, u_5, u_7, \cdots, u_{2m-3}, u_{2m}\}$$
$$W_j = \{u_j, u_{j+2}, u_{j+4}, \cdots, u_{j+2m-4}\} \text{ for any } j \in [4, n-1].$$

In this case the edges $u_1 u_{2m-1}, u_2 u_{2m}, u_3 u_{2m}$ *and the edges* $u_j u_{j+2m-4}$ *for every* $j \in [4, n-1]$ *are proper.* □

From Lemmas 4.3-4.4 and Theorem 2.3 we have

Theorem 4.5. *For the complementary graph G_n of the cycle C_n with $n \geq 5, i(G_n) = \theta_m(G_n)$, and $i(G_n) = n$ if n is odd and $i(G_n) = n-1$ if n is even.* $\square$

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