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ON THE HV-CURVATURE TENSORS OF FINSLER SPACES

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In his recent papers [7], [8] M. Matsumoto has treated the interesting Finsler spaces with the curvature tensors of some special forms. In the h-isotropic and C^h-recurrent Finsler spaces [7][1] and the $P2$-like Finsler spaces [8][2] among those spaces, the hv-curvature tensor P^2 is symmetric in the last two indices:

(1) $$P_{ijkl}=P_{ijlk}\,,$$

and the $(v)hv$-torsion tensor P^1 is proportional to the $(h)hv$-torsion tensor C:

(2) $$P_{jkl}=\lambda\cdot C_{jkl} \qquad (\lambda : \text{a scalar})\,.$$

These conditions (1), (2) are also satisfied in all 2-dimensional Finsler spaces.[3] So, in this note we shall generally consider the above conditions and show that the conditions (1), (2) yield $P_{ijkl}=0$ or $S_{ijkl}=0$ (Theorem A), and so especially, in the h-isotropic Finsler spaces endowed with the condition (2) it follows $R_{ijkl}=P_{ijkl}=0$ or $S_{ijkl}=0$ (Theorem B).

Throughout the present note we shall use the terminologies and notations described in M. Matsumoto [6]. The used Finsler connection is the one given by E. Cartan [5].

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1°. In the first place, we shall treat the condition (1). With respect to the Finsler connection given by E. Cartan, the components P_{ijkl} of the hv-curvature tensor P^2 are written in the form

(3) $$P_{ijkl}=(C_{jkl|i}-C_{ikl|j})+(C_{ikm}P^m_{jl}-C_{jkm}P^m_{il})\,,$$

where C_{jkl} and P_{jkl} $(=g_{jm}P^m_{kl})$ are the components of the $(h)hv$-torsion tensor C and the $(v)hv$-torsion tensor P^1 respectively. So, we have

(4) $$P_{ijkl}-P_{ijlk}=(C_{ikm}P^m_{jl}-C_{jkm}P^m_{il})-(C_{ilm}P^m_{jk}-C_{jlm}P^m_{ik})\,.$$

On the other hand, the components S_{ijkl} of the v-curvature tensor S^2 are written in the form

(5) $$S_{ijkl}=-(C_{ikm}C^m_{jl}-C_{jkm}C^m_{il})\,.$$

By the h-differentiation and the contraction by the supporting element y^i, we have

(6) $$-S_{ijkl|0}=(C_{ikm}P^m_{jl}-C_{jkm}P^m_{il})+(P_{ikm}C^m_{jl}-P_{jkm}C^m_{il})\,,$$

from which it follows

$$P_{ijkl}-P_{ijlk}=-S_{ijkl|0}\,. \tag{7}$$

Thus, we have

PROPOSITION 1. *The condition* (1) *is equivalent to*

$$S_{ijkl|0}=0\,. \tag{8}$$

And this condition is satisfied if it holds

$$C_{ikm}P^{\,m}_{j\,l}-C_{jkm}P^{\,m}_{i\,l}=0\,. \tag{9}$$

Next, let us consider the relations between the (*v*)*hv*-tensor P^1 and the *hv*-curvature tensor P^2. Since it holds

$$P_{jkl}=y^iP_{ijkl}\,, \tag{10}$$

$P_{ijkl}=0$ implies $P_{jkl}=0$. Conversely, we have

PROPOSISION 2. *If* $P_{jkl}\equiv 0$ *identically on a domain* U *in the tangent space* T_x *at a point* x *then* $P_{ijkl}\equiv 0$ *on* U.

The proof is immediately obtained from the formula[4)]

$$P_{ijkl}=\frac{\partial P_{jkl}}{\partial y^i}-\frac{\partial P_{ikl}}{\partial y^j}+C_{jlm}P^{\,m}_{i\,k}-C_{ilm}P^{\,m}_{j\,k}\,. \tag{11}$$

Now, if we glance at the formula (6), we shall recognize that (1) and (2) are the conditions easily melted together. Substituting (2) into (6), we have

$$-S_{ijkl|0}=2\lambda S_{ijkl}\,. \tag{12}$$

Hence, we know that the conditions (1), (2) yield

$$\lambda S_{ijkl}=0\,. \tag{13}$$

If $S_{ijkl}\neq 0$ at $(x, y)\in T_x$, then $S_{ijkl}\neq 0$ on a domain $U\ni(x, y)$ in T_x, which implies $\lambda\equiv 0$ on U and so $P_{jkl}\equiv 0$ on U. By Proposition 2, $P_{ijkl}\equiv 0$ on U. Thus, we have proved

THEOREM A. *Suppose that the hv-curvature tensor* P^2 *be symmetric in the last two indices and the* (*v*)*hv-torsion tensor* P^1 *be proportional to the* (*h*)*hv-torsion tensor* C. *Then, the hv-curvature tensor* P^2 *or the v-curvature tensor* S^2 *vanishes.*

2°. We shall here suppose that the Finsler space be *h*-isotropic. According to L. Berwald [4], we may easily conclude as follows.

PROPOSITION 3. *In order that the space be h-isotropic*:

$$R_{ijkl}=R(g_{ik}g_{jl}-g_{il}g_{jk})\quad (R: \textit{a constant}^{5)})\,, \tag{14}$$

it is necessary and sufficient to hold the following two conditions:

(15) $$K_{ijkl}=R(g_{ik}g_{jl}-g_{il}g_{jk}) \quad (R: a\ constant)$$

and

(16) $$P_{ikm}P^{m}_{jl}-P_{jkm}P^{m}_{il}=0\,,$$

where R_{ijkl} and K_{ijkl} denote the components of the h-curvature tensors of the connections given by E. Cartan and by L. Berwald respectively. And, in this case it holds

(17) $$R_{ijkl}=K_{ijkl}\,.$$

PROOF. We know the relation[6)]

(18) $$R_{ijkl}=1/2(K_{ijkl}-K_{jikl})-(P_{ikm}P^{m}_{jl}-P_{jkm}P^{m}_{il})\,.$$

When the condition (15) is satisfied, we have $K_{ijkl}+K_{jikl}=0$ and so (18) is reduced to

(19) $$R_{ijkl}=K_{ijkl}-(P_{ikm}P^{m}_{jl}-P_{jkm}P^{m}_{il})\,.$$

Hence (15) and (16) jield (14).

Conversely, let us assume that (14) holds. Since R^{m}_{kl} becomes

(20) $$R^{m}_{kl}=R(g_{ak}y^{a}\delta^{m}_{l}-g_{al}y^{a}\delta^{m}_{k})\,,$$

we have

$$K_{i}{}^{m}{}_{kl}=\frac{\partial R^{m}_{kl}}{\partial y^{i}}=R\left(\frac{\partial(g_{ak}y^{a})}{\partial y^{i}}\delta^{m}_{l}-\frac{\partial(g_{al}y^{a})}{\partial y^{i}}\delta^{m}_{k}\right)$$
$$=R(g_{ik}\delta^{m}_{l}-g_{il}\delta^{m}_{k})$$

and so (15) and (17). Thus, (16) follows from (19).

The condition (16) is also well combined with (2) and is equivalent to (9) under the condition (2). By Proposition 1, we may conclude that in the h-isotropic Finsler spaces the condition (1) is satisfied if we impose the condition (2). Therefore, the h-isotropic Finsler space endowed with the condition (2) belongs to our spaces and it holds $P_{ijkl}=0$ or $S_{ijkl}=0$.

On the other hand, M.H. Akbar-Zadeh [1] has proved that in the h-isotropic Finsler space $R_{ijkl}=0$ or $S_{ijkl}=0$. Thus, we have

THEOREM B. *Suppose that the Finsler space be h-isotropic. If the (v)hv-torsion tensor P^1 be proportional to the (h)hv-tortion tensor C, then the h-curvature tensor R^2 and the hv-curvature tensor P^2 vanish or the v-curvature tensor S^2 vanishes.*

3°. We shall finally give some remarks about the condition (1) in the h-isotropic Finsler spaces. In these spaces, it is proved by M.H. Akbar-Zadeh[1] that the condition (1) is satisfied, but it seems that it needs the condition $R_{ijkl}\neq 0$. So, we have proved the condition (1) under the condition (2) in order to include the case of $R_{ijkl}=0$.

On the other hand, M. Matsumoto has recently pointed out that in the case of $R_{ijkl}=0$

it holds as a similar condition to (1) that

$$P_{ijk|l}=P_{ijl|k}\,. \tag{21}$$

It is easily shown that in the h-isotropic non-Riemannian Finsler spaces the condition (21) implies $R_{ijkl}=0$ conversely. For convenience sake, we shall explain the above situation in the following proposition.

PROPOSITION 4. *For the h-isotropic non-Riemannian*[7] *Finsler spaces, the following conditions are mutually equivalent*:

(i) $R_{ijkl}=0\,,$

(ii) $C_{ijm}R^{m}_{kl}=0\,,$

(iii) $P_{ijk|l}=P_{ijl|k}\,,$

(iv) $P_{ijk|0}=0$[8].

PROOF. The implication (i)→(ii) is evident.

Here, we shall be concerned with the relation[9]

$$K_{ijkl}=R_{ijkl}-C_{ijm}R^{m}_{kl} +(P_{ikm}P^{m}_{jl}-P_{jkm}P^{m}_{il})+(P_{ijk|l}-P_{ijl|k})\,. \tag{22}$$

Since the conditions (16), (17) hold in the h-isotropic Finsler spaces, (22) is reduced to

$$P_{ijk|l}-P_{ijl|k}=C_{ijm}R^{m}_{kl}\,. \tag{23}$$

Hence, (iii) is equivalent to (ii).

(iv) follows from (iii) by the contraction by the supporting element.

Finally, if we substitute (20) into (23) and contract by the supporting element, we have

$$P_{ijk|0}=-RL^2C_{ijk}\,, \tag{24}$$

where L is the fundamental function. Thus, (i) follows form (iv) if we assume the space to be non-Riemannian.

L. Berwald [2] has introduced the *stretch-curvature tensor* T_{ijkl} and showed that this tensor vanishes if and only if the length of a vector remains unchanged under the parallel displacement (in the sence of L. Berwald) along an infinitesimal parallelogram[10]. In our notations the components T_{ijkl} are expressed as

$$\frac{1}{2}T_{ijkl}=P_{ijk|l}-P_{ijl|k}\,. \tag{25}$$

[11]

Thus, Proposition 4 gives us

THEOREM C. *In the h-isotropic non-Riemannian Finsler spaces, the h-curvature tensor* R^2 *vanishes if and only if the stretch-curvature tensor vanishes.*

Notes

1) A Finsler space is called *h-isotropic* if there exists such a constant R that $R_{ijkl}=R(g_{ik}g_{jl}-g_{il}g_{jk})$. Also, a Finsler space is called *C^h-recurrent* if there exists such a covariant vector λ_i that $C_{jkl|i}=\lambda_i C_{jkl}$, where the solidus denotes the h-covariant differentiation.
2) A Finsler space is called *P2-like* if there exists such a covariant vector λ_i that $P_{ijkl}=\lambda_i C_{jkl}-\lambda_j C_{ikl}$.
3) In 2-dimensional Finsler spaces we have $P_{ijkl}=C_{jkl|i}-C_{ikl|j}$ and also $P_{jkl}=(J_{|0}/J)C_{jkl}$ at such a point that $J\neq 0$, where J denotes the main scalar [9] and the contraction by the supporting element y^i(not l^i) is indicated by a zero. If $J=0$ at some point, then $C_{jkl}=0$ at that point. In the condition (2) we suppose that $\lambda \in [-\infty, \infty]$, and $\lambda=\pm\infty$ means $C_{jkl}=0$.
4) See (26) in [3].
5) It is known by M.H. Akbar-Zadeh [1] that the assumption R to be a scalar implies that R is a constant, provided $n>2$.
6) See (11.6) in [4].
7) The assumption that the spaces be non-Riemannian is required only for the implication (iv)→(i).
8) Due to M.H. Akbar-Zadeh [1] this condition may be replaced by the condition $C_{i|0|0}=0$.
9) See (11.8) in [4].
10) See (53) in [2], where T_{ijkl} are written as S_{ijkl}.
11) See (25) in [3].

References

[1] Akbar-Zadeh, M.H.: Les espaces de Finsler et certaines de leurs généralisations. Ann. scient Éc. Norm. Sup. (3) **80** (1963), 1–79.

[2] Berwald, L.: Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus. Math. Z. **25** (1926), 40–73.

[3] Berwald, L.: Über die Beziehungen zwischen den Theorien der Parallelübertragung in Finslerschen Räumen. Nederl. Akad. Wetensch. Proc. Ser. A **49** (1946), 642–647.

[4] Berwald, L.: Über Finslersche und Cartansche Geometrie IV. Projektivkrümmung allgemeiner affiner Räume und Finslersche Räume skalarer Krümmung. Ann. Math. (2) **48** (1947), 755–781.

[5] Cartan, E.: Les espaces de Finsler. Actualités **79**, Paris, 1934.

[6] Matsumoto, M.: The theory of Finsler connections. Publ. of the Study Group of Geometry **5**, 1970.

[7] Matsumoto, M.: On h-isotropic and C^h-recurrent Finsler spaces. J. Math. Kyoto Univ. **11** (1971), 1-9.

[8] Matsumoto, M.: On Finsler spaces with curvature tensors of some special forms. Tensor (N.S.) **22** (1971), 201-204.

[9] Rund, H.: The differential geometry of Finsler spaces. Springer, 1959.