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ON SOME SPECIAL (α, β)-METRICS

By

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An affinely connected space defined by L. Berwald [1, 2] is also called a *Berwald space*, which is defined as the Finsler space such that Berwald's connection coefficients depend on position alone. If we obey the Cartan connection [3], such a space is also the one in which Cartan's connection coefficients $\Gamma^{*}{}_{j}{}^{i}{}_{k}$ depend on position alone, and is characterized by the well-known condition $C_{ijk|l}=0$. V. Wagner [16] has generalized the notion of a Berwald space, and has called a Finsler space as a *generalized Berwald space* if it is possible to introduce a *generalized Cartan connection*, with torsion $({}^{*}\Gamma_{j}{}^{i}{}_{k}-{}^{*}\Gamma_{k}{}^{i}{}_{j}\neq 0)$, in such a way that the connection coefficients ${}^{*}\Gamma_{j}{}^{i}{}_{k}$ depend on position alone. And in a two-dimensional case he has shown that a Finsler space is a generalized Berwald space if and only if $\partial A/\partial\theta$ is a function of A, where A and θ are the *main scalar* and the *Landsberg angle* respectively (Berwald [2]). It would seem that an aim of his study is to search the geometrical class to which the two-dimensional Finsler space with the interesting metric $ds=(a_{ijk}(x)dx^{i}dx^{j}dx^{k})^{1/3}$ belongs. In fact, for this metric it holds $\partial A/\partial\theta=-3A^{2}-3/2$, and so a generalized Berwald space is thought to be an important model of a Finsler space.

In his recent paper [6], M. Hashiguchi, one of the authors, has investigated in detail various axioms imposed on a Finsler connection, to clarify a meaning of the generalized Cartan connection used by Wagner, and to characterize Wagner's generalized Berwald spaces of general dimensions. For the purpose of these considerations, a generalized Berwald space is defined in a broader sense than Wagner's, while Wagner's generalized Berwald space is called a *Wagner space.* The generalized Cartan connection used by Wagner is thought to be a semi-symmetric metrical connection without deflection, which is called a *Wagner connection.* And it is shown that a Finsler space is a Wagner space if and only if it is possible to introduce a Wagner connection satisfying the condition $C_{ijk|l}=0$ formally similar to the one for a Berwald space.

The purpose of the present paper is to give examples of a Berwald space, a generalized Berwald space and a Wagner space (§3). Y. Ichijyō, the other author, has given an effective method as shown in §2 to obtain such spaces, which comes from the study [8] about *Finsler spaces modeled on a Minkowski space.* Our examples rise naturally from his theory by recalling the (α, β)-metric introduced by M. Matsumoto [13].

On the other hand, the conditions that a Finsler space becomes a generalized Berwald space in the broader sense have already been given in [6]. It was pointed out by Y. Ichijyō that one complicated condition (3.5) in [6] may be replaced by a trivial condition (4.6) in the present paper. As additional remarks we shall provide §4 to improve some results obtained in [6].

Throughout the present paper we shall use the terminologies and notations described in M. Matsumoto [12] and M. Hashiguchi [6]. For convenience' sake we shall devote §1 to sketching the materials necessary for our discussions.

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§1. Preliminaries.

1.1. Given a differentiable manifold M, we denote by $L(M)$ and $T(M)$ the linear frame bundle and the tangent bundle respectively. The standard fibre V is assumed that a base $\{e_a\}$ is fixed. The *Finsler bundle* $F(M)$ is defined as the induced bundle $\tau^{-1}L(M)=\{(y, z)\in T(M)\times L(M)\,|\,\tau(y)=\pi(z)\}$, where τ and π are the projections of $T(M)$ and $L(M)$ respectively.

Since a point of $F(M)$ is a pair of a tangent vector y and a linear frame $z=(z_a)$ at a point x of the base manifold M, a coordinate system (x^i) in M induces a coordinate system $(x^i, y^i, z_a{}^i)$ in $F(M)$ by $y=y^i(\partial/\partial x^i)_x$ and $z_a=z_a{}^i(\partial/\partial x^i)_x$. As a coordinate system in $F(M)$ we shall use such a one in the following.

1.2. The *Finsler connection* of M is by the third definition of M. Matsumoto a triad (Γ_V, N, Γ^v) of a *V-connection* Γ_V in $L(M)$, a *non-linear connection* N in $T(M)$ and a *vertical connection* Γ^v in $F(M)$.

If a Finsler connection is given, the *h-* and *v-basic vector fields* $B^h(v)$ and $B^v(v)$ $(v\in V)$ are defined in $F(M)$. They are expressed by

(1.1) $$B^h(v) = v^a z_a{}^k(\partial/\partial x^k - N^i{}_k\partial/\partial y^i - z_b{}^j F_j{}^i{}_k\partial/\partial z_b{}^i)$$

and

(1.2) $$B^v(v) = v^a z_a{}^k(\partial/\partial y^k - z_b{}^j C_j{}^i{}_k\partial/\partial z_b{}^i)$$

respectively, where $v=v^a e_a$, and $F_j{}^i{}_k$, $N^i{}_k$ and $C_j{}^i{}_k$ are called the *coefficients* of the Finsler connection.

Let K be a Finsler tensor field. The *h-* and *v-covariant derivatives* of K are defined by $B^h(v)K$ and $B^v(v)K$ respectively. If K is assumed, for instance, to be of type (1, 1), i.e.,

(1.3) $$K = z'_i{}^a z_b{}^j K^i{}_j e_a\otimes e^b,$$

where $(z'_i{}^a)=(z_a{}^i)^{-1}$, and $\{e^b\}$ is the dual base of $\{e_a\}$, their components are expressed as follows:

(1.4) $$K^i{}_{j|k} = \delta K^i{}_j/\partial x^k + K^m{}_j F_m{}^i{}_k - K^i{}_m F_j{}^m{}_k,$$

(1.5) $$K^i{}_j|_k = \partial K^i{}_j/\partial y^k + K^m{}_j C_m{}^i{}_k - K^i{}_m C_j{}^m{}_k,$$

where $\delta/\partial x^k=\partial/\partial x^k - N^m{}_k\partial/\partial y^m$.

1.3. Now, we shall treat Finsler spaces. Let $L(x, y)$ be the fundamental function, whose metric tensor field is defined by

(1.6) $$g_{ij} = \partial^2(L^2/2)/\partial y^i\partial y^j.$$

As a famous Finsler connection of a Finsler space there exists the Cartan connection, which is uniquely determined by the following four axioms due to M. Matsumoto.

(C1) The Finsler connection is *metrical*, i.e.,

(C1h) $g_{ij|k} = 0$, (C1v) $g_{ij}|_k=0$.

(C2) The *deflection tensor field* D vanishes identically, i.e.,

$$D^i{}_k = y^j F_j{}^i{}_k - N^i{}_k = 0, \text{ equivalently } y^i{}_{|k} = 0.$$

(C3) The *(h)h-torsion tensor field* T vanishes identically, i.e.,

$$T_j{}^i{}_k = F_j{}^i{}_k - F_k{}^i{}_j = 0.$$

(C4) The *(v)v-torsion tensor field* S^1 vanishes identically, i.e.,

$$S_j{}^i{}_k = C_j{}^i{}_k - C_k{}^i{}_j = 0.$$

1.4. To investigate Wagner's generalized Berwald space, M. Hashiguchi [6] has tried to replace the axioms (C2) and (C3) by some weaker conditions. To suit our convenience we shall restate the definitions.

DEFINITION 1. A Finsler connection satisfying the axioms (C1) and (C4) is called a *generalized Cartan connection.*

As shown in [6], given a Finsler (1, 1)-tensor field D and a skew-symmetric Finsler (1,2)-tensor field T in a Finsler space, there exists a unique generalized Cartan connection with the given D and T as the respective deflection and $(h)h$-torsion tensor fields. So, various generalized Cartan connections are introduced in a Finsler space.

The $C_j{}^i{}_k$ is uniquely determined by the axioms (C1v) and (C4), and it follows

$$C_j{}^i{}_k = 1/2\, g^{ih} \partial g_{jh} / \partial y^k, \tag{1.7}$$

where $(g^{ih}) = (g_{jh})^{-1}$. Thus, a Finsler connection with the above $C_j{}^i{}_k$ is a generalized Cartan connection if and only if $g_{ij|k}=0$. Some treatments proceed smoothly even if we use a generalized Cartan connection, because each covariant differentiation commutes with the raising and lowering of indices, and the so-called C_1-*condition* $y^j C_j{}^i{}_k = 0$ is satisfied. However, since the condition $y^i{}_{|k}=0$ is attractive, a non-linear connection is ordinarily chosen such that the axiom (C2) is satisfied, i.e., $N^i{}_k = y^j F_j{}^i{}_k$.

On the other hand, as an axiom giving a typical Finsler connection with torsion $T_j{}^i{}_k$, we know

(C3*) The Finsler connection is *semi-symmetric*, i.e.,

$$T_j{}^i{}_k = F_j{}^i{}_k - F_k{}^i{}_j = \delta_j^i s_k - \delta_k^i s_j$$

for some covariant vector field s_k.

Thus, for some covariant vector field s_k we have a typical generalized Cartan connection used by Wagner as follows.

DEFINITION 2. A Finsler connection satisfying the axioms (C1), (C2), (C3*) and (C4) is called a *Wagner connection.*

1.5. A Finsler space is called a Berwald space if the coefficients $F_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k$ of the Cartan connection depend on position alone. Corresponding to Definitions 1 and 2 we have the following generalizations of a Berwald space.

DEFINITION 3. A Finsler space is called a *generalized Berwald space* if it is possible to introduce a generalized Cartan connection in such a way that the coefficients $F_j{}^i{}_k$ depend on position alone.

DEFINITION 4. A Finsler space is called a *Wagner space* if it is possible to introduce a Wagner connection in such a way that the coefficients $F_j{}^i{}_k$ depend on position alone.

1.6. In the following it is usually assumed that a Finsler connection satisfies the axiom (C2) and its coefficients $F_j{}^i{}_k$ depend on position alone. With respect to such a Finsler connection it is easily verified that the h-covariant differentiation commutes with the partial differentiation by the supporting element y. Hence, we have $g_{ij|k}=\partial^2(LL_{|k})/\partial y^i\partial y^j$ from (1.6). On the other hand, it holds $2LL_{|k}=g_{ij|k}y^iy^j$ because of $L^2=g_{ij}y^iy^j$, and so we have

PROPOSITION 1. *Assumed that a Finsler connection satisfies the axiom* (C2) *and its coefficients* $F_j{}^i{}_k$ *depend on position alone. Then,* $g_{ij|k}=0$ *if and only if* $L_{|k}=0$.

§2. The Finsler connection associated with a linear connection.

2.1. On a Finsler space M we have especially interested in the Finsler connection (Γ_V, N, Γ^v) such that the coefficients $F_j{}^i{}_k$ depend on position alone, that is, the corresponding V-connection Γ_V falls into a linear connection in the linear frame bundle $L(M)$. Conversely, let a linear connection Γ be given in $L(M)$. Denoting the coefficients by $\Gamma_j{}^i{}_k$, a Finsler connection $^*\Gamma$ of M is introduced by

(2.1) $$F_j{}^i{}_k=\Gamma_j{}^i{}_k,\quad N^i{}_k=y^jF_j{}^i{}_k,\quad C_j{}^i{}_k=1/2\,g^{ih}\partial g_{jh}/\partial y^k.$$

We shall call this connection the *Finsler connection associated with the linear connection* Γ.

2.2. We shall first notice a feature of the h-covariant differentiation of a tensor field K. If K is assumed, for instance, to be of type (1,1), the h-covariant derivative with respect to the $^*\Gamma$ is expressed by

(2.2) $$K^i{}_{j|k}=\delta K^i{}_j/\partial x^k+K^m{}_j\Gamma_m{}^i{}_k-K^i{}_m\Gamma_j{}^m{}_k.$$

If $K^i{}_j$ depend on position alone, $\delta K^i{}_j/\partial x^k$ become $\partial K^i{}_j/\partial x^k$ and it holds

(2.3) $$K^i{}_{j|k}=\nabla_kK^i{}_j,$$

where ∇_k denotes the covariant differentiation with respect to Γ, i.e.,

(2.4) $$\nabla_kK^i{}_j=\partial K^i{}_j/\partial x^k+K^m{}_j\Gamma_m{}^i{}_k-K^i{}_m\Gamma_j{}^m{}_k.$$

Thus we have

PROPOSITION 2. *Let a linear connection* Γ *be given. For a tensor field depending on position alone, the h-covariant differentiation with respect to the associated Finsler connection* $^*\Gamma$ *coincides with the covariant differentiation with respect to the given* Γ.

2.3. Next, we shall examine the various axioms stated in the previous section. The axioms (C1v), (C2) and (C4) are always satisfied by the $^*\Gamma$. Since $F_j{}^i{}_k$ depend on position alone, Proposition 1 tells us that the axiom (C1h) is satisfied if and only if $L_{|k}=0$. On the other hand, it is easily seen that the axiom (C3) is satisfied if and only if Γ is symmetric, i.e., $\Gamma_j{}^i{}_k-\Gamma_k{}^i{}_j=0$, and the axiom (C3*) is satisfied if and only if Γ is semi-symmetric, i.e., $\Gamma_j{}^i{}_k-\Gamma_k{}^i{}_j=\delta_j{}^is_k-\delta_k{}^is_j$ for some covariant vector field $s_k(x)$. Thus we have

THEOREM 1. *Let a linear connection* Γ *be given in a Finsler space* M. *The Finsler connection* $^*\Gamma$ *associated with* Γ *is a generalized Cartan connection of* M *if and only if* $L_{|k}=0$.

In this case, if the linear connection Γ *is symmetric* (*resp. semi-symmetric*), *the Finsler connection* $^*\Gamma$ *is the Cartan connection* (*resp. a Wagner connection*) *of* M.

2.4. Since $F_j{}^i{}_k$ depend on position alone, we have from Theorem 1

THEOREM 2. *Let a linear connection* Γ *be given in a Finsler space* M. *If it* *holds that $L_{|k}=0$ with respect to the Finsler connection $*\Gamma$ associated with Γ, the Finsler space M is a generalized Berwald space.*

In this case, if the linear connection Γ is symmetric (resp. semi-symmetric), the Finsler space M is a Berwald space (resp. a Wagner space).

§3. Special (α, β)-metrics.

3.1. To obtain an example for the theorems stated in the previous section, we shall treat the Finsler space with an (α, β)-metric, which is defined as follows.

DEFINITION 5. A Finsler space is called to be with an *(α, β)-metric* when the fundamental function $L=L(\alpha, \beta)$ is positively homogeneous of degree one in

$$\alpha = (a_{ij}(x)y^i y^j)^{1/2} \text{ and } \beta = b_i(x)y^i,$$

where α is a Riemannian metric and b_i is a non-zero covariant vector field.

Well-known examples are the *Randers metric* $L=\alpha+\beta$ [15] and the *Kropina metric* $L=\alpha^2/\beta$ [9, 10].

3.2. A Finsler space M with an (α, β)-metric $L(\alpha, \beta)$ has two metrics. The one is the Finsler metric $L(\alpha, \beta)$ itself, and the other is the Riemannian metric α. A linear connection Γ of M is called to be *metrical* if it is metrical with respect to the latter, i.e.,

$$(3.1) \qquad \nabla_k a_{ij} = \partial a_{ij}/\partial x^k - a_{mj}\Gamma_i{}^m{}_k - a_{im}\Gamma_j{}^m{}_k = 0.$$

Now, let the vector field b_i be parallel with respect to a metrical linear connection Γ, i.e.,

$$(3.2) \qquad \nabla_k b_i = \partial b_i/\partial x^k - b_m \Gamma_i{}^m{}_k = 0.$$

With respect to the Finsler connection $*\Gamma$ associated with Γ, we have $\alpha_{|k}=0$, $\beta_{|k}=0$, because it holds $a_{ij|k}=\nabla_k a_{ij}=0$, $b_{i|k}=\nabla_k b_i=0$ from Proposition 2, and $y^i{}_{|k}=0$ from (C2). Thus, we have $L_{|k}=(\partial L/\partial\alpha)\alpha_{|k}+(\partial L/\partial\beta)\beta_{|k}=0$, and from Theorems 1 and 2 we obtain

EXAMPLE 1. *Let a Finsler space M be with an (α, β)-metric $L(\alpha, \beta)$, where $\beta=b_i y^i$. If the covariant vector field b_i is parallel with respect to some metrical linear connection Γ of M, a generalized Cartan connection is introduced by*

$$(3.3) \qquad F_j{}^i{}_k = \Gamma_j{}^i{}_k, \quad N^i{}_k = y^j F_j{}^i{}_k, \quad C_j{}^i{}_k = 1/2 g^{ih}\partial g_{jh}/\partial y^k,$$

where $\Gamma_j{}^i{}_k$ denote the coefficients of Γ, and the space M becomes a generalized Berwald space.

3.3. Especially, if the metrical linear connection Γ is symmetric, that is, Γ is the Riemannian connection, we obtain

EXAMPLE 2. *Let a Finsler space M be with an (α, β)-metric $L(\alpha, \beta)$, where $\beta= b_i y^i$. If the covariant vector field b_i is parallel with respect to the Riemannian connection determined by the Riemannian metric α, the Cartan connection is given by*

$$(3.4) \qquad F_j{}^i{}_k = \{{}_j{}^i{}_k\}, \quad N^i{}_k = y^j F_j{}^i{}_k, \quad C_j{}^i{}_k = 1/2 g^{ih}\partial g_{jh}/\partial y^k,$$

where $\{{}_j{}^i{}_k\}$ denote the Christoffel symbols formed with respect to α, and the space M is a Berwald space.

It should be remarked that the Cartan connection in the above example is obtained from the geometrical axioms without any artificial techniques. If we calculate $F_j{}^i{}_k$ from the fundamental function $L(\alpha, \beta)$ directly, we might be at once lost in a maze.

3.4. We shall lastly give examples for a Wagner space. We may easily conclude that the semi-symmetric linear connection Γ, with $\Gamma_j{}^i{}_k - \Gamma_k{}^i{}_j = \delta_j{}^i s_k - \delta_k{}^i s_j$, is metrical if and only if $\Gamma_j{}^i{}_k$ has a form

(3.5) $$\Gamma_j{}^i{}_k = \{{}_j{}^i{}_k\} + a_{jk}s^i - s_j\delta_k{}^i,$$

where $(a^{im}) = (a_{mj})^{-1}$ and $s^i = a^{im}s_m$. With respect to such a linear connection Γ, b_i is parallel if and only if

(3.6) $$b_{i,k} = a_{ik}b_m s^m - s_i b_k,$$

where we use the comma to denote the covariant differentiation with respect to the Riemannian connection, i.e.,

(3.7) $$b_{i,k} = \partial b_i/\partial x^k - b_m\{{}_i{}^m{}_k\}.$$

We shall call a covariant vector field $b_i(x)$ satisfying (3.6) for some $s_i(x)$ to be *semi-parallel*. If a semi-parallel field b_i is given, the linear connection Γ defined by (3.5) is semi-symmetric and metrical, and b_i becomes parallel with respect to Γ. Thus we obtain

EXAMPLE 3. *Let a Finsler space M be with an* (α, β)*-metric* $L(\alpha, \beta)$*, where* $\alpha = (a_{ij}y^iy^j)^{1/2}$, $\beta = b_iy^i$. *If the covariant vector field* b_i *is semi-parallel by satisfying* (3.6) *for some* $s_i(x)$*, a Wagner connection of M is introduced by*

(3.8) $$F_j{}^i{}_k = \{{}_j{}^i{}_k\} + a_{jk}s^i - s_j\delta_k{}^i, \quad N^i{}_k = y^jF_j{}^i{}_k, \quad C_j{}^i{}_k = 1/2g^{ih}\partial g_{jh}/\partial y^k,$$

and the space M becomes a Wagner space.

Putting $s_i = \lambda b_i$ for some scalar field $\lambda(x)$, (3.6) becomes

(3.9) $$b_{i,k} = \lambda(a_{ik}b_m b^m - b_i b_k),$$

where $b^m = a^{im}b_i$, and we have an example of a semi-parallel field. In this case b_i has a constant length $b = (a_{ij}b^ib^j)^{1/2}$ and the unit vector field $u_i = b_i/b$ satisfies

(3.10) $$u_{i,k} = \rho(a_{ik} - u_iu_k),$$

where $\rho = \lambda b$, and so such a field b_i is nothing but the *torse-forming* one due to K. Yano [17], which was also treated by C.M. Fulton [4] and M. Hashiguchi [5] to characterize Riemannian spaces of negative constant curvature. Thus we obtain

EXAMPLE 4. *Let a Finsler space M be with an* (α, β)*-metric* $L(\alpha, \beta)$*, where* $\alpha = (a_{ij}y^iy^j)^{1/2}$, $\beta = b_iy^i$. *If the covariant vector field* b_i *is torse-forming by satisfying* (3.9) *for some* $\lambda(x)$*, a Wagner connection is introduced by*

(3.11) $$F_j{}^i{}_k = \{{}_j{}^i{}_k\} + \lambda(a_{jk}b^i - b_j\delta_k{}^i), \quad N^i{}_k = y^jF_j{}^i{}_k, \quad C_j{}^i{}_k = 1/2g^{ih}\partial g_{jh}/\partial y^k,$$

and the space M becomes a Wagner space.

3.5. The (α, β)-metrics offer good examples of a Berwald space, a generalized Berwald space and a Wagner space. It is a pair of a Riemannian metric and a covariant vector field, and so it might contribute to the unified treatments of a Riemannian metric α and a covariant vector field $b_i(x)$ satisfying some properties with respect to α. Therefore, we should discuss the converse problem, that is, for a Finsler space with a special (α, β)-metric, what properties the β enjoys with respect to the α.

As to the Randers space with $L=\alpha+\beta$, it is known by M. Matsumoto [14] that the space is a Landsberg space if and only if b_i is parallel with respect to α, and such a space becomes a Berwald space. A *Landsberg* space [11] means here a Finsler space satisfying the condition $C_{ijk|l}y^l=0$ with respect to the Cartan connection, where $C_{ijk}=1/2\partial g_{ij}/\partial y^k$. Since a Berwald space is characterized by $C_{ijk|l}=0$, a Landsberg space is also a generalization of a Berwald space. Thus, the converse of Example 2 holds in the case of the Randers space.

M. Hashiguchi, S. Hōjō and M. Matsumoto [7] gives the conditions that a two-dimensional Finsler space with an (α, β)-metric becomes a Landsberg space, and finds the concrete form of the fundamental function L for each of the Landsberg spaces with the Randers metric, the Kropina metric and the *generalized Kropina metric* $L=\alpha^{m+1}/\beta^m$ $(m\neq 0, -1)$.

§4. Additional remarks.

4.1. Returning to general Finsler spaces, we shall give some remarks for the conditions found in [6] that a Finsler space becomes a generalized Berwald space. It has been a matter of regret that the conditions contain the very troublesome tensor field A_{ijkl} defined by

$$(4.1)\qquad 2A_{ijkl}=(T_{ijk}-T_{jik}+T_{ikj})|_l+(T_{ijm}-T_{jim}+T_{jmi})C_k{}^m{}_l \\ +\mathfrak{S}_{ij}\{(T_{kim}+T_{ikm}+T_{kmi})C_j{}^m{}_l\},$$

where $T_{ijk}=g_{jm}T_i{}^m{}_k$ and $\mathfrak{S}_{ij}\{\dots\}$ denotes, for instance, $\mathfrak{S}_{ij}\{A_{ij}\}=A_{ij}-A_{ji}$. This A_{ijkl} is, however, easily rewritten in the form

$$(4.2)\qquad 2A_{ijkl}=g_{jm}(\partial T_i{}^m{}_k/\partial y^l)-g_{im}(\partial T_j{}^m{}_k/\partial y^l)-g_{km}(\partial T_j{}^m{}_i/\partial y^l),$$

from which we notice that $A_{ijkl}=0$ is equivalent to $\partial T_j{}^i{}_k/\partial y^l=0$, because $A_{ijkl}=0$ is represented by

$$(4.3)\qquad g_{jm}(\partial T_i{}^m{}_k/\partial y^l)=g_{im}(\partial T_j{}^m{}_k/\partial y^l)+g_{km}(\partial T_j{}^m{}_i/\partial y^l),$$

whose left- and right-hand members are skew-symmetric and symmetric with respect to the indices i and k respectively. Thus, it is better to define the A_{ijkl} by (4.2) and to restate the first theorem in [6] as follows.

THEOREM 3. *In a generalized Cartan connection the coefficients $F_j{}^i{}_k$ depend on position alone if and only if it holds that*

$$(4.4)\qquad C_{ijk}D^i{}_l=0,$$

$$(4.5)\qquad C_{ijk|l}=C_{ijm}|_kD^m{}_l,$$

and

$$(4.6)\qquad \partial T_j{}^i{}_k/\partial y^l=0.$$

4.2. The condition (4.6) means that $T_j{}^i{}_k$ depend on position alone. From Theorem 3 we have

THEOREM 4. *A Finsler space is a generalized Berwald space if and only if it is possible to introduce a generalized Cartan connection with torsion $T_j{}^i{}_k(x)$ in such a way that it satisfies the conditions* (4.4) *and* (4.5).

Especially, a Finsler space is a generalized Berwald space if it is possible to introduce a generalized Cartan connection without deflection and with torsion $T_j{}^i{}_k(x)$ in such a way that it satisfies the condition

(4.7) $$C_{ijk|l} = 0.$$

If we consider a Wagner connection as a typical generalized Cartan connection, we have

THEOREM 5. *A Finsler space is a Wagner space if and only if it is possible to introduce a Wagner connection with respect to $s_i(x)$ in such a way that it satisfies the condition* (4.7).

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References

[1] Berwald, L.: *Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus*, Math. Z. **25** (1926), 40–73.

[2] Berwald, L.: *Über zweidimensionale allgemeine metrische Räume, I, II*, J. reine angew. Math. **156** (1927), 191–210, 211–222.

[3] Cartan, E.: *Les espaces de Finsler*, Actualités **79**, Paris, 1934.

[4] Fulton, C.M.: *Parallel vector fields*, Proc. Amer. Math. Soc. **16** (1965), 136–137.

[5] Hashiguchi, M.: *Semi-symmetric connections and spaces of negative constant curvature*, Sci. Rep. Kagoshima Univ. **15** (1966), 7–11.

[6] Hashiguchi, M.: *On Wagner's generalized Berwald space*, J. Korean Math. Soc. **12** (1975), 51–61.

[7] Hashiguchi, M., S. Hōjō and M. Matsumoto: *On Landsberg spaces of two dimensions with (α,β)-metric*, J. Korean Math. Soc. **10** (1973), 17–26.

[8] Ichijyō, Y.: *Finsler manifolds modeled on a Minkowski space*, to appear in J. Math. Kyoto Univ..

[9] Kropina, V.K.: *On projective Finsler spaces with a certain special form*, Naučn. Doklady vysš. Skoly, fiz.-mat. Nauki **1959** (1960), 38–42 (Russian).

[10] Kropina, V.K.: *Projective two-dimensional Finsler spaces with special metric*, Trudy Sem. Vektor. Tenzor. Anal. **11** (1961), 277–292 (Russian).

[11] Landsberg, G.: *Über die Krümmung in der Variationsrechnung*, Math. Ann. **65** (1908), 313–349.

[12] Matsumoto, M.: *The theory of Finsler connections*, Publ. of the Study Group of Geometry, vol. **5**, 1970.

[13] Matsumoto, M.: *On C-reducible Finsler spaces*, Tensor, N.S. **24** (1972), 29–37.

[14] Matsumoto, M.: *On Finsler spaces with Randers' metric and special forms of important tensors*, J. Math. Kyoto Univ. **14** (1974), 477–498.

[15] Randers, G.: *On an asymmetrical metric in the four-space of general relativity*, Phys. Rev. (2) **59** (1941), 195–199.

[16] Wagner, V.: *On generalized Berwald spaces*, C.R. (Doklady) Acad. Sci. URSS (N.S.) **39** (1943), 3–5.

[17] Yano, K.: *On the torse-forming directions in riemannian spaces*, Proc. Imp. Acad. Tokyo **20** (1944), 340–345.