COMPACT T1-SPACE AND THE FAMILY OF FINITE TOPOLOGICAL SPACES

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COMPACT T_1-SPACE AND THE FAMILY OF FINITE TOPOLOGICAL SPACES

By

Mitsunobu SHIRAKI*

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§ 1. Introduction.

We first note that given a finite open covering of a topological space, there is a finite T_0-space corresponds to the covering. And if we consider all finite open coverings of the space, we have a family of finite T_0-spaces. From the family we can obtain an inverse system by defining an order relation on the family. These shall be given in §2. In §3 we shall show that any T_0-space can be embedded in the inverse limit space of the associated inverse system. Moreover in §3 it will be proved that any compact T_1-space is homeomorphic to a subsapce in the inverse limit space which will be called the maximal inverse limit space here.

§ 2. The family of finite T_0-spaces.

Let (X, τ) be any T_0-space and suppose $\{\Delta_\alpha | \alpha \in M\}$ is the family of all finite open covering of (X, τ), where Δ_α is an open covering of X for all $\alpha \in M$. For Δ_α let τ_α be the topology on X for which Δ_α is a subbase. Then we have the family $\{(X, \tau_\alpha) | \alpha \in \Lambda\}$ of topological spaces.

Now let us consider a topological space (X, τ_α) of the family. We define the following relation $\sim$ on X with respect to τ_α: for $x, y \in X$, let $x \sim y$ mean that $G \in \tau_\alpha$ and $G \cap (x \cup y) \neq \phi$ imply $G \supset (x \cup y)$. Then the relation $\sim$ is a equivalence relation.

In fact, $x \sim x$ and $x \sim y \Rightarrow y \sim x$ are immediate consequence of its definition.

To prove that $\sim$ is transitive, let $x \sim y$ and $y \sim z$, and suppose that $G \in \tau_\alpha$ and $G \cap (x \cup z) \neq \phi$. If $G \ni x$, then $G \ni y$ since $y \sim z$. Hence $G \ni z$, so that $G \supset (x \cup z)$. If $G \ni z$, then $G \ni y$ since $y \sim z$. Hence $G \ni x$ since $x \sim y$. Hence $G \supset (x \cup z)$. Thus $x \sim z$, whence $\sim$ is transitive.

By the equivalence relation $\sim$ we may obtain the quotient space $(X, \tau_\alpha)/\sim$ of (X, τ_α) with respect to $\sim$, which we denote by $(Y_\alpha, \sigma_\alpha)$ (or merely Y_α). Then the space $(Y_\alpha, \sigma_\alpha)$ is a finite T_0-space.

In fact, let p: $(X, \tau_\alpha) \rightarrow (Y_\alpha, \sigma_\alpha)$ be the natural projection, and $y_1, y_2 \in Y_\alpha$, $y_1 \neq y_2$. Then $p^{-1}(y_1) \cap p^{-1}(y_2) = \phi$ since $y_1 \neq y_2$, so taking $x_i \in p^{-1}(y_i)$ $(i=1, 2)$, x_1 is not $\sim$ related to x_2. Hence there exists an open set $G \in \tau_\alpha$ such that either $x_1 \in G$ and $x_2 \notin G$ or

* Institute of Mathematics, Faculty of Science, Kagoshima University Kagoshima, Japan.

$x_1 \notin G$ and $x_2 \in G$. Now suppose that $x_1 \in G$ and $x_2 \notin G$. Then $y_1 = p(x_1) \in p(G)$ and $y_2 = p(x_2) \notin p(G)$. On the other hand, $p(G)$ is open in $(Y_\alpha, \sigma_\alpha)$ since $p^{-1}(p(G)) = G$. Therefore $(Y_\alpha, \sigma_\alpha)$ is a finite T_0-space.

From the above observations it follows that there exists the family $\{(Y_\alpha, \sigma_\alpha) | \alpha \in \Lambda\}$ of finite T_0-spaces associated with a given topological space (X, τ). Now we shall show that the family $\{(Y_\alpha, \sigma_\alpha) | \alpha \in \Lambda\}$ can be made into a inverse system in a natural way.

We first define the order relation on Λ as follows: for $\alpha, \beta \in \Lambda$, $\beta \leq \alpha$ if and only if $\tau_\beta \subset \tau_\alpha$. Then $(\Lambda, \leq)$ is a directed set. Because for $\alpha, \beta \in \Lambda$, taking the topology τ_γ for which $\Delta_\alpha \cup \Delta_\beta$ is a subbase, clearly $\tau_\alpha \subset \tau_\gamma$ and $\tau_\beta \subset \tau_\gamma$. Hence $\alpha \leq \gamma$ and $\beta \leq \gamma$.

We next define a mapping $f_{\alpha\beta}: Y_\beta \to Y_\alpha$ for each pair $\alpha, \beta \in \Lambda$ such that $\alpha \leq \beta$. Let $i_\alpha: (X, \tau) \to (X, \tau_\alpha)$ be the identity mapping and $p_\alpha: (X, \tau_\alpha) \to (Y_\alpha, \sigma_\alpha)$ be the natural projection. Then putting

$$f_\alpha = p_\alpha \cdot i_\alpha ,$$

$f_\alpha: (X, \tau) \to Y_\alpha$ is clearly continuous onto. We now define $f_{\alpha\beta}: Y_\beta \to Y_\alpha$ by the following manner: for $y_\beta \in Y_\beta$ we take $x \in X$ such that $f_\beta(x) = y_\beta$ and put

$$f_{\alpha\beta}(y_\beta) = f_\alpha(x) ,$$

that is

$$f_{\alpha\beta}(f_\beta(x)) = f_\alpha(x) .$$

This mapping is well defined. In fact, suppose that $f_\beta(x_1) = f_\beta(x_2)$, then there exists the open set G_β in (X, τ_β) such that G_β is the minimal neighborhood of x_1 and also of x_2. Hence $x_1 \sim x_2$ that is $U \in \tau_\beta$ and $U \cap (x_1 \cup x_2) \neq \phi$ imply $U \supset (x_1 \cup x_2)$. But $\tau_\alpha \subset \tau_\beta$ since $\alpha < \beta$. Accordingly, if $V \in \tau_\alpha$ and $V \cap (x_1 \cup x_2) \neq \phi$, then $V \in \tau_\beta$ and $V \cap (x_1 \cup x_2) \neq \phi$, so that $V \supset (x_1 \cup x_2)$. Hence $x_1 \sim x_2$, that is $f_\alpha(x_1) = f_\alpha(x_2)$. Thus $f_{\alpha\beta}(f_\beta(x_1)) = f_{\alpha\beta}(f_\beta(x_2))$, which show that $f_{\alpha\beta}$ is well defined.

Futher it follows immediately from the definition that these mappings satisfy

(i) $f_{\alpha\alpha}$ is the identity mapping for each $\alpha \in \Lambda$

and

ii) $f_{\beta\alpha} f_{\beta\gamma} = f_{\alpha\gamma}$ whenever $\alpha < \beta < \gamma$.

In summary, we have seen that $\{Y_\alpha, f_{\alpha\beta}\}$ is the inverse system of finite T_0-spaces.

§3. Embedding of a T_0-space.

Let (X, τ) be a T_0-space and let suppose $\{Y_\alpha | \alpha \in \Lambda\}$ be the inverse system of finite T_0-spaces associated with (X, τ), described above. And we denote by Y_∞ the inverse limit space of this system. Then we have the following theorem.

THEOREM 1. *A T_0-space (X, τ) can be embedded in Y_∞.*

PROOF. Define $f: X \to Y_\infty \subset \Pi Y_\alpha$ by

$$f(x) = \{f_\alpha(x)\}_{\alpha \ni \Lambda} .$$

Since $\{f_\alpha(x)\} \in Y_\infty$, we have $f(X) \subset Y_\infty$.

We shall show that f is embedding.

(i) f is continuous. This is obvious since f_α is continuous for all $\alpha \in \Lambda$.

(ii) f is one to one. To prove this, let $x, y \in X$ such that $x \neq y$. Since (X, τ) is a T_0-space, there is an open neighborhood U of at least one, say x which does not contain the other y. Let now

$$\tau_\alpha = \{\phi, U, X\} ,$$

then $\alpha \in \Lambda$. Clearly

$$f_\alpha(x) \neq f_\alpha(y) .$$

in the space Y_α, whence

$$\{f_\alpha(x)\} \neq \{f_\alpha(y)\} ,$$

that is

$$f(x) \neq f(y) .$$

(iii) f is open. To prove this, it suffies to show that $\{f_\alpha\}_{\alpha \in \Lambda}$ distinguishes points and closed sets. Suppose that F is closed in X and $x \notin F$. Since X is T_0 there is an open nbd V of x such that $V \cap F = \phi$. We now take the topology

$$\tau_\gamma = \{\phi, V, X\} .$$

and consider the space Y_γ, then clearly

$$f_\gamma(x) \notin f_\gamma(F) = \overline{f_\gamma(F)} .$$

Thus (i), (ii) and (iii) have been established. Therefore by the embedding lemma f is embedding. [1].

§ 4. Embedding of a compact T_1-space.

We first observe that every finite T_0-space can be made into a partially ordered set by defining a suitable order relation on this space. [2].

Let $Y = \{y_1, y_2, \cdots, y_n\}$ be a finite topological space and $\{U_1, U_2, \cdots, U_n\}$ be the system of minimal basic neighborhoods, where U_i is the minimal neighborhood of y_i.

We define an order relation $\leq$ on Y, by saying

$$y_1 \leq y_j \text{ whenever } U_i \subset U_j .$$

Then it follows that Y is a T_0-space if and only if $(Y, \leq)$ is a partially ordered set.

Next let (X, τ) be a topological space, $\{Y_\alpha, f_{\alpha\beta}\}$ be its associated inverse system and Y_∞ be the inverse limit space of the family.

We define an ordering on the product space $\prod_{\alpha \in \Lambda} Y_\alpha$ as follows: for $\{x_\alpha\}, \{y_\alpha\} \in \prod_{\alpha \in \Lambda} Y_\alpha$ let $\{x_\alpha\} \leq \{y_\alpha\}$ mean that $x_\alpha \leq y_\alpha$ in $(Y_\alpha, \leq)$ for all $\alpha \in \Lambda$. Then $(\prod_{\alpha \in \Lambda} Y_\alpha, \leq)$ is a partially ordered set.

Here the subspace

$$M(Y_\infty) = \{\{z_\alpha\} \in Y_\infty \mid \{z_\alpha\} \text{ is maximal in } Y_\infty\}$$

will be called the *maximal inverse limit space* of the inverse system $\{Y_\alpha, f_{\alpha\beta}\}$.

THEOREM 2. *A compact T_1-space (X, τ) is homeomorphic to $M(Y_\infty)$.*

PROOF. Let (X, τ) be a compact T_1-space. The notation of Theorem 1 will be used. Let $f\colon X \to Y_\infty$ be as in Theorem 1. Then it suffies to show that

(1) $f(x) \in M(Y_\infty)$ for all $x \in X$.

and

(2) $f\colon X \to M(Y_\infty)$ is onto.

(1) Suppose that there exists $x \in X$ such that $f(x) \notin M(Y_\infty)$. Since (X, τ) is T_1, $\{x\}$ is closed in (X, τ), so that $U=X-\{x\}$ is open. Then taking the topology

$$\tau_\beta = \{\phi, U, X\},$$

we consider the finite T_0-space Y_β. Since $f(x) \notin M(Y_\infty)$, $f(x)=\{f_\alpha(x)\}$ is not maximal in Y_∞, whence there exists $\{z_\alpha\} \in Y_\infty$ such that $\{f_\alpha(x)\}<\{z_\alpha\}$, that is, there exists Y_γ such that $f_\gamma(x)<z_\gamma$.

We now choose the topology τ_ξ generated by the subbase $\tau_\beta \cup \tau_\xi$, then we have

$$\beta, \gamma \leq \xi.$$

Since $\{f_\alpha(x)\}<\{z_\alpha\}$ it follows that $f_\xi(x)\leq z_\xi$, hence $f_\xi(x)<z_\xi$ or $f_\xi(x)=z_\xi$.

Suppose $f_\xi(x)<z_\xi$. Now there exists $y \in X$ such that $z_\xi=f_\xi(y)$, then $x \neq y$ since $f_\xi(x) \neq z_\xi$. Hence $f_\beta(x) \neq f_\beta(y)$, so that

$$f_\beta(y) < f_\beta(x).$$

Since $z_\beta = f_\beta(y)$, we have

$$z_\beta < f_\beta(x),$$

which contradicts $\{f_\alpha(x)\}<\{z_\alpha\}$.

Next suppose $f_\xi(x)=z_\xi$. Then $f_{\gamma\xi}(f_\xi(x))=f_\gamma(x)$ and $f_{\gamma\xi}(z_\xi)=z_\gamma$, whence

$$f_\gamma(x) = z_\gamma,$$

which contradicts $f_\gamma(x)<z_\gamma$.

Therefore

$$f(X) \subset M(Y_\infty).$$

(2) If $f(X)=M(Y_\infty)$, then there is $\{z_\alpha\} \in M(Y_\infty)$ such that $\{f_\alpha(x)\} \neq \{z_\alpha\}$ for all $x \in X$. Since $\{z_\alpha\}$ is maximal, for each $x \in X$ there exists $\beta=\beta(x) \in \Lambda$ such that

$$f_\beta(x) \gneqq z_\beta.$$

Now let U_β be the minimal neighborhood of x in (X, τ_β), then there exists $y \in X$ such that $z_\beta = f_\beta(y)$ and $U_\beta \not\ni y$.

Taking the topology

$$\tau_\gamma = \{\phi, U, X\}$$

and the finite T_0-space Y_γ,

$$\gamma < \beta .$$

Also, since $U_\beta \not\ni y$, $f_\gamma(x) < f_\gamma(y)$ and $f_\gamma(y) = z_\gamma$. Hence

$$f_\gamma(x) < z_\gamma .$$

Now since $\{U_{\beta(x)} \mid x \in X\}$ is an open covering of X and X is compact, this covering has a finite subcovering, which we denote by

$$D = \{U_{\beta_1}(x_1) , U_{\beta_2}(x_2) , \cdots , U_{\beta n}(x_n)\} .$$

Then let τ_δ be the topology on X generated by D, and consider the space Y_δ. Then

$$\gamma_i \leq \delta \ (i = 1,2, \cdots, n) .$$

For $z_\delta \in Y_\delta$, take $y \in X$ such that $f_\delta(y) = z_\delta$. Since D is a covering of X, there exists $U_{\beta k}(x_k) \in D$ such that $y \in U_{\beta k}(x_k)$. Hence

$$f_{\gamma_k}(x_k) \geq f_{\gamma_k}(y)$$

in Y_{γ_k}, also $\gamma_k \leq \delta$, $z_{\gamma_k} = f_{\gamma_k}(y)$, whence

$$f_{\gamma_k}(x_k) \geq z_{\gamma_k} .$$

But this a contradiction.

References

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