CUBIC SPLINE INTERPOLATION AND TWO-SIDED DIFFERENCE METHODS TO TWO-POINT BOUNDARY VALUE PROBLEMS

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CUBIC SPLINE INTERPOLATION AND TWO-SIDED DIFFERENCE METHODS TO TWO-POINT BOUNDARY VALUE PROBLEMS

By

Manabu SAKAI*

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Abstract

In the present paper we consider the two-sided approximations by the use of cubic spline function. A selection of numerical results is presented in the Tables 1–5.

1. Introduction

We shall consider here the two-sided approximations of the solution of the following nonlinear two-point boundary value problem:

$$y''=f(x,y)\,,\qquad 0\le x\le 1 \tag{1}$$

with boundary conditions

$$a_0y(0)-b_0y'(0)=c_0\,, \tag{2}$$

$$a_1y(1)+b_1y'(1)=c_1\,, \tag{3}$$

where $f(x, y)$ is three-continuously differentiable with respect to x and y in the closed bounded domain D of xy-space intercepted by two-lines $x=0$ and $x=1$.

We assume that the problem (1)–(3) has an isolated solution $\hat{y}(x)$ satisfying the internality condition

$$U=\{(x,y)\,|\,|y-\hat{y}(x)|\le\delta\,,\ x\in[0,1]\}\subset D\qquad \textit{for some }\delta>0\,.$$

By the use of B-spline $Q_4(x)$, we shall consider the cubic spline function

$$y_h(x)=\sum a_iQ_4(x/h-i)\qquad (nh=1) \tag{4}$$

such that

$$y_h''=P_kf(x,y_h)\,,\qquad 0\le x\le 1\,, \tag{5}$$

$$a_0y_h(0)-b_0y_h'(0)=c_0\,, \tag{6}$$

$$a_1y_h(1)+b_1y_h'(1)=c_1\,. \tag{7}$$

Here the operator $P_k(k=1,2)$ is defined as follows:

* Department of Mathematics, Kagoshima University.

(i) $$(P_1 f)(x) = \Sigma f(x_i) L_i(x)$$

with the piecewise linear function $L_i(x)$ such that

$$L_i(x_j) = L_i(jh) = \delta_{ij} ,$$

(ii) $$(P_2 f)(x) = \Sigma \beta_i L_i(x)$$

such that the coefficient $\beta_i (i=0,1,\cdots,n)$ is determined by

$$(2\beta_0+\beta_1)/6 = (2f(x_0)+f(x_1))/6 ,$$
$$(\beta_{i+1}+4\beta_i+\beta_{i-1})/6 = f(x_i) , \qquad (i = 1,2, \cdots, n-1) ,$$
$$(2\beta_n+\beta_{n-1})/6 = (2f(x_n)+f(x_{n-1}))/6 .$$

For the operator P_1, equation (5) becomes

$$(y_{i+1}-2y_i+y_{i-1})/h^2$$
$$= (f(x_{i+1}, y_{i+1})+4f(x_i,y_i)+f(x_{i-1},y_{i-1}))/6 , \qquad (i = 1,2, \cdots, n-1) .$$

If ristricted on $[x_0,x_1]$, $y_h(x)$ is the polynomial of degree 3. Therefore $y_h(x_0)$, $y_h(x_1)$, $y_h'(x_0)$, $y_h''(x_0)$, and $y_h''(x_1)$ cannot be linearly independent, *i.e.*, there is a unique linear relation

$$(y_1-y_0)/h-y_0' = h(2y_0''+y_1'')/6 ,$$

from which follows

$$(y_1-y_0)/h-y_0' = h(2f(x_0,y_0)+f(x_1, y_1))/6 . \tag{9}$$

Similarily we have

$$-(y_n-y_{n-1})/h+y_n' = h(2f(x_n, y_n)+f(x_{n-1}, y_{n-1}))/6 . \tag{10}$$

For the operator P_2, equation (5) becomes

$$(y_{i+1}-2y_i+y_{i-1})/h^2 = f(x_i, y_i) , \qquad (i = 1,2, \cdots, n-1) ,$$

with (9) and (10).

In the present paper, we shall prove the following asymptotic expansion

$$e_k(x) = \hat{y}(x)-y_k(x) = (-1)^k h^2 \psi(x)/12+O(h^3) \ (h \to 0) \qquad (k = 1,2) ,$$

where $y_k(x)$ is the solution of (5)–(7), and $\psi(x)$ is the solution of the variation equation of (1)–(3):

$$\Psi'' = f_y(x, y)\psi+D^2 f(x, \hat{y}(x)) , \qquad 0 \le x \le 1 ,$$

with boundary conditions

$$a_0\psi(0)-b_0\psi'(0) = 0 ,$$
$$a_1\psi(1)+b_1\psi'(1) = 0 .$$

2. Asymtotic Expansion of Error Function $e_k(x)$ (k=1, 2)

In [2],[3] we have shown the following theorem.

THEOREM. *In a sufficiently small neighbourhood of the isolated solution $\hat{y}(x)$, there* *exists the approximate solution* $y_k(x)$ $(k=1,2)$ *of* (5)–(7) *such that*

$$|D^j(\hat{y}(x)-y_1(x))| = O(h^2) \ (h \to 0) \qquad (j=0,1,2)$$

and

$$|D^j(\hat{y}(x)-y_2(x))| = O(h^2) \ (h \to 0) \qquad (j=0,1,2).$$

Before we proceed with analysis, we shall require

LEMMA. *If* $g(x)$ *is twice continuously differentiable, then we have*

$$\|(I-P_k)g\| \leq ch^2\|g''\| \qquad (k=1,2),$$

where c *is a constant independent of* h *and for any continuous function* $\varphi(t)$ *we denote its maximum norm by* $\|\varphi\|$.

Combining Theorem and this Lemma, we have

$$e_k'' = f_y(x,\hat{y})e_k + (I-P_k)f(x,\hat{y}) + O(h^4), \tag{11}$$

$$a_0e_k(0)-b_0e_k'(0) = 0,$$

$$a_1e_k(1)+b_1e_k'(1) = 0.$$

Since $\hat{y}(x)$ is isolated, there exists the Green function $H(x, t)$ such that

$$e_k(x) = \int H(x,t)(I-P_k)g(t)dt + O(h^4), \tag{12}$$

with $$g(t) = f(t,\hat{y}(t)).$$

By Taylor series expansion, we have

$$(I-P_1)g(t) = g''(x_i)(t-x_i)(t-x_{i+1})/2 + O(h^3) \text{ on } [x_i, x_{i+1}] \tag{13}$$

from which follows

$$\int H(x,t)(I-P_1)g(t)dt = 1/2\sum\int H(x,t)g''(x_i)(t-x_i)(t-x_{i+1})dt + O(h^3)$$

$$= 1/2\sum H(x,\eta_i)g''(x_i)\int(t-x_i)(t-x_{i+1})dt + O(h^3)$$

$$= -h^3/12\sum H(x,\eta_i)g''(x_i) + O(h^3)$$

$$= -h^2/12\int H(x,t)g''(t)dt + O(h^3).$$

On the other hand, we have

$$e_2(x) = \int H(x,t)(I-P_2)g(t)dt$$

$$= \sum\int \{H(x,x_i)L_i(t)+H(x,x_{i+1})L_{i+1}(t)\}\ \{g(t)-\beta_iL_i(t)-\beta_{i+1}L_{i+1}(t)\}\ dt + O(h^3)$$

with $(P_2g)(t) = \sum\beta_iL_i(t)$.

Therefore we have

$$e_2(x) = \sum_{i=1}^{n-1} H(x,x_i) \left\{\int g(t)L_i(t)dt - h(\beta_{i+1}+4\beta_i+\beta_{i-1})/6\right\} \tag{14}$$

$$+ H(x,x_0) \left\{\int g(t)L_0(t)dt - h(2\beta_0+\beta_1)/6\right\}$$

$$+ H(x,x_n) \left\{\int g(t)L_n(t)dt - h(2\beta_n+\beta_{n-1})/6\right\} + O(h^3)$$

$$= \sum H(x_1x_i) \left\{\int g(t)L_i(t)dt - hg(x_i)\right\}$$

$$+ H(x,x_0) \left\{\int g(t)L_0(t)\,dt - h(2g(x_0)+g(x_1))/6\right\}$$

$$+ H(x,x_n) \left\{\int g(t)L_n(t)dt - h(2g(x_n)+g(x_{n-1}))/6\right\} + O(h^3)$$

$$= h^3/12 \sum H(x,x_i)g''(x_i) - h^3/24\{H(x,x_0)g''(x_0)+H(x,x_n)g''(x_n)\} + O(h^3)$$

$$= h^2/12 \int H(x,t)g''(t)dt + O(h^3)\,.$$

Thus we have

$$e_k(x) = (-1)^k h^2/12 \int H(x,t)g''(t)dt + O(h^3) \qquad (k=1,2)\,. \tag{15}$$

Next we consider the following two-point boundary value problem

$$y'' = f(x,y,y')\,, \qquad 0 \leq x \leq 1 \tag{16}$$

with boundary conditions

$$\varphi_l(y(0), y'(0), y(1), y'(1)) = 0 \qquad (l=1,2)\,. \tag{17}$$

For this problem (16) and (17), we consider the cubic spline function

$$y_h(x) = \sum \alpha_i Q_4(x/h-i)$$

such that

$$y_h'' = P_k f(x,y_h,y_h')\,, \qquad 0 \leq x \leq 1 \tag{18}$$

with

$$\varphi_l(y_h(0), y_h'(0), y_h(1), y_h'(1)) = 0 \qquad (l=1,2)\,. \tag{19}$$

In a similar way as in the case when $f(x, y, y') \equiv f(x, y, 0)$, we can prove the asymptotic expansion

$$e_k(x) = (-1)^k h^2 \psi(x)/12 + O(h^3) \;\; (h \to 0) \qquad (k=1,2)\,,$$

where $\psi(x)$ is the solution of the following equation

$$\psi'' = f_y(x,\hat{y},\hat{y}')\psi + f_{y'}(x,\hat{y},\hat{y}')\psi' + D^2 f(x,\hat{y}(x),\hat{y}'(x))$$

subject to boundary conditions

$$\varphi_{l1}(\hat{y}(0),\hat{y}'(0),\hat{y}(1),\hat{y}'(1))\psi(0)+\cdots$$
$$+\varphi_{l4}(\hat{y}(0),\hat{y}'(0),\hat{y}(1),\hat{y}'(1))\psi'(1)=0 \qquad (l=1,2)$$

with
$$\varphi_{lk}(x_1,x_2,x_3,x_4)=\frac{\partial\varphi_l}{\partial x_k}.$$

3. Numerical Examples

Example 1. As our first example, we consider the following linear problem:

$$y''=100y\,,\qquad 0\le x\le 1\,,$$
$$y(0)=y(1)=1\,.$$

Its exact solution is $y(x)=\cosh(10x\text{–}5)/\cosh 5$.

Table 1. ($h=1/20$)

x	$e_1(x)$	$e_2(x)$	$\hat{y}(x)$	$(y_1+y_2)/2$
0. 1	3. 93(−3)	−3. 85(−3)	0. 36799	0. 36790
0. 2	2. 89(−3)	−2. 80(−3)	0. 13566	0. 13562
0. 3	1. 64(−3)	−1. 60(−3)	0. 05070	0. 15068
0. 4	0. 88(−3)	−0. 91(−3)	0. 02079	0. 02079
0. 5	0. 71(−3)	−0. 69(−3)	0. 01348	0. 01347

Example 2. (Stetter [4]). We now take as our next example the boundary value problem as follows:

$$y''=6\sqrt{y-x-1}\,,\qquad 0\le x\le 1$$

subject to the boundary conditions

$$y'(0)-y^2(0)=7/16\,,$$
$$y'(1)-54/y(1)=0\,.$$

Sample values obtained for $h=1/20$, $1/40$, $1/100$ are shown in the Tables 2.1 and 2.2.

Table 2.1 ($e_1(x)$)

x	$h=1/20$	$h=1/40$	$h=1/100$
0	1. 334(−4)	3. 33(−5)	5. 3(−6)
0. 2	1. 913(−4)	4. 78(−5)	7. 7(−6)
0. 4	2. 309(−4)	5. 77(−5)	9. 2(−6)
0. 6	2. 485(−4)	5. 21(−5)	9. 9(−6)
0. 8	2. 395(−4)	5. 99(−5)	9. 6(−6)
1	1. 909(−4)	4. 95(−5)	7. 9(−6)

Table 2.2 ($e_2(x)$)

x	$h=1/20$	$h=1/40$	$h=1/100$
0	−1. 167(−4)	−3. 13(−5)	−5. 2(−6)
0. 2	−1. 770(−4)	−4. 60(−5)	−7. 5(−6)
0. 4	−2. 166(−4)	−5. 59(−5)	−9. 1(−6)
0. 6	−2. 326(−4)	−6. 01(−5)	−9. 8(−6)
0. 8	−2. 203(−4)	−5. 75(−5)	−9. 4(−6)
1	−1. 743(−4)	−4. 65(−5)	−7. 7(−6)

Example 3. Let us consider the following nonlinear problem:

$$y'' = 1.5y^2, \qquad 0 \leq x \leq 1,$$

with $y(0)=4$ and $y(1)=1$.
This problem has two isolated solutions such that

$$y(0.5) = 16/9 \text{ and } y(0.5) \doteqdot -10.53.$$

Table 3.1 ($e_1(x)$ for $\hat{y}(0.5)=16/9$)

x	$h=1/20$	$h=1/40$	$h=1/100$
0.1	−8.2329(−4)	−2.0495(−4)	−3.2754(−5)
0.2	−1.1376(−3)	−2.8336(−4)	−4.5291(−5)
0.3	−1.2001(−3)	−2.9903(−4)	−4.7801(−5)
0.4	−1.1350(−3)	−2.8289(−4)	−4.5224(−5)
0.5	−1.0037(−3)	−2.5023(−4)	−4.0006(−5)
0.6	−8.3682(−4)	−2.0865(−4)	−3.3360(−5)
0.7	−6.4876(−4)	−1.6179(−4)	−2.5868(−5)
0.8	−4.4581(−4)	−1.1119(−4)	−1.7779(−5)
0.9	−2.2977(−4)	−5.7314(−5)	−9.1644(−6)

Table 3.2 ($e_2(x)$ for $\hat{y}(0.5)=16/9$)

x	$h=1/20$	$h=1/40$	$h=1/100$
0.1	8.1414(−4)	2.0438(−4)	3.2740(−5)
0.2	1.1265(−3)	2.8266(−4)	4.5273(−5)
0.3	1.1189(−3)	2.9837(−4)	4.7818(−5)
0.4	1.1258(−3)	2.8231(−4)	4.5209(−5)
0.5	9.9629(−4)	2.4976(−4)	3.9994(−5)
0.6	8.3100(−4)	2.0829(−4)	3.3351(−5)
0.7	6.4450(−4)	1.6152(−4)	2.5861(−5)
0.8	4.4303(−4)	1.1101(−4)	1.7774(−5)
0.9	2.2839(−4)	5.7228(−5)	9.1622(−6)

Table 3.3 ($y_1(x)$ for $\hat{y}(0.5)\doteqdot-10.53$)

x	$h=1/20$	$h=1/40$	$h=1/100$
0.1	0.4610 8789	0.474 4466	0.478 1951
0.2	−3.036 0000	−3.015 1148	−3.009 2359
0.3	−6.359 5030	−6.338 1672	−6.332 1475
0.4	−9.059 3094	−9.043 7869	−9.039 4076
0.5	−10.54 7647	−10.53 9087	−10.53 6684
0.6	−10.41 8460	−10.41 2260	−10.41 0524
0.7	−8.709 2591	−8.700 5180	−8.698 0532
0.8	−5.875 7206	−5.864 6259	−5.861 4933
0.9	−2.503 0694	−2.494 4919	−2.492 0752

Table 3.4 ($y_2(x)$ for $\hat{y}(0.5) \doteqdot -10.53$)

x	$h=1/20$	$h=1/40$	$h=1/100$
0.1	0.496 8104	0.483 3772	0.479 6240
0.2	−2.979 9213	−3.001 0952	−3.006 9927
0.3	−6.302 0388	−6.323 8017	−6.329 8490
0.4	−9.017 5047	−9.033 3360	−9.037 7354
0.5	−10.52 4745	−10.53 3361	−10.53 5768
0.6	−10.40 1925	−10.40 8126	−10.40 9862
0.7	−8.085 7360	−8.694 6371	−8.697 1123
0.8	−5.854 8110	−5.857 2487	−5.860 2970
0.9	−2.480 0101	−2.488 7272	−2.491 1529

Example 4. We consider the following van der Pol's equation:

$$y'' = (1-y^2)y'/2 - y/4, \qquad 0 \leq x \leq 1,$$

with $y(0)=0$ and $y(1)=2$.

Table 4.1 ($y_1(x)$)

x	$h=1/20$	$h=1/40$	$h=1/100$
0.1	0.1989 1973	0.1988 7210	0.1988 5874
0.2	0.4070 1570	0.4069 3478	0.4069 1206
0.3	0.6228 1925	0.6227 2447	0.6226 9784
0.4	0.8436 2224	0.8435 3603	0.8435 1179
0.5	1.0653 913	1.0653 343	1.0653 182
0.6	1.2929 327	1.2829 168	1.2829 123
0.7	1.4903 804	1.4904 034	1.4904 098
0.8	1.6819 670	1.6820 110	1.6820 234
0.9	1.8528 893	1.8529 261	1.8529 364

Table 4.2 ($y_2(y)$)

x	$h=1/20$	$h=1/40$	$h=1/100$
0.1	0.1987 9234	0.1988 4026	0.1988 5364
0.2	0.4067 9909	0.4068 8063	0.4069 0340
0.3	0.6225 6532	0.6226 6099	0.6226 8768
0.4	0.8433 9105	0.8434 7824	0.8435 0254
0.5	1.0652 381	1.0652 960	1.0653 121
0.6	1.2828 898	1.2829 061	1.2829 106
0.7	1.4904 419	1.4904 188	1.4904 123
0.8	1.6820 849	1.6820 405	1.6820 281
0.9	1.8529 877	1.8529 507	1.8529 403

Example 5 (Ciarlet [1]). As our final example, consider

$$y'' = y^3 - (\cos x + 1)^3 - \cos x, \qquad 0 \leq x \leq 1$$

with $y'(0)=0$ and $y'(1) = -y^3(1)\, sin\, 1/(cos\, 1+1)^3$

The unique solution is $y(x) = cos\, x + 1$.

The numerical results are listed in Tables 5.1 and 5.2.

Table 5.1 ($y_1(x)$)

x	$h=1/20$	$h=1/40$	$h=1/100$	exact solution
0	1.9999 832	... 958	... 993	2.000 000
0.1	1.9949 874	50 000	50 035	50 042
0.2	1.9800 499	624	659	666
0.3	1.9553 200	324	358	365
0.4	1.9210 448	569	603	610
0.5	1.8775 667	786	819	826
0.6	1.8253 203	318	350	356
0.7	1.7648 276	385	416	422
0.8	1.6966 931	7 033	7 062	7 067
0.9	1.6215 977	6 069	6 095	6 100
1	1.5402 916	996	3 019	3 023

Table 5.2 ($y_2(x)$)

x	$h=1/20$	$h=1/40$	$h=1/100$	exact solution
0	2.0000 137	... 038	... 006	... 000
0.1	1.9950 187	081	048	042
0.2	1.9800 816	751	672	666
0.3	1.9553 517	405	371	365
0.4	1.9210 762	649	616	610
0.5	1.8775 975	864	832	826
0.6	1.8253 501	393	362	356
0.7	1.7648 559	457	428	422
0.8	1.6967 193	100	072	067
0.9	1.6216 211	129	105	100
1	1.5403 116	048	027	023

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