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ON THE RAMIFICATION NUMBERS OF A TOWER OF THE MAXIMAL ABELIAN EXTENSION OF $\mathfrak{p}$-ADIC NUMBER FIELDS WITH EXPONENT p^m

By

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1. Introduction.

Let m be a natural number which we fix throughout this note, and let k be a $\mathfrak{p}$-adic number field and p the characteristic of the residue class field of k. We define a chain of fields $K_0=k$, K_1, K_2, $\cdots$, which has the property such that K_i is the maximal abelian extension of K_{i-1} with exponent p^m for each $i\geqq 1$.

I.R. Šafarevič [6] has given the detailed structure of such fields, when k does not contain the p-th roots of unity and $m=1$. For this case, E. Maus [5] has given the upper ramification numbers and J. Idt [3] has given the explicit values of the ramification numbers.

In this note, we compute the ramification numbers and the orders of the ramification groups of K_i/k for general m.

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2. Preliminaries.

Throughout this note, k denotes a $\mathfrak{p}$-adic number field, k^* the multiplicative group of k, p the characteristic of the residue class field of k, and μ_k the order of the group of p power roots of unity in k. Let $K_0=k$ and let K_i be the maximal abelian extension of K_{i-1} with exponent p^m for each positive integer i. Let $\mathfrak{e}_s$, $\mathfrak{f}_s$, and $\mathfrak{n}_s$ denote the absolute ramification index, the absolute residue class degree, and the absolute degree of K_s, respectively. Then we have a following

Lemma 1. *Suppose that the notations are the same as in the above.*

(1) *If* $\mu_k=1$, *then*

$$\mathfrak{n}_s=\mathfrak{n}_{s-1}p^{m(\mathfrak{n}_{s-1}+1)},\ \mathfrak{f}_s=p^m\mathfrak{f}_{s-1}\ \textit{and}\ \mathfrak{e}_s=\mathfrak{e}_{s-1}p^{m\mathfrak{n}_{s-1}}.$$

(2) *If* $\mu_k\geqq p^m$, *then*

$$\mathfrak{n}_s=\mathfrak{n}_{s-1}p^{m(\mathfrak{n}_{s-1}+2)},\ \mathfrak{f}_s=p^m\mathfrak{f}_{s-1}\ \textit{and}\ \mathfrak{e}_s=\mathfrak{e}_{s-1}p^{m(\mathfrak{n}_{s-1}+1)}.$$

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This follows easily from the local class field theory and the structure of the multiplicative group of $\mathfrak{p}$-adic number fields.

Let π be a prime element in k and let $\mathfrak{e}$ be the absolute ramification index of k. Let U_k be the group of units of k. We define the usual filtration by $U_0=U_k$, $U_i=\{x\in k^*;\ \mathrm{ord}_k(x-1\geqq i\}$ for $i>0$, where ord_k is the normalized additive valuation of k. Iet λ denote the function defined by

$$\lambda(n)=\min\{pn, n+\mathfrak{e}\}\,, n\in \boldsymbol{Z}, n>0\,.$$

For each positive integer i, let $R_i\subseteqq U_i$ denote a complete set of representatives for the factor group U_i/U_{i+1}. It is shown by Hasse [2] that these representative sets can be chosen in such a manner that

$$R_i=R_j^p \qquad \text{whenever } i=\lambda(j)$$

except in case $\mu_k>1$ and $i=\dfrac{p\mathfrak{e}}{p-1}\cdot$ It is shown also that, even in the exceptional case, R_j^p $\left(\text{where } j=\dfrac{\mathfrak{e}}{p-1}\right)$ is a subgroup mod π of R_i with index p.

LEMMA 2. *Let t be a natural number* $\leqq \mathfrak{e}\left(m+\dfrac{1}{p-1}\right)$ *with*

$$t\not\equiv 0 \text{ mod } p^m \qquad \text{for } 0<t\leqq \frac{\mathfrak{e}}{p-1}+\mathfrak{e}$$

$$t\not\equiv l\mathfrak{e} \text{ mod } p^{m-l} \qquad \text{for } \frac{\mathfrak{e}}{p-1}+l\mathfrak{e}<t\leqq\frac{\mathfrak{e}}{p-1}+(l+1)\mathfrak{e}\,, \quad l=1,\cdots,m-1\,.$$

Then

$$U_t\cap k^{*p^m}\subseteqq U_{t+1}\,.$$

PROOF. Suppose that there is an element η_t in $U_t\cap k^{*p^m}$ but not in U_{t+1}. Then we have $\eta_t=a^{p^m}$ where a is a principal unit. Assume $a=a_\nu\in U_\nu-U_{\nu+1}$. Then we may assume $a_\nu\in R_\nu$. If $0<\nu<\dfrac{\mathfrak{e}}{p^{m-1}(p-1)}$ then $a_\nu^{p^m}\in U_{p^m\nu}-U_{p^m\nu+1}$ from the above result of Hasse. Therefore we have $t=p^m\nu<\dfrac{\mathfrak{e}}{p-1}+\mathfrak{e}$, which is a contradiction.

If $\dfrac{\mathfrak{e}}{p^{m-l}(p-1)}<\nu<\dfrac{\mathfrak{e}}{p^{m-l-1}(p-1)}$, $l=1,\cdots,m-1$, then $p^{m-l-1}\nu<\dfrac{\mathfrak{e}}{p-1}$ and $\dfrac{\mathfrak{e}}{p-1}<p^{m-l}\nu$. Thus, by the above result of Hasse, we have

$$a_\nu^{p^m}\in U_{p^{m-l}\nu+l\mathfrak{e}}-U_{p^{m-l}\nu+l\mathfrak{e}+1}\,.$$

Therefore we have

$$\frac{\mathfrak{e}}{p-1}+l\mathfrak{e}<t=p^{m-l}\nu+l\mathfrak{e}<\frac{\mathfrak{e}}{p-1}+(l+1)\mathfrak{e}\,,$$

which is a contradiction.

If $p^{m-l-1}(p-1)\,|\,\mathfrak{e}$, $l=0,1,\cdots,m-1$, then we see $\dfrac{\mathfrak{e}}{p-1}+(l+1)\mathfrak{e}\equiv l\mathfrak{e} \text{ mod } p^{m-l}$. Thus the consideration for the case $\nu=\frac{\mathfrak{e}}{p^{m-l}(p-1)}$, $l=1,\cdots,m$ is not necessary. The proof is complete.

3. The upper ramification numbers and the ramification numbers of K_s/K_{s-1}.

For a finite Galois extension L/k with Galois group G, let $T(L/k)$ denote the set of the upper ramification numbers, i.e. the set of jumps in the upper numbering of the ramification groups of L/k, and let $v_i=v_i(L/k)$ denote the ramification number, i.e. the jump in the usual numbering of the ramification groups of L/k. For real $x\geqq 0$, the symbol $\{x\}$ will denote the least integer $\geqq x$. The next theorem was given by Maus [4], [5]. We set for brevity $\bar{\nu}_0=0$ and $\bar{\nu}_l=l\mathfrak{e}+\left[\frac{\mathfrak{e}}{p-1}\right]-\left[\frac{\mathfrak{e}}{p^{m-l}(p-1)}\right]$ for $l=1,\cdots,m$-1.

THEOREM 3 (E. Maus). *Let L be the maximal abelian extension of k with exponent p^m, and let $\mathfrak{e}$ be the absolute ramification index of k.*

(1) *If $\mu_k=1$, then $T(L/k)=\{t_\nu;\nu=0,1,\cdots,m\mathfrak{e}-1\}$, where*

$$t_\nu=\nu+\left\{\frac{\nu-l\mathfrak{e}+1}{p^{m-l}-1}\right\}\qquad \text{for } \bar{\nu}_l\leqq\nu<\bar{\nu}_{l+1},\ l=0,1,\cdots,m-1\,.$$

(2) *If $\mu_k\geqq p^m$, then $T(L/k)=\{t_\nu;\nu=0,1,\cdots,m\mathfrak{e}+m-1\}$ where*

$$t_\nu=\begin{cases}\nu-l+\left\{\dfrac{\nu-l-l\mathfrak{e}+1}{p^{m-l}-1}\right\} & \text{for } \bar{\nu}_l+l\leqq\nu<\bar{\nu}_{l+1}+l,\ l=0,1,\cdots,m-1\\[2ex] l\mathfrak{e}+\dfrac{\mathfrak{e}}{p-1} & \text{for } \nu=\bar{\nu}_l+l-1,\ l=1,2,\cdots,m\,.\end{cases}$$

PROOF. Let $\omega\colon k^*\to G=\mathrm{Gal}(L/k)$ be the reciprocity law map of the local class field theory corresponding to the extension L/k. Then, it is known that $\omega(U_n)=G^n$ for all $n\geqq 0$ (cf. Serre [7], Theorem 2, p. 235). Therefore, for each integer $n\geqq 0$, we have

$$G^n/G^{n+1}\cong U_n/U_{n+1}(U_n\cap k^{*p^m})\,,$$

because L/k is the maximal abelian extension of exponent p^m.

(1) First, if t satisfies the hypotheses in Lemma 2, then, $G^t/G^{t+1}\cong U_t/U_{t+1}\cong\bar{k}$, where $\bar{k}$ is the residue class field of k. Therefore such t is an upper ramification number of L/k. Next, let $t\equiv 0 \bmod p^m$ and $0<t\leqq\frac{p\mathfrak{e}}{p-1}$. We put $t=p^m\nu$. If $\eta_t\in U_t$, then η_t is uniquely written as

$$\eta_t=(1+a_t\pi^t)\eta_{t+1}\ (a_t \bmod \mathfrak{p}\,,\ \eta_{t+1}\in U_{t+1})$$

where π, $\mathfrak{p}$ is a prime element and the prime ideal in k (cf. Hasse [2], p. 206). We may assume $1+a_t\pi^t\in R_t$. From the result of Hasse in **2**, there is a suitable element η_ν in U_ν such that $\eta_\nu^{p^m}=(1+a_t\pi^t)\eta'_{t+1}$ where η'_{t+1} is some element in U_{t+1}. Hence

$U_t \subseteqq U_{t+1}(U_t \cap k^{*p^m})$. Therefore $G^t = G^{t+1}$.

Finally, let $t \equiv l\mathfrak{e} \mod p^{m-l}$ and $\frac{\mathfrak{e}}{p-1} + l\mathfrak{e} < t \leqq \frac{\mathfrak{e}}{p-1} + (l+1)\mathfrak{e}$, $l = 1, \cdots, m-1$. We put $t = l\mathfrak{e} + p^{m-l}\nu$. Similarly, for any $\eta_t = (1+\alpha_t\pi^t)\eta_{t+1} \in U_t$, there exists a suitable $\eta_\nu \in U_\nu$ such that $\eta_\nu^{p^m} = (1+\alpha_t\pi^t)\eta'_{t+1}$. Therefore $U_{t+1}(U_t \cap k^{*p^m}) = U_t$, so $G^t = G^{t+1}$. This completes the proof of (1).

(2) Since $\mu_k \geqq p^m$, $p^{m-1}(p-1) | \mathfrak{e}$. Put $\nu_l = \frac{\mathfrak{e}}{p^{m-l}(p-1)}$ for $l = 1, \cdots, m$. Let $\eta_{\frac{\mathfrak{e}}{p-1}+l\mathfrak{e}} \in U_{\frac{\mathfrak{e}}{p-1}+l\mathfrak{e}}$, $l = 1, \cdots, m$. Then

$$\eta_{\frac{\mathfrak{e}}{p-1}+l\mathfrak{e}} = \left(1+\alpha_{\frac{\mathfrak{e}}{p-1}+l\mathfrak{e}}\pi^{\frac{\mathfrak{e}}{p-1}+l\mathfrak{e}}\right)\eta_{\frac{\mathfrak{e}}{p-1}+l\mathfrak{e}+1}, \alpha_{\frac{\mathfrak{e}}{p-1}+l\mathfrak{e}} \bmod \mathfrak{p}, \eta_{\frac{\mathfrak{e}}{p-1}+l\mathfrak{e}+1} \in U_{\frac{\mathfrak{e}}{p-1}+l\mathfrak{e}+1}.$$

From the result of Hasse in 2, there exist some $\eta_{\nu_l} \in U_{\nu_l}$ and a suitable $\eta^* \in U_{\frac{p\mathfrak{e}}{p-1}}$ such that

$$\eta_{\nu_l}^{p^m}\eta^{*p^{l-1}} = \left(1+\alpha_{\frac{\mathfrak{e}}{p-1}+l\mathfrak{e}}\pi^{\frac{\mathfrak{e}}{p-1}+l\mathfrak{e}}\right)\eta'_{\frac{\mathfrak{e}}{p-1}+l\mathfrak{e}+1}.$$

Therefore $U_{\frac{\mathfrak{e}}{p-1}+l\mathfrak{e}} \Big/ U_{\frac{\mathfrak{e}}{p-1}+l\mathfrak{e}+1}\left(U_{\frac{\mathfrak{e}}{p-1}+l\mathfrak{e}} \cap k^{*p^m}\right)$ is a cyclic group of order p. The proof is complete.

In the proof of Theorem 3, we obtain the orders of the ramification groups of L/k. These give the ramification numbers together with the Hasse's function $\psi_{L/k}$, i.e. the function defined by $\psi_{L/k}(x) = \int_0^x (G^0 : G^t)dt$ for real $x \geqq 0$.

Let $\mathfrak{e} = p^{m-l}(p-1)q_{m-l} + r_{m-l}$, $0 \leqq r_{m-l} < p^{m-l}(p-1)$, and let $r_{m-l} = (p-1)\bar{q}_{m-l} + \bar{r}_{m-l}$, $0 \leqq \bar{r}_{m-l} < p-1$, for $l = 1, \cdots, m$.

COROLLARY. *Suppose L/k is the same as in Theorem 3 and $\mathfrak{f}$ is the absolute residue class degree of k. Let $\mu_k = 1$ and let v_i, $0 \leqq i \leqq m\mathfrak{e}-1$, be the ramification number of L/k.*

(1) *If $0 \leqq j < \bar{v}_1$, then*

$$v_j = \frac{(1-p^{p^m\mathfrak{f}})(1-p^{q_{m,j}(p^m-1)\mathfrak{f}})}{(1-p^{\mathfrak{f}})(1-p^{\mathfrak{f}(p^m-1)})} + p^{q_{m,j}(p^m-1)\mathfrak{f}}\frac{1-p^{\mathfrak{f}(r_{m,j}+1)}}{1-p^{\mathfrak{f}}},$$

where $j = (p^m-1)q_{m,j} + r_{m,j}$, $0 \leqq r_{m,j} < p^m - 1$.

(2) *If $\bar{v}_l \leqq j < \bar{v}_{l+1}$, $1 \leqq l \leqq m-1$, then*

$$v_j = v_{\bar{v}_l-1} + p^{f\bar{v}_l}\left(\delta_{m-l} + \frac{1-p^{\mathfrak{f}(j-\bar{v}_l+1)}}{1-p^{\mathfrak{f}}} + p^{\mathfrak{f}(p^{m-l}-1-\bar{q}_{m-l})}\frac{1-p^{\mathfrak{f}q_{m-l,j}(p^{m-l}-1)}}{1-p^{\mathfrak{f}(p^{m-l}-1)}}\right)$$

where $q_{m-l,j} = \left[\frac{j-\bar{v}_l+\bar{q}_{m-l}}{p^{m-l}-1}\right]$ and $\delta_{m-l} = \begin{cases} 1 & \text{if } p^{m-l}(p-1) | \mathfrak{e} \\ 0 & \text{otherwise.} \end{cases}$

PROOF. Let $t_{-1}=0$ and t_j be the upper ramification number of L/k. Then we have

$$v_j = \psi_{L/k}(t_j) = \sum_{i=0}^{j} \frac{g_0}{g_i}(t_i - t_{i-1})$$

where $g_i=(G_{v_i}:1)$. Therefore the corollary follows easily from Theorem 3.

REMARK. For the case such that $\mu_k \geqq p^m$, we obtain the similar results. In fact, if $\bar{\nu}_l + l \leqq j < \bar{\nu}_{l+1}+l$ then we have

$$v_j = \sum_{i=0}^{\bar{\nu}_1} p^{\mathfrak{f}i}(t_i - t_{i-1}) + \sum_{i=\nu_1+1}^{\bar{\nu}_2+1} p^{1+\mathfrak{f}(i-1)}(t_i - t_{i-1}) +$$

$$\cdots + \sum_{i=\bar{\nu}_l+l}^{j} p^{l+\mathfrak{f}(i-l)}(t_i - t_{i-1})\,.$$

4. The ramification numbers of K_s/k.

In this and the next sections, suppose that the notations are the same as in **2**. Furthermore, let $v_j^{(s)}, \bar{v}_j^{(s)}$ be the ramification number of $K_s/k, K_s/K_{s-1}$, respectively. Let $\varphi_{s,s-1}=\varphi_{K_s/K_{s-1}}$ be the inverse function of $\psi_{s,s-1}=\psi_{K_s/K_{s-1}}$.

THEOREM 4. *For each j, $v_j^{(s)}=\bar{v}_j^{(s)}$.*

PROOF. We prove the theorem by induction on s. Let $\mathrm{Gal}(K_s/k)=G^{(s)}$ and $\mathrm{Gal}(K_s/K_{s-1})=N^{(s)}$. Then we have, for all j, $N_j^{(s)}=G_j^{(s)}\cap N^{(s)}$ and

$$G_j^{(s)}N^{(s)}/N^{(s)} = (G^{(s)}/N^{(s)})_{\varphi_{s,s-1}(j)} \cong G^{(s-1)}_{\varphi_{s,s-1}(j)}$$

from the Herbrand's theorem (cf. Serre [7], Ch. IV, §.3). Hence

$$(G_j^{(s)}:G_{j+1}^{(s)}) = (G^{(s-1)}_{\varphi_{s,s-1}(j)}:G^{(s-1)}_{\varphi_{s,s-1}(j+1)})(N_j^{(s)}:N_{j+1}^{(s)})\,.$$

Now, suppose $N_j^{(s)}=N_{j+1}^{(s)}$. Then $N^{(s)\varphi_{s,s-1}(j)} = N^{(s)\varphi_{s,s-1}(j+1)}$.

If $\varphi_{s,s-1}(j+1) \leqq \dfrac{\mathfrak{e}_{s-1}}{p-1}+\mathfrak{e}_{s-1}$, then we have, from Theorem 3,

$$\{\varphi_{s,s-1}(j)\} = \{\varphi_{s,s-1}(j+1)\}$$

or

$$p^m h_j - 1 < \varphi_{s,s-1}(j) < \varphi_{s,s-1}(j+1) \leqq p^m h_j + 1\,,$$

where h_j is a suitable integer. If $\{\varphi_{s,s-1}(j)\}=\{\varphi_{s,s-1}(j+1)\}$, then $G^{(s-1)}_{\varphi_{s,s-1}(j)} = G^{(s-1)}_{\{\varphi_{s,s-1}(j)\}} = G^{(s-1)}_{\varphi_{s,s-1}(j+1)}$, so $G_j^{(s)} = G_{j+1}^{(s)}$.

By the induction hypothesis and Theorem 3 we have

$$v_0^{(s-1)} = \bar{v}_0^{(s-1)} = \psi_{s-1,s-2}(\bar{t}_0^{(s-1)}) = \psi_{s-1,s-2}(1) = 1\,,$$

where $\bar{t}_j^{(s-1)}$ is the upper ramification number of K_{s-1}/K_{s-2}. Hence $v_i^{(s-1)} \equiv 1 \bmod p$ for $i \geqq 1$. Therefore, if $p^m h_j - 1 < \varphi_{s,s-1}(j) < \varphi_{s,s-1}(j+1) \leqq p^m h_j + 1$, then $\varphi_{s,s-1}(j)$ is not the ramification number of K_{s-1}/k. Thus we obtain

$$G^{(s-1)}_{\varphi_{s,s-1}(j)} = G^{(s-1)}_{\varphi_{s,s-1}(j+1)}, \text{ so } G^{(s)}_j = G^{(s)}_{j+1}.$$

For all $i \geqq 0$, by the theorem of Dedekind-Hensel-Ore, we have

$$v^{(s-1)}_i \leqq \frac{p\mathfrak{e}_{s-1}}{g^{(s-1)}_i(p-1)} \leqq \frac{\mathfrak{e}_{s-1}}{p-1}$$

where $g^{(s-1)}_i = (G^{(s-1)}_{v^{(s-1)}_i} : 1)$ (cf. Maus [5], **1**.4). On the other hand, we have from Theorem 3 $0<\varphi_{s,s-1}(j+1)-\varphi_{s,s-1}(j)<2$. Therefore, if $\varphi_{s,s-1}(j+1) > \frac{p\mathfrak{e}_{s-1}}{p-1}$, then we obtain

$$G^{(s-1)}_{\varphi_{s,s-1}(j)} = 1 = G^{(s-1)}_{\varphi_{s,s-1}(j+1)}, \text{ so } G^{(s)}_j = G^{(s)}_{j+1}.$$

This completes the proof of our assertion.

By the corollary to Theorem 3 and Theorem 4 we can compute an explicit value of $v^{(s)}_j$. For $s=1$, we may replace $\mathfrak{f}$ in the corollary to Theorem 3 with $\mathfrak{f}_0$. Let $s \geqq 2$ and $\mu_k=1$. Put $\mathfrak{e}_0=(p-1)q_0 + r_0$, $0 \leqq r_0 < p-1$. Then, in Corollary to Theorem 3 we have

$$\bar{q}_{m-l} = r_0(p^{m-l-1}+p^{m-l-2}+ \cdots +p+1)$$

for all l and we may replace $\mathfrak{f}$, $\mathfrak{e}$ with $\mathfrak{f}_{s-1}=p^{s-1}\mathfrak{f}_0$, $\mathfrak{e}_{s-1}$, respectively. Similarly, in the case such that $\mu_k \geqq p^m$, we may replace $\mathfrak{f}$, $\mathfrak{e}$ in Remark to Theorem 3 with $\mathfrak{f}_{s-1}=p^{s-1}\mathfrak{f}_0$, $\mathfrak{e}_{s-1}$, respectively.

REMARK. In the case where $m=1$ our result coincides with the result of J. Idt [3].

5. The orders of the ramification groups of K_s/k.

The next theorem is a generalization of the result of J. Idt [3] and from this theorem we can compute inductively the orders of the ramification groups of K_s/k.

We put $g^{(s)}_j = \left(G^{(s)}_{v^{(s)}_j} : 1\right)$.

THEOREM 5. *Assume that* $\mu_k=1$, $\mathfrak{f}_0 \geqq m$ *and* $s \geqq 2$.

(1) *If* $j \neq \frac{p^m-1}{p^m}(v^{(s-1)}_i - 1)$ *for all* $i = 0,1, \cdots, m\mathfrak{e}_{s-2}-1$, *then*

$$g^{(s)}_j / g^{(s)}_{j+1} = p^{\mathfrak{f}_{s-1}}.$$

(2) *If* $j = \frac{p^m-1}{p^m}(v^{(s-1)}_i - 1)$ *for some* i, *then*

$$g^{(s)}_j / g^{(s)}_{j+1} = (g^{(s-1)}_i / g^{(s-1)}_{i+1}) p^{\mathfrak{f}_{s-1}}.$$

PROOF. For each j, we have

$$\left(G^{(s)}_{v^{(s)}_j} : G^{(s)}_{v^{(s)}_{j+1}}\right) = \left(G^{(s-1)}_{\varphi_{s,s-1}(v_j)} : G^{(s-1)}_{\varphi_{s,s-1}(v_{j+1})}\right)\left(N^{(s)}_{v^{(s)}_j} : N^{(s)}_{v^{(s)}_{j+1}}\right).$$

Now, assume $\varphi_{s,s-1}(v_j^{(s)})=v_i^{(s-1)}$ for some i. Then we have $v_j^{(s)}=\psi_{s,s-1}(v_i^{(s-1)})=\psi_{s,s-1}(\bar{v}_j^{(s-1)})$, because $v_i^{(s-1)}=\bar{v}_i^{(s-1)}$ by Theorem 4. On the other hand, we have $v_j^{(s)}=\psi_{s,s-1}(\bar{l}_j^{(s)})$ where $\bar{l}_j^{(s)}$ is the upper ramification number of K_s/K_{s-1}. Therefore we obtain $v_i^{(s-1)}=\bar{l}_j^{(s)}$. Since $v_i^{(s-1)}\leqq\frac{\mathfrak{e}_{s-1}}{p-1}$, we have

$$j<\bar{v}_1 \text{ and } v_i^{(s-1)}=j+\left\{\frac{j+1}{p^m-1}\right\}.$$

We put $j=(p^m-1)q_j+r_j$, with $0\leqq r_j<p^m-1$. Then we have $v_i^{(s-1)}=p^m q_j+r_j+1$. From the theorem of Hasse-Arf we have

$$\bar{v}_{i+1}^{(s-1)}\equiv\bar{v}_i^{(s-1)} \mod \bar{g}_0^{(s-1)}/\bar{g}_i^{(s-1)},$$

where $\bar{g}_i^{(s-1)}=(N^{(s-1)}_{\bar{v}_j^{(s-1)}}:1)$, $N^{(s-1)}=\mathrm{Gal}(K_{s-1}/K_{s-2})$. Hence, from Theorem 3 and Theorem 4 we see $v_i^{(s-1)}\equiv 1 \mod p^{\mathfrak{f}_{s-2}}$. Therefore we have $p^m q_j+r_j\equiv 0 \mod p^{\mathfrak{f}_{s-2}}$. Because $r_j<p^m-1\leqq p^{\mathfrak{f}_{s-2}}-1$, we obtain $r_j=0$. Therefore we have $j=\frac{p^m-1}{p^m}(v_i^{(s-1)}-1)$. This completes the proof of our assertion.

Remark. In Theorem 5, the restriction such that $\mu_k=1$ and $\mathfrak{f}_0\geqq m$ is not essential. Similar results hold for more general case.

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