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TWO-SIDED QUINTIC SPLINE APPROXIMATIONS FOR TWO-POINT BOUNDARY VALUE PROBLEMS

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Two-sided quintic spline approximations for two-point boundary value problems are considered. A selection of numerical results is illustrated in Tables 1–3.

I. Introduction

Splines are of much use for approximating solutions of simple two-point boundary value problems for both linear and nonlinear ordinary differential equations. Recently we have considered the two-sided cubic spline approximations of second order ([7]). This paper discusses the two-sided quintic spline approximations of fourth order.

The two-point boundary value problems to be solved is

$$x''(t) = f(t, x(t), x'(t)) \qquad (0 \leq t \leq 1) \tag{1.1}$$

with boundary conditions

$$A_0x(0) - B_0x'(0) = a\,, \tag{1.2}$$

$$A_1x(1) + B_1x'(1) = b\,, \tag{1.3}$$

where $f(t, x, y)$ is defined and four times continuously differentiable in a region D of (t, x, y)-space intercepted by two planes $t=0$ and $t=1$.

Now making use of B-spline $Q_6(t)=(1/120)\sum(-1)^i\binom{6}{i}(t-i)_+^5$, we consider spline function $x_k(t)$ $(k=1, 2)$ of the form

$$x_1(t) = \sum \alpha_i Q_6(t/h-i) \qquad (nh=1)$$

and

$$x_2(t) = \sum \beta_i Q_6(t/h-i)$$

with undetermined coefficients α_i and β_i $(i=-5, -4, \cdots, n\text{-}1)$.
The above $x_k(t)$ $(k=1,2)$ will be the approximate solution to the problem (1. 1)–(1. 3) if it satisfies

$$x_k''(t) = P_k f(t, x_k(t), x_k'(t)) \qquad (0 \leq t \leq 1) \tag{1.4}$$

with boundary conditions

$$A_0x_k(0) - B_0x_k'(0) = a\,, \tag{1.5}$$

$$A_1x_k(1) + B_1x_k'(1) = b\,. \tag{1.6}$$

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Here the operator $P_k (k=1, 2)$ is defined as follows:

(i) $(P_1 g)(t)$ is the cubic spline function with the node $t_i (i=0, 1, \cdots, n)$ such that

$$(P_1 g)(t_i) = g(t_i) \qquad (t_i = ih,\ i = 0, 1, \cdots, n),$$
$$(P_1 g)'(t_i) = g'(t_i) \qquad (i = 0, n).$$

(ii) $(P_2 g)(t)$ is the cubic spline function with the node $t_i (i=0, 1, \cdots, n)$ such that

$$(P_2 g, L_i) = \begin{cases} h(g_{i-1}+10g_i+g_{i+1})/12 & (i = 1, 2, \cdots, n-1) \\ h(7g_0+3g_1)/20+h^2(3g_0'-2g_1')/60 & (i = 0) \\ h(7g_n+3g_{n-1})/20-h^2(3g_n'-2g_{n-1}')/60 & (i = n). \end{cases}$$

and $\qquad (P_2 g)'(t_i) = g'(t_i) \qquad (i = 0, n),$

where for any $\varphi(t)$ and $\psi(t) \in L^2[0, 1]$, let us denote

$$\int_0^1 \varphi(t)\,\psi(t)\,dt \quad \text{by} \quad (\varphi, \psi).$$

For $k=1$, it follows that Eqs. (1. 4)-(1. 6) are equivalent to the following system of $(n+5)$ equations:

$$F_{-2}(a) = A_0(a_{-5}+26a_{-4}+66a_{-3}+26a_{-2}+a_{-1})/120$$
$$\qquad -B_0(a_{-1}+10a_{-2}-10a_{-4}-a_{-5})/24h-a = 0,$$

$$F_{-1}(a)/h = (-a_{-5}+2a_{-4}-2a_{-2}+a_{-1})/2h^3$$
$$-f_1(0, (a_{-5}+26a_{-4}+66a_{-3}+26a_{-2}+a_{-1})/120, (-a_{-5}-10a_{-4}+10a_{-2}+a_{-1})/24h)$$
$$-f_2(0, \cdots)(a_{-1}+10a_{-2}-10a_{-4}-a_{-5})/24h$$
$$-f_3(0, \cdots)(a_{-1}+2a_{-2}-6a_{-3}+2a_{-4}+a_{-5})/6h^2 = 0,$$

$$F_i(a) = (a_{i-5}+2a_{i-4}-6a_{i-3}+2a_{i-2}+a_{i-1})/6h^2$$
$$-f(t_i, (a_{i-5}+26a_{i-4}+66a_{i-3}+26a_{i-2}+a_{i-1})/120,$$
$$\qquad (a_{i-1}+10a_{i-2}-10a_{i-4}-a_{i-5})/24h) = 0,$$

$$F_{n+1}(a)/h = (a_{n-1}-2a_{n-2}+2a_{n-4}-a_{n-5})/2h^3$$
$$-f_1(t_n, (a_{n-1}+26a_{n-2}+66a_{n-3}+26a_{n-4}+a_{n-5})/120,$$
$$\qquad (a_{n-1}+10a_{n-2}-10a_{n-4}-a_{n-5})/24h)$$
$$-f_2(t_n, \cdots)(a_{n-1}+10a_{n-2}-10a_{n-4}-a_{n-5})/24h$$
$$-f_3(t_n, \cdots)(a_{n-1}+2a_{n-2}-6a_{n-3}+2a_{n-4}+a_{n-5})/6h^2 = 0,$$

$$F_{n+2}(a) = A_1(a_{n-1}+26a_{n-2}+66a_{n-3}+26a_{n-4}+a_{n-5})/120$$
$$\qquad + B_1(a_{n-1}+10a_{n-2}-10a_{n-4}-a_{n-5})/24h-b = 0,$$

where $\qquad f_k(x_1, x_2, x_3) = \dfrac{\partial f(x_1, x_2, x_3)}{\partial x_k} \qquad (k = 1, 2 \text{ and } 3).$

For $k=2$, Eq. (1. 4) is equivalent to the following system of $(n+1)$ equations:

$$G_0(\beta) = (x_2(h)-x_2(0))/h-x_2'(0)-h(7f_0+3f_1)/20-h(3f_0'-2f_1')/60\,,$$

$$G_i(\beta) = (x_2(t_{i+1})-2x_2(t_i)+x_2(t_{i-1}))/h^2-(f_{i+1}+10f_i+f_{i-1})/12 \qquad (i=1,2,\cdots,n-1)\,,$$

$$G_n(\beta) = (x_2(1)-x_2(1-h))/h-x_2'(1)+h(7f_n+3f_{n-1})/20-h(3f_n'-2f_{n-1}')/60\,,$$

where $x_2(t) = \sum \beta_i Q_6(t/h-i)$, $f_i = f(t_i, x_2(t_i), x_2'(t_i))$ and

$$f_i' = f_1(t_i, x_2(t_i), x_2'(t_i)) + f_2(t_i, x_2(t_i), x_2'(t_i))\, x_2'(t_i) + f_3(t_i, x_2(t_i), x_2'(t_i))\, x_2''(t_i)\,.$$

In the present paper we assume that the problem (1.1)–(1.3) has the isolated solution $\hat{x}(t)$ satisfying the internalty condition

$$U = \{(t,x,y)\,|\,|x-\hat{x}(t)|+|y-\hat{x}'(t)|\le\delta,\, t\in[0,1]\}\subset D \qquad \text{for some } \delta>0\,.$$

The object of this paper is to show the following asymptotic expansion of the error function:

$$e_k(t)(=\hat{x}(t)-\bar{x}_k(t)) = d_k h^4\psi(t)+o(h^4) \quad (h\to 0) \qquad (k=1,2)\,,$$

where $d_1=1/720$, $d_2=-1/240$, $\bar{x}_k(t)$ $(k=1, 2)$ is the solution of the approximate problem (1.4)–(1.6) and $\psi(t)$ is the solution of the following variation equation of (1.1)–(1.3):

$$\psi''(t) = f_2(t,\hat{x}(t),\hat{x}'(t))\,\psi(t)+f_3(t,\hat{x}(t),\hat{x}'(t))\,\psi'(t)+\hat{x}^{(6)}(t) \qquad (0\le t\le 1)\,,$$

subject to the boundary conditions

$$A_0\psi(0)-B_0\psi'(0)=0\,,$$
$$A_1\psi(1)+B_1\psi'(1)=0\,.$$

2. Some Properties of Spline Functions

In what follows, for any continuous function $\varphi(t)$, we shall denote its maximum norm by $\|\varphi\|_\infty$, and for any finite dimensional vector c, we shall denote its maximum norm by $\|c\|_\infty$. For any square matrix A, we shall denote the norm induced by the maximum vector norm by $\|A\|_\infty$.

Lemma 1 ([1]). *Let $B=(b_{ij})$ be diagonally dominant, then*

$$\|B^{-1}\|_\infty \le \max\,[(|b_{ii}|-\sum_{j\ne i}|b_{ij}|)^{-1}]\,.$$

As an applciation of this Lemma, we shall prove the following Lemmas 2–9.

Lemma 2. *Let $g(t)\in C^4[0,1]$, then*

$$\|(I-P_k)\,g\|_\infty \le ch^4\,\|g^{(4)}\|_\infty \qquad (k=1,2)\,,$$

where I is the unit operator and c is a constant independent of h.

Proof. For $k=1$, see [1].

For $k=2$, let us rewrite $(I-P_2)g$ in the form:

$$(I-P_2)\,g = (g-P_1 g)+(P_1-P_2)\,g\,, \tag{2.1}$$

where
$$\|g-P_1g\|_\infty \le ch^4 \|g^{(4)}\|_\infty . \tag{2.2}$$
For the second part of (2. 1), we have
$$(P_1-P_2)\,g = \sum (\beta_i-\gamma_i)\, Q_4(t/h-i) \tag{2.3}$$
with
$$P_1 g = \sum \beta_i Q_4(t/h-i) \quad \text{and} \quad P_2 g = \sum \gamma_i Q_4(t/h-i) .$$
From the definition of the operator P_2, it follows that
$$(\sum \gamma_i Q_4(\cdot/h-i), L_k) = h(g_{k+1}+10g_k+g_{k-1})/12 \qquad (k=1,2,\cdots,n-1) .$$
Thus we have
$$(\gamma_k+26\gamma_{k-1}+66\gamma_{k-2}+26\gamma_{k-3}+\gamma_{k-4})/120$$
$$= (g_{k+1}+10g_k+g_{k-1})/12 \qquad (k=1,2,\cdots,n-1) . \tag{2.4}$$
Similarly we have
$$\begin{aligned}(\sum \gamma_i Q_4(\cdot/h-i), L_0)/h &= (4\gamma_{-3}+32\gamma_2+23\gamma_{-1}+\gamma_0)/60 \\ &= (7g_0+3g_1)/20+h(3g_0'-2g_1')/60 , \\ (\sum \gamma_i Q_4(\cdot/h-i), L_n)/h &= (4\gamma_{n-1}+32\gamma_{n-2}+23\gamma_{n-3}+\gamma_{n-4})/60 \\ &= (7g_n+3g_{n-1})/20+h(-3g_n'+2g_{n-1}')/60 .\end{aligned} \tag{2.5}$$
Since $\varphi(t)=(P_1\,g)(t)$ is a polynomial of degree 3 on $[t_i,\ t_{i+1}]$, $\varphi(t)$ is represented as follows:
$$\varphi(t) = \varphi_i L_i(t)+\varphi_{i+1} L_{i+1}(t)+\varphi_i'' T_i(t)+\varphi_{i+1}'' T_{i+1}(t),$$
where
$$T_i(t) = \begin{cases} (t-t_i)^2/2-(t-t_i)^3/6h-h(t-t_i)/3 & (t_i \le t \le t_{i+1}) \\ (t-t_{i-1})^3/6h-h(t-t_{i-1})/6 & (t_{i-1} \le t \le t_i) \\ 0 & (\text{otherwise}). \end{cases}$$
Thus we have
$$(\varphi, L_k)/h = (\varphi_{k+1}+4\varphi_k+\varphi_{k-1})/6-h^2(7\varphi_{k+1}''+16\varphi_k''+7\varphi_{k-1}'')/360$$
$$(k=1,2,\cdots,n-1) . \tag{2.7}$$
Similarly we have
$$(\varphi, L_0)/h = (7\varphi_0+3\varphi_1)/20+h(3\varphi_0'-2\varphi_1')/60 , \tag{2.8}$$
$$(\varphi, L_n)/h = (7\varphi_n+3\varphi_{n-1})/20+h(-3\varphi_n'+2\varphi_{n-1}')/60 . \tag{2.9}$$
Since $(P_2g)'(t_k)=g_k'=\varphi_k'$ $(k=0,\ n)$, we have two equations for $\theta_k(=\gamma_k-\beta_k)$:
$$\theta_{-3}-\theta_{-1}=0 \qquad \text{and} \qquad \theta_{n-1}-\theta_{n-3}=0 . \tag{2.10}$$
From (2. 4)–(2. 10), we have the following system of equations for $\theta_k(k=-2, -1, \cdots, n-2)$:
$$(\theta_k+26\theta_{k-1}+66\theta_{k-2}+26\theta_{k-3}+\theta_{k-4})/120$$
$$= -(g_{k+1}-2g_k+g_{k-1})/12+h^2(7\varphi_{k+1}''+16\varphi_k''+7\varphi_{k-1}'')/360$$
$$(k=2,3,\cdots,n-2) , \tag{2.11}$$

$$(32\theta_{-2}+27\theta_{-1}+\theta_0)/60 = -h(g_1'-\varphi_1')/30\,,$$
$$(32\theta_{n-2}+27\theta_{n-3}+\theta_{n-4})/60 = h(g_{n-1}'-\varphi_{n-1}')/30\,.$$

Since $|g_i^{(m)}-\varphi_i^{(m)}|\leq ch^{4-m}\|g^{(4)}\|_\infty$ ($i=0, 1, \cdots, n$ and $m=1, 2$), by Lemma 1 and (2. 10) we have

$$|\theta_i| \leq ch^4 \|g^{(4)}\|_\infty \qquad (i=0,1,\cdots,n)\,.$$

Thus we have

$$\|(P_1-P_2)\,g\|_\infty \leq \|\theta\|_\infty \leq ch^4 \|g^{(4)}\|_\infty\,. \tag{2.12}$$

Combining (2.2) and (2.12) yields the desired result.

LEMMA 3. *Let $g(t)\in C[0, 1]$, then we have*

$$\|(I-P_k)\,g\|_\infty \leq ch\,\|g'\|_\infty \qquad (k=1,2)\,.$$

PROOF. For $k=1$, we have

$$(\varphi_{i+1}'+4\varphi_i'+\varphi_{i-1}')/6 = (g_{i+1}-g_{i-1})/2h$$

with
$$\varphi(t) = (P_1 g)(t)\,.$$

Thus we have

$$\begin{aligned}&[(\varphi_{i+1}'-g_{i+1}')+4(\varphi_i'-g_i')+(\varphi_{i-1}'-g_{i-1}')]/6\\&\quad= (g_{i+1}-g_{i-1})/2h-(g_{i+1}'+4g_i'+g_{i-1}')/6\,.\end{aligned} \tag{2.13}$$

From the definition of the operator P_1, we have two additional equations:

$$\varphi_0'-g_0' = 0 \qquad \text{and} \qquad \varphi_n'-g_n' = 0\,. \tag{2.14}$$

Applying Lemma 1 to (2. 13) and (2. 14), we have the desired result. For $k=2$, let us rewrite $(I-P_2)g$ in the form:

$$(I-P_2)\,g = (g-P_1 g)+(P_1-P_2)\,g\,,$$

where
$$\|(I-P_1)\,g\|_\infty \leq ch\,\|g'\|_\infty\,. \tag{2.15}$$

In a similar manner analogous to Lemma 2, we have only to show:

$$\begin{aligned}((P_1-P_2)\,g, L_k) &= (P_1 g, L_k)-h(g_{k+1}+10g_k+g_{k-1})/12\\&= h(3\varphi_{k+1}+4\varphi_k+3\varphi_{k-1})/20-h^2(\varphi_{k+1}'-\varphi_{k-1}')/30-h(g_{k+1}+10g_k+g_{k-1})/12\\&= h(g_{k+1}-2g_k+g_{k-1})/15-h^2(\varphi_{k+1}'-\varphi_{k-1}')/30 \qquad (k=1,2,\cdots,n-1)\,.\end{aligned} \tag{2.16}$$

A simple calculation shows:

$$((P_1-P_2)\,g, L_0) = 0 \qquad \text{and} \qquad ((P_1-P_2)\,g, L_n) = 0\,. \tag{2.17}$$

Thus we have the desired result.

LEMMA 4 ([4]). *Let $g(t)\in C^6\,[0, 1]$, then there exists a quintic spline function of the form*

$$\varphi(t) = \sum a_i Q_6(t/h-i)$$

so that

(i) $\varphi(t_i)=g(t_i) \qquad (i=0, 1,\cdots, n)$

$\varphi^{(m)}(t_i)=g^{(m)}(t_i) \qquad (i=0, n \text{ and } m=1, 2)$

and

(ii) $\|\varphi^{(m)}-g^{(m)}\|_\infty=O(h^{6-m}) \qquad (m=0, 1, 2, 3) \qquad (h\to 0)$.

PROOF. By the use of consistency conditions, we shall prove this Lemma. In the case of the quntic spline, there are the following relationships between the first and second derivatives of the spline:

$$(\varphi'_{i+2}+26\varphi'_{i+1}+66\varphi'_i+26\varphi'_{i-1}+\varphi'_{i-2})/120$$
$$=(\varphi_{i+2}+10\varphi_{i+1}-10\varphi_{i-1}-\varphi_{i-2})/24h \qquad (i=2, 3,\cdots, n-2) \tag{2.18}$$

and

$$(\varphi''_{i+2}+26\varphi''_{i+1}+66\varphi''_i+26\varphi''_{i-1}+\varphi''_{i-2})/120$$
$$=(\varphi_{i+2}+2\varphi_{i+1}-6\varphi_i+2\varphi_{i-1}+\varphi_{i-2})/6h^2 \qquad (i=2, 3,\cdots, n-2)\,. \tag{2.19}$$

If ristricted on $[0, t_3]$, $\varphi(t)$ depends upon 8 parameters. Therefore 9 quantities φ'_0, φ_i and φ''_i, (i=0, 1, 2, 3) are not mutually independent, there is a unique linear relation:

$$37\varphi_0-54\varphi_1+9\varphi_2+8\varphi_3+12h\varphi'_0$$
$$=h^2(-23\varphi''_0+354\varphi''_1+201\varphi''_2+8\varphi''_3)/20 \tag{2.20}$$

Similarly we have

$$37\varphi_n-54\varphi_{n-1}+9\varphi_{n-2}+8\varphi_{n-3}-12h\varphi'_n$$
$$=h^2(-23\varphi''_n+354\varphi''_{n-1}+201\varphi''_{n-2}+8\varphi''_{n-3})/20\,. \tag{2.21}$$

Also we have

$$-235\varphi_0+65\varphi_1+155\varphi_2+15\varphi_3$$
$$=16h^2\varphi''_0+h(111\varphi'_0+227\varphi'_1+79\varphi'_2+3\varphi'_3) \tag{2.22}$$

and

$$235\varphi_n-65\varphi_{n-1}-155\varphi_{n-2}-15\varphi_{n-2}$$
$$=-16h^2\varphi''_n+h(111\varphi'_n+227\varphi'_{n-1}+79\varphi'_{n-2}+3\varphi'_{n-3})\,. \tag{2.23}$$

Since $\varphi_i^{(m)}-g_i^{(m)}=0$ (i=0, n and m=1, 2), by Taylor series expansion and Lemma 1 we have

$$|\varphi'_i-g'_i|=O(h^5) \qquad (i=0, 1,\cdots, n) \qquad \text{from (2.18), (2.22), and (2.23)} \tag{2.24}$$

and

$$|\varphi''_i-g''_i|=O(h^4) \qquad (i=0, 1,\cdots, n) \qquad \text{from (2.19), (2.20) and (2.21).} \tag{2.25}$$

Thus we have the desired result.

LEMMA 5. *Let* $\varphi(t)=\sum a_iQ_4(t/h-i)$, *then*

$$\|a\|_\infty \le c\|\varphi\|_\infty\,.$$

PROOF. As is readily seen, we have

$$3\alpha_{-1}+\alpha_0=6\varphi_1-8\varphi_{1/2}/3+(\varphi_0+\varphi_1)/3$$

$$(\alpha_{i-3}+4\alpha_{i-2}+\alpha_{i-1})/6=\varphi_i \qquad (i=2,3,\cdots,n-2)$$

$$3\alpha_{n-3}+\alpha_{n-4}=6\varphi_n-8\varphi_{n-1/2}/3+(\varphi_n+\varphi_{n-1})/3\,.$$

From Lemma 1, it follows that

$$|\alpha_i|\leq c\|\varphi\|_\infty \qquad (i=-1,0,\cdots,n-3)\,.$$

Since

$$\alpha_{i-3}=6\varphi_i-4\alpha_{i-2}-\alpha_{i-1} \qquad (i=1,0)$$

$$\alpha_{i-1}=6\varphi_i-4\alpha_{i-2}-\alpha_{i-3} \qquad (i=n-1,n)\,,$$

we have

$$\|\alpha\|_\infty\leq c\|\varphi\|_\infty\,.$$

LEMMA 6. *Let* $\varphi(t)=\sum a_i Q_4(t/h-i)$ *such that*

$$\varphi(t_i)=\eta_i \quad (i=0,1,\cdots,n), \qquad \varphi'(t_0)=\eta_{-1}/h \quad \text{and} \quad \varphi'(t_n)=\eta_{n+1}/h\,.$$

Then we have

$$\|\varphi\|_\infty\leq c\|\eta\|_\infty\,.$$

PROOF. By consistency relations, it follows that

$$(\varphi'_{i+1}+4\varphi'_i+\varphi'_{i-1})/6=(\varphi_{i+1}-\varphi_{i-1})/2h$$

$$=(\eta_{i+1}-\eta_{i-1})/2h \qquad (i=1,2,\cdots,n-1)\,,$$

$$\varphi'_0=\eta_{-1}/h \qquad \text{and} \qquad \varphi'_n=\eta_{n+1}/h\,.$$

According to Lemma 1, we have

$$|\varphi'_i|\leq c\|\eta\|_\infty/h \qquad (i=0,1,\cdots,n)\,. \tag{2.26}$$

Since $|\varphi_i|\leq\|\eta\|_\infty$, by (2. 26) we have

$$\|\varphi\|_\infty\leq c\,\|\eta\|_\infty\,.$$

The following Lemma 7 follows through the same arguments as in the proof of the latter part of Lemma 2.

LEMMA 7. *Let* $\varphi(t)$ *be cubic spline function with the node* t_i $(i=0,1,\cdots,n)$ *such that*

$$(\varphi,L_i)=h\eta_i \quad (i=0,1,\cdots,n)\,, \qquad \varphi'(t_0)=\eta_{-1}/h \quad \text{and} \quad \varphi'(t_n)=\eta_{n+1}/h\,.$$

Then we have

$$\|\varphi\|_\infty\leq c\|\eta\|_\infty\,.$$

LEMMA 8 ([3]). *Let* $g(t)\in C^4[0,1]$, *then*

$$(P_1g)''(t_i)=g''(t_i)-h^2g_i^{(4)}/12+o(h^2) \qquad (h\to 0)\,.$$

PROOF. Let us denote $\varphi(t)=(P_1g)(t)$, then we have

$$(\varphi''_{i+1}+4\varphi''_i+\varphi''_{i-1})/6=(\varphi_{i+1}-2\varphi_i+\varphi_{i-1})/h^2$$

$$=(g_{i+1}-2g_i+g_{i-1})/h^2 \qquad (i=1,2,\cdots,n-1)\,. \tag{2.27}$$

Thus we have

$$[(\varphi''_{i+1}-g''_{i+1})+4(\varphi''_i-g''_i)+(\varphi''_{i-1}-g''_{i-1})]/6$$
$$=-h^2g_i^{(4)}/12+o(h^2) \qquad (i=1,2,\cdots,n-1). \tag{2.28}$$

Since $\varphi(t)$ is cubic on $[t_0, t_1]$, we have

$$h(2\varphi''_0+\varphi''_1)/6=(\varphi_1-\varphi_0)/h-\varphi'_0,$$

from which follows

$$[2(\varphi''_0-g''_0)+(\varphi''_1-g''_1)]/3=-h^2g_0^{(4)}/12+o(h^2). \tag{2.29}$$

Similarly we have

$$[2(\varphi''_n-g''_n)+(\varphi''_{n-1}-g''_{n-1})]/3=-h^2g_n^{(4)}/12+o(h^2). \tag{2.30}$$

From (2.28)–(2.30), we have the desired result.

LEMMA 9. *Let $g(t)$ be continously differentiable on* $[0, 1]$, *then we have*

$$\|P_1g\|_\infty \le c(\|g\|_\infty+h\|g'\|_\infty).$$

PROOF. Let $(P_1g)(t)=\sum \alpha_iQ_4(t/h-i)$, then

$$\|P_1g\|_\infty \le \|\alpha\|_\infty.$$

It follows from the definition of operator P_1 that

$$(\alpha_{-1}-\alpha_{-3})/2h=g'_0,$$
$$(\alpha_{i-1}+4\alpha_{i-2}+\alpha_{i-3})/6=g_i \qquad (i=0,1,\cdots,n),$$
$$(\alpha_{n-1}-\alpha_{n-3})/2h=g'_n.$$

Hence, from Lemma 1, we have

$$\|\alpha\|_\infty \le c(\|g\|_\infty+h\|g'\|_\infty).$$

Thus we have the desired result.

3. Existence and Convergence of Spline Approximations

In this section, using Kantorovich's theorem on Newton's method, we shall prove the solvability of the determining equations $F(\alpha)=(F_{-2}(\alpha), F_{-1}(\alpha), \cdots, F_{n+1}(\alpha), F_{n+2}(\alpha))=0$ and $G(\beta)=(F_{-2}(\beta), F_{-1}(\beta), G_0(\beta), \cdots, G_n(\beta), F_{n+1}(\beta), F_{n+2}(\beta))=0$. Corresponding to $\hat{x}(t)$, one can determine uniquely a qunitic spline function $\hat{x}_h(t)$, of the form

$$\hat{x}_h(t)=\sum \hat{\alpha}_iQ_6(t/h-i)$$

so that

$$\hat{x}_h(t_i)=\hat{x}(t_i) \qquad (i=0,1,\cdots,n)$$
$$\hat{x}_h^{(m)}(t_i)=\hat{x}^{(m)}(t_i) \qquad (i=0, n \text{ and } m=1,2).$$

Since $\hat{x}(t)\in C^6[0,1]$ due to the assumption $f(t, x, y)\in C^4(D)$, it is valid that

$$\|\hat{x}_h^{(m)}-\hat{x}^{(m)}\|_\infty=O(h^{6-m}) \qquad (m=0,1,2,3) \qquad (h\to 0).$$

For simplicity, we shall consider the special case when $f(t, x, y)$ is independent of y. Let $J_1(\alpha)$ be the Jacobian matrix with respect to α_i ($i=-5, -4, \cdots, n-1$). In order to investigate the properties of $J_1(\hat{\alpha})$, let us consider a linear system

$$J_1(\hat{\alpha})\xi = \eta\,, \tag{3.1}$$

where $\xi = (\xi_{-5}, \xi_{-4}, \cdots, \xi_{n-1})$ and $\eta = (\eta_{-2}, \eta_{-1}, \cdots, \eta_{n+2})$.

Corresponding to ξ and η, we consider quintic and cubic splines $y_1(t)$ and $y_2(t)$ defined by

$$y_1(t) = \sum \xi_i Q_6(t/h-i)$$

and

$$y_2(t_i) = \eta_i \qquad (i = 0, 1, \cdots, n)$$

$$y_2'(t_0) = \eta_{-1}/h, \quad y_2'(t_n) = \eta_{n+1}/h\,.$$

From (3.1), we have

$$\begin{aligned} y_1^{(3)}(t_0) &= [f_{21}(t_0, \hat{x}_h(t_0)) + f_{22}(t_0, \hat{x}_h(t_0))\,\hat{x}_h'(t_0)]\, y_1(t_0) \\ &\quad + f_2(t_0, \hat{x}_h(t_0))\, y_1'(t_0) + y_2'(t_0)\,, \\ y_1''(t_i) &= f_2(t_i, \hat{x}_h(t_i))\, y_1(t_i) + y_2(t_i) \qquad (i = 0, 1, \cdots, n)\,, \\ y_1^{(3)}(t_n) &= [f_{21}(t_n, \hat{x}_h(t_n)) + f_{22}(t_n, \hat{x}_h(t_n))\,\hat{x}_h'(t_n)]\, y_1(t_n) \\ &\quad + f_2(t_n, \hat{x}_h(t_n))\, y_1'(t_n) + y_2'(t_n)\,, \\ A_0 y_1(0) &- B_0\, y_1'(0) = \eta_{-2}\,, \\ A_1 y_1(1) &+ B_1\, y_1'(1) = \eta_{n+2}\,. \end{aligned}$$

Since $\hat{x}_h(t_i) = \hat{x}(t_i)$ $(i = 0, 1, \cdots, n)$, $\hat{x}(t_i) = \hat{x}'(t_i)$ $(i = 0, n)$,

thus we have

$$y_1''(t) = P_1[f_2(t, \hat{x}(t))\, y_1(t)] + y_2(t) \qquad (0 \le t \le 1)\,. \tag{3.2}$$

Equation (3.2) can be rewritten as follows:

$$y_1'' - f_2(t, \hat{x})\, y_1 = y_2 + R\,,$$

where $R=-[I-P_1](f_2\, y_1)$.

Since $\hat{x}(t)$ is isolated, there exists the Green function $H(t, s)$ such that

$$y_1 = \int_0^1 H(\cdot, s)\,[y_2(s) + R(s)]\, ds + \varphi\,, \tag{3.3}$$

where $\varphi(t)$ is the solution of the following equation:

$$\varphi''(t) - f_2(t, \hat{x}(t))\, \varphi(t) = 0 \qquad (0 \le t \le 1)$$

subject to the boundary conditions

$$A_0\varphi(0) - B_0\varphi'(0) = \eta_{-2}\,,$$

$$A_1\varphi(1) + B_1\varphi'(1) = \eta_{n+2}\,.$$

A simple calculation shows that

$$\|\varphi\|_\infty \leq c_1\|(\eta_{-2}, \eta_{n+2})\| \,.$$

Throughout this section, c_i ($i=1,2, \cdots, 15$) will denote the constant independent of h. From (3.3) we have the inequality of the form:

$$\|y_1\|_\infty \leq c_2(\|y_2\|_\infty + \|R\|_\infty) + c_3\|\varphi\|_\infty \,.$$

Application of Lemma 3 yields

$$\|R\|_\infty \leq c_4 h(\|y_1'\|_\infty + \|y_1\|_\infty) \,.$$

By virute of the well-known inequality:

$$\|y_1'\|_\infty \leq c_5\|y_1\|_\infty + c_6\|y_1''\|_\infty \qquad \text{for some } c_5 \text{ and } c_6$$

we have

$$\begin{aligned}\|R\|_\infty &\leq c_7 h(\|y_1\|_\infty + \|y_1''\|_\infty)\\ &\leq c_8 h(\|y_1\|_\infty + \|y_2\|_\infty + \|R\|_\infty) \,.\end{aligned}$$

Hence we have the estimate of $\|R\|_\infty$ of the form

$$\|R\|_\infty \leq c_9 h(\|y_1\|_\infty + \|y_2\|_\infty) \qquad \text{for sufficiently small } h.$$

Thus we have

$$\|y_1\|_\infty \leq c_{10}(\|y_2\|_\infty + \|(\eta_{-2}, \eta_{n+2})\|) \,.$$

Since $\|y_1\|_\infty \geq c_{11}\|\xi\|_\infty$ and $\|y_2\|_\infty \leq c_{12}\|\eta\|_\infty$ we finally have the inequality of the form

$$\|\xi\|_\infty \leq c_{13}\|\eta\|_\infty \qquad \text{for any} \quad h \leq h_0 \,. \tag{3.4}$$

provided h_0 is sufficiently small.
By (3. 1), inequality (3. 4) implies the non-singularity of $J_1(\hat{\alpha})$ and in addition the inequality

$$\|J_1^{-1}(\hat{\alpha})\|_\infty \leq c_{14}(=c_{13}) \qquad \text{for any} \quad h \leq h_0 \,.$$

Let $\alpha = (\alpha_{-5}, \alpha_{-4}, \cdots, \alpha_{n-1})$ and $\beta = (\beta_{-5}, \beta_{-4}, \cdots, \beta_{n-1})$ be arbitrary vectors such that

$$\begin{aligned}&|(\alpha_{i-5}+26\alpha_{i-4}+66\alpha_{i-3}+26\alpha_{i-2}+\alpha_{i-1})/120-\mathfrak{x}(t_i)| \leq \delta \,,\\ &|(\beta_{i-5}+26\beta_{i-4}+66\beta_{i-3}+26\beta_{i-2}+\beta_{i-1})/120-\mathfrak{x}(t_i)| \leq \delta \,,\\ &(i=0, 1, \cdots, n) \,.\end{aligned}$$

Hence by the means of the mean value theorem we have

$$\|J_1(\alpha)-J_1(\beta)\|_\infty \leq c_{15}\|\alpha-\beta\|_\infty \tag{3.5}$$

By Lemma 2, we have the equality of the form

$$\|F(\hat{\alpha})\|_\infty = O(h^4) \,. \tag{3.6}$$

The expressions (3.4)–(3.6) show that all the conditions of Kantorovich's theorem on Newton's method are fulfilled ([5]). Therefore the determining equation $F(\alpha)=0$ has the solution $\bar{\alpha}$ such that

$$\|\bar{\alpha}-\hat{\alpha}\|_\infty = O(h^4)\,.$$

Thus we have

$$\|\bar{x}_1-\hat{x}\|_\infty \leq \|\bar{x}_1-\hat{x}_h\|_\infty+\|\hat{x}_h-\hat{x}\|_\infty = O(h^4) \qquad (h\to 0)\,,$$

where $\bar{x}_1(t) = \sum \bar{\alpha}_i Q_6(t/h-i)\,.$

For the second derivative, we have

$$\bar{x}_1''-\hat{x}'' = P_1 f(t, \bar{x}_1)-f(t, \hat{x})$$
$$= P_1[f(t, \bar{x}_1)-f(t, \hat{x})]-(I-P_1)f(t, \hat{x})\,.$$

By the means of Lemma 9 we have

$$\|\bar{x}_1''-\hat{x}''\|_\infty = O(h^4)\,.$$

For the first derivative, it follows that

$$\|\bar{x}_1'-\hat{x}'\|_\infty \leq c_5\|\bar{x}_1-\hat{x}\|_\infty+c_6\|\bar{x}_1''-\hat{x}''\|_\infty = O(h^4) \qquad (h\to 0)\,.$$

Therefore we have

$$\|\bar{x}_1^{(m)}-\hat{x}^{(m)}\|_\infty = O(h^4) \qquad (m=0, 1, 2) \qquad (h\to 0)\,.$$

In the general case when $f(t, x, y)$ contains the component y, we can show the same result by reducing the equations (1.1)–(1.3) into the system of first order differential equations ([6,8]).

Thus we have the following

THEOREM 1. *In the sufficiently small neighbourhood of the isolated solution* $\hat{x}(t)$ *of the problem (1. 1)–(1. 3), there exists a quintic spline function* $\bar{x}_1(t)$ *of the form:*

$$\bar{x}_1(t) = \sum \bar{\alpha}_i Q_6(t/h-i)$$

such that

$$\|\bar{x}_1^{(m)}-\hat{x}^{(m)}\|_\infty = O(h^4) \qquad (m=0,1,2) \qquad (h\to 0)\,.$$

In analogy with Theorem 1, we now have Theorem 2.

THEOREM 2. *Let the hypotheses of Theorem 1 hold. There exists a quinitic spline function* $\bar{x}_2(t)$ *of the form*

$$\bar{x}_2(t) = \sum \bar{\beta}_i Q_6(t/h-i)$$

so that

(i) *the coefficient* $\bar{\beta}=(\bar{\beta}_{-5}, \bar{\beta}_{-4}, \cdots, \bar{\beta}_{n-1})$ *satisfies the determining equation* $G(\beta)=0$,

(ii) $\|\bar{x}_2^{(m)}-\hat{x}^{(m)}\|_\infty = O(h^4) \qquad (m=0, 1, 2) \qquad (h\to 0)\,.$

PROOF. For the operator P_2, let $G(\beta)=(F_{-2}(\beta), F_{-1}(\beta), G_0(\beta), \cdots, G_n(\beta), F_{n+1}(\beta), F_{n+2}(\beta))$ and $J_2(\beta)$ be the Jacobian matrix with respect to β_i ($i=-5, -4, \cdots, n-1$). For simplicity, we shall consider the special case when $f(t, x, y)=f(t, x, 0)$. In a similar manner analogous to Theorem 1, let us consider a linear system:

$$J_2(\hat{\alpha})\xi = \eta\,. \tag{3.7}$$

Corresponding to ξ and η, we consider quintic and cubic spline functions $z_1(t)$ and $z_2(t)$ defined by

$$z_1(t) = \sum \xi_i Q_6(t/h-i)$$

and

$$(z_2, L_i) = h\eta_i \qquad (i = 0, 1, \cdots, n)$$

$$z(t_0) = \eta_{-1}/h\,, \qquad z(t_n) = \eta_{n+1}/h\,.$$

From (3.7), we have

$$(z_1'', L_i) = (P_2(f_2 z_1), L_i) + (z_2, L_i) \qquad (i = 0, 1, \cdots, n)\,, \tag{3.8}$$

$$z_1^{(3)}(t_i) = (f_2 z_1)'(t_i) + z_2'(t_i) \qquad (i = 0, n)\,. \tag{3.9}$$

Applying Lemma 7 to the equations (3.8) and (3.9) we have

$$z_1'' = P_2(f_2 z_1) + z_2 \qquad (0 \le t \le 1)\,.$$

Thus in a similar way as in Theorem 1, we have the desired result.

4. Asymptotic Expansion of Error Function $E_k(t)$ (k=1, 2)

By the means of the results of the previous sections, we shall prove in this section the asymptotic expansion of the error function $e_k(t) = \hat{x}(t) - \bar{x}_k(t)$ $(k=1, 2)$.

THEOREM 3. *With the hypotheses of Theorem 1, then we have the asymptotic expansion*:

$$e_k(t) = d_k h^4 \psi(t) + o(h^4) \qquad (h \to 0) \qquad (k = 1, 2)\,.$$

PROOF. From Theorem 1, we have

$$e_k'' = f_2(t, \hat{x}, \hat{x}')\, e_k + f_3(t, \hat{x}, \hat{x}')\, e_k' + (I - P_k)\, f(t, \hat{x}, \hat{x}') + O(h^5)\,,$$
$$(k = 1, 2) \qquad (h \to 0)\,, \tag{4.1}$$

$$A_0 e_k(0) - B_0 e_k'(0) = 0\,, \tag{4.2}$$

$$A_1 e_k(1) + B_1 e_k'(1) = 0\,. \tag{4.3}$$

Since $\hat{x}(t)$ is isolated, equations (4.1)–(4.3) can be rewritten in the form:

$$e_k(t) = \int_0^1 H(t, s)(I - P_k)\, g(s)\, ds + O(h^5) \qquad (k = 1, 2) \qquad (h \to 0)\,.$$

with $\quad g(t) = \hat{x}''(t)$. For $k = 1$, we have

$$(I - P_1)\, g(t) = g(t) - g_3(t) + g_3(t) - (P_1 g)(t) \qquad \text{for} \quad t \in [t_i, t_{i+1}]\,,$$

where $\quad g_3(t) = g_i L_i(t) + g_{i+1} L_{i+1}(t) + g_i'' T_i(t) + g_{i+1}'' T_{i+1}(t)\,.$

According to Lemma 8, we have

$$g_3(t) - (P_1 g)(t) = [g_i'' - (P_1 g)_i'']\, T_i(t) + [g_{i+1}'' - (P_1 g)_{i+1}'']\, T_{i+1}(t)$$
$$= (1/12)\, h^2 (g_i^{(4)} T_i(t) + g_{i+1}^{(4)} T_{i+1}(t)) + o(h^4)\,,$$

from which follows, using the second mean value theorem on the definite integral,

$$\int H(t,s)\,[g_3(s)-(P_1g)(s)]\,ds$$

$$= (h^2/12)\Big[g_i^{(4)}\int H(t,s)\,T_i(s)\,ds + g_{i+1}^{(4)}\int H(t,s)\,T_{i+1}(s)\,ds\Big] + o(h^5)$$

$$= -(1/288)\,h^5[g_i^{(4)}H(t,\xi_i)+g_{i+1}^{(4)}H(t,\eta_i)]+o(h^5)$$

for some ξ_i and $\eta_i \in [t_i, t_{i+1}]$.

Thus we have

$$\int_0^1 H(t,s)\,[g_3(s)-P_1g(s)]\,ds = -(1/144)\,h^4\int_0^1 H(t,s)\,g^{(4)}(s)\,ds+o(h^4) \tag{4.4}$$

Futhermore, a simple calculation shows:

$$g(t)-g_3(t) = [\{(t-t_i)^4-h^3(t-t_i)\}/24-\{h(t-t_i)^3-h^3(t-t_i)\}/12]\,g_i^{(4)}+o(h^4)$$

for any $t \in [t_i, t_{i+1}]$.

Thus it follows that

$$\int_0^1 H(t,s)\,(g(s)-g_3(s))\,ds = (h^4/120)\int_0^1 H(t,s)\,g^{(4)}(s)\,ds+o(h^4) \tag{4.5}$$

From (4.4) and (4.5), we have the following asymptotic expansion:

$$e_1(t) = \int_0^1 H(t,s)\,(I-P_1)\,g(s)\,ds = (h^4/720)\int_0^1 H(t,s)\,g^{(4)}(s)\,ds+o(h^4)\,. \tag{4.6}$$

On the other hand, we have

$$e_2(t) = \int_0^1 H(t,s)(I-P_2)\,g(s)\,ds = \sum I_i\,,$$

where, for each $i=0, 1, \cdots, n-1$,

$$I_i = \int H(t,s)(I-P_2)\,g(s)\,ds$$

$$= \int [H(t,t_i)\,L_i(s)+H(t,t_{i+1})\,L_{i+1}(s)]\,(I-P_2)\,g(s)\,ds+o(h^6)$$

$$= H(t,t_i)\int L_i(s)(I-P_2)\,g(s)\,ds+H(t,t_{i+1})\int L_{i+1}(s)(I-P_2)\,g(s)\,ds+o(h^6)\,.$$

Thus we have

$$e_2(t) = H(t,t_0)\int L_0(s)(I-P_2)\,g(s)\,ds$$

$$+\sum_{i=0}^{n-1} H(t,t_i)\int L_i(s)(I-P_2)\,g(s)\,ds+H(t,t_n)\int L_n(s)(I-P_2)\,g(s)\,ds+o(h^5)$$

$$= H(t, t_0)\left[\int L_0(s)\, g(s)\, ds - h(7g_0+3g_1)/20 - h^2(3g_0'-2g_1')/60)\right]$$

$$+ \sum H(t, t_i)\left[\int L_i(s)\, g(s)\, ds - h(g_{i+1}+10g_i+g_{i-1})/12\right]$$

$$+ H(t, t_n)\left[\int L_n(s)\, g(s)\, ds - h(7g_n+3g_{n-1})/20 + h^2(3g_n'-2g_{n-1}')/60\right].$$

By Taylor expansion, we have

$$e_2(t) = -(h^4/240)\int_0^1 H(t, s)\, g^{(4)}(s)\, ds + o(h^4). \tag{4,7}$$

Thus we have the desired result.

As an immediate consequence of Theorem 3, we have the

COROLLARY. *With the hypotheses of Theorem 1, then we have*

$$(3\bar{x}_1(t)+\bar{x}_2(t))/4 = \hat{x}(t)+o(h^4) \qquad (h\to 0).$$

Finally it should be remarked that for the type of the first derivatives absent the approximate problem ($k=2$) (1.4)–(1.6) is identical with the well-known differnce scheme as the Numerov formula. Thus the collocation method using quintic spline function gives the opposite approximation to the solution of the problem (1.1)–(1.3) as compared with the Numerov difference method.

5. Numerical Examples

In this section, we discuss numerical results obtained from some concrete examples. These numerical results conform the theoretical accuracies established in previous sections. In the case of examples 1 and 2, the approximate problems ($k=2$) (1.4)–(1.6) are identical with the Numerov difference schemes. We now consider the numerical solutions of particular examples (1.1)–(1.3).

Example 1 ([1]). As our first example, we consider the linear problem:

$$x'' = 100x \qquad (0\le t\le 1)$$

$$x(0) = x(1) = 1.$$

The unique solution is $x(t)=cosh\ (10t-5)/\cosh 5$.

Table 1.1 ($e_1(t)$)

t	$h=1/20$	$h=1/40$	$h=1/80$
0.1	−1.5388(−5)	−9.8931(−7)	−6.2338(−8)
0.2	−1.1333(−5)	−7.3228(−7)	−4.6184(−8)
0.3	−6.4387(−6)	−4.1708(−7)	−2.6314(−8)
0.4	−3.6378(−6)	−2.3598(−7)	−1.4891(−8)
0.5	−2.7793(−6)	−1.8040(−7)	−1.1385(−8)

1.5 (−5)=1.5×10^{-5}.

Table 1.2 ($e_2(t)$)

t	$h=1/20$	$h=1/40$	$h=1/80$
0.1	4.7568(−5)	2.9940(−6)	1.8747(−7)
0.2	3.5245(−5)	2.2185(−6)	1.3891(−7)
0.3	2.0081(−5)	1.2641(−6)	7.9151(−8)
0.4	1.1363(−5)	7.1537(−7)	4.4793(−8)
0.5	8.6870(−6)	5.4691(−7)	3.4245(−8)

For this simple problem we have

$$(3\bar{x}_1(0.5)+\bar{x}_2(0.5))/4 = \hat{x}(0.5)-2.25(-11) \qquad \text{for } h = 1/80\,.$$

Example 2. Let us consider the nonlinear differential equation:

$$x'' = 1.5x^2 \qquad (0 \le t \le 1)$$

$$x(0) = 4\,, \qquad x(1) = 1\,.$$

This problem has two isolated solutions such that

$$\hat{x}(t) = 4/(t+1)^2 \qquad \text{and} \qquad \hat{x}(0.5) \doteqdot -10.53\,.$$

Table 2.1 ($e_1(t)$ for $\hat{x}(0.5)=16/9$)

t	$h=1/20$	$h=1/40$	$h=1/80$
0.1	−1.0389(−6)	−6.5916(−8)	−4.1385(− 9)
0.2	−1.3221(−6)	−8.4604(−8)	−5.3121(− 9)
0.3	−1.3207(−6)	−8.3876(−8)	−5.2656(− 9)
0.4	−1.1856(−6)	−7.5272(−8)	−4.7248(− 9)
0.5	−1.0034(−6)	−6.3684(−8)	−3.9970(− 9)
0.6	−8.0610(−7)	−5.1147(−8)	−3.2098(− 9)
0.7	−6.0576(−7)	−3.8427(−8)	−2.4113(− 9)
0.8	−4.0552(−7)	−2.5719(−8)	−1.6138(− 9)
0.9	−2.0447(−7)	−1.2967(−8)	−8.1361(−10)

Table 2.2 ($e_2(t)$ for $\hat{x}(0.5)=16/9$)

t	$h=1/20$	$h=1/40$	$h=1/80$
0.1	3.1666(−6)	1.9871(−7)	1.2432(−8)
0.2	4.0657(−6)	2.5507(−7)	1.5957(−8)
0.3	4.0309(−6)	2.5284(−7)	1.5817(−8)
0.4	3.6174(−6)	2.2687(−7)	1.4192(−8)
0.5	3.0605(−6)	1.9192(−7)	1.2005(−8)
0.6	2.4580(−6)	1.5412(−7)	9.6405(−9)
0.7	1.8466(−6)	1.1578(−7)	7.2422(−9)
0.8	1.2360(−6)	7.7489(−8)	4.8469(−9)
0.9	6.2313(−7)	3.9066(−8)	2.4435(−9)

As in the previous exmaple, we have

$$(3\bar{x}_1(0.5)+\bar{x}_2(0.5))/4-\hat{x}(0.5) = -3.5(-12) \ \text{for } h = 1/80\,.$$

Table 2.3 ($\bar{x}_1(t)$ for $\hat{x}(0.5) \fallingdotseq -10.53$)

t	$h=1/20$	$h=1/40$	$h=1/80$
0.1	0.47888 2153	0.47890 7800	0.47890 9356
0.2	−3.00817 147	−3.00811 801	−3.00811 477
0.3	−6.33106 629	−6.33100 275	−6.33099 890
0.4	−9.03861 971	−9.03857 472	−9.03857 195
0.5	−10.53624 51	−10.53622 75	−10.53622 63
0.6	−10.41020 88	−10.41019 40	−10.41019 30
0.7	−8.69761 980	−8.69758 515	−8.69758 301
0.8	−5.86094 255	−5.86089 822	−5.86089 552
0.9	−2.49164 475	−2.49161 604	−2.49161 429

Tєble 2.4 ($\bar{x}_2(t)$ for $\hat{x}(0.5) \fallingdotseq -10.53$)

t	$h=1/20$	$h=1/40$	$h=1/80$
0.1	0.47898 8923	0.47891 4396	0.47890 9796
0.2	−3.00794 894	−3.00810 426	−3.00811 391
0.3	−6.33080 168	−6.33098 639	−6.33099 788
0.4	−9.03843 017	−9.03856 294	−9.03857 122
0.5	−10.53616 63	−10.53622 24	−10.53622 60
0.6	−10.41014 19	−10.41018 97	−10.41019 27
0.7	−8.69747 345	−8.69757 604	−8.69758 244
0.8	−5.86075 761	−5.86088 678	−5.86089 481
0.9	−2049152 480	−2.49160 861	−2.49161 382

Example 3 ([2]). As our final exmaple, we consider the following nonlinear differential equation:

$$x'' = x^3-(1+\cos t)^3-\cos t\,,$$

$$x'(0) = 0\,, \qquad x'(1) = -x^3(1)\sin 1/(1+\cos 1)^3\,.$$

The unique solution is $\hat{x}(t)=1+\cos t$.

Table 3.1 ($e_1(t)$)

t	$h=1/8$	$h=1/16$	$h=1/32$
0	2.7463(−8)	1.7128(−9)	1.0700(−10)
1/8	2.7378(−8)	1.7076(−9)	1.0067(−10)
1/4	2.7110(−8)	1.6912(−9)	1.0565(−10)
3/8	2.6615(−8)	1.6607(−9)	1.0375(−10)
1/2	2.5822(−8)	1.6119(−9)	1.0071(−10)
5/8	2.4626(−8)	1.5384(−9)	9.6138(−11)
3/4	2.2909(−8)	1.4326(−9)	8.9551(−11)
7/8	2.0504(−8)	1.2857(−9)	8.0405(−11)
1	1.7350(−8)	1.0890(−9)	6.8148(−11)

Table 3.2 ($e_2(t)$)

t	$h=1/8$	$h=1/16$	$h=1/32$
0	−5.6945(−8)	−4.3398(−9)	−2.9604(−10)
1/8	−6.5089(−8)	−4.5891(−9)	−3.0336(−10)
1/4	−6.9507(−8)	−4.7043(−9)	−3.0544(−10)
3/8	−7.1104(−8)	−4.7102(−9)	−3.0280(−10)
1/2	−7.0293(−8)	−4.6113(−9)	−2.9523(−10)
5/8	−6.7118(−8)	−4.4060(−9)	−2.8192(−10)
3/4	−6.1347(−8)	−4.0695(−9)	−2.6156(−10)
7/8	−5.2553(−8)	−3.5778(−9)	−2.3253(−10)
1	−4.0197(−8)	−2.9010(−9)	−1.9307(−10)

For this example, we have

$$(3\bar{x}_1(0.5)+\bar{x}_2(0.5))/4-\hat{x}(0.5) = -1.7(-12) \qquad \text{for } h=1/32\,.$$

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