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CUBIC SPLINE INTERPOLATION AND CHOPPING PROCEDURE FOR TWO-POINT BOUNDARY VALUE PROBLEMS

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Chopping procedure for two-point boundary value problems is considered. A selection of numerical results is illustrated in Tables 1–4.

1. Introduction and Description of Method

Russell and Christiansen in [1] have treated various adaptive mesh selection strategies for two-point boundary value problems. One of the major methods is the numerical integration using chopping procedure. In the present paper we describe this procedure based on cubic spline interpolation and its asymptotic expansion. The problems to be solved is

$$x'' = f(t, x, x') \qquad (0 \leq t \leq 1) \tag{1}$$

$$a_0 x(0) - b_0 x'(0) = c_0 \tag{2}$$

$$a_1 x(1) + b_1 x'(1) = c_1 . \tag{3}$$

By using B-spline $Q_4(t)$, let us consider a cubic spline function $x_k(t)$ of the form:

$$x_k(t) = \sum \alpha_i Q_4(t/h - i) \qquad (nh = 1)$$

with undertermined α_i ($i=-3, -2, \cdots, n-1$).
The above $x_k(t)$ will be an approximate solution if it satisfies

$$x_k'' = P_k f(t, x_k, x_k') \qquad (0 \leq t \leq 1) \tag{4}$$

$$a_0 x_k(0) - b_0 x_k'(0) = c_0 \tag{5}$$

$$a_1 x_k(1) + b_1 x_k'(1) = c_1 . \tag{6}$$

Here the operator P_k ($k=1, 2$) is defined as follows:

(i) $$(P_1 f)(t) = \sum f_i L_i(t)$$

with the piecewise linear function $L_i(t)$ such that

$$L_i(t_j) = L_i(jh) = \delta_{ij} ,$$

(ii) $$(P_2 f)(t) = \sum \beta_i L_i(t)$$

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such that the coefficient β_i ($i=0, 1, \cdots, n$) is determined by

$$(2\beta_0+\beta_1)/6 = (2f(t_0)+f(t_1))/6$$
$$(\beta_{i+1}+4\beta_i+\beta_{i-1})/6 = f(t_i) \qquad (i=1, 2, \cdots, n-1)$$
$$(2\beta_n+\beta_{n-1})/6 = (2f(t_n)+f(t_{n-1}))/6 .$$

For the approximate problem (4)–(6), we have

THEOREM ([5]). In a sufficiently small neighbourhood of the isolated solution $\hat{x}(t)$, there exists the approximate solution $x_k(t)$ ($k=1, 2$) such that

$$\hat{x}(t)-x_k(t) = (-1)^k h^2 \psi(t)/12+O(h^3)$$
$$= (-1)^k(x_1(t)-x_2(t))/2+O(h^3) \qquad (h \to 0) .$$

If $x_k(t)$ satisfies the inequalty:

$$|x_1(t)-x_2(t)| \le 2\varepsilon \quad (\varepsilon \text{ is a desired tolerance})$$

for $t \in [0, a]$, $[b, 1](a = n_1h, b = n_2h)$, let us consider the following approximate problem on the remaining interval [a, b]:

$$x_k'' = P_k f(t, x_k, x_k') \qquad (a \le t \le b;\ h: = h/2) \tag{7}$$
$$x_k(a): = (x_1(a)+x_2(a))/2 \tag{8}$$
$$x_k(b): = (x_1(b)+x_2(b))/2 , \tag{9}$$

where if $a=0$, the boundary condition (8) is replaced by (5) and if $b=1$, the boundary condition (9) is replaced by (6). If the following homogeneous problem

$$\varphi'' = f_2(t, \hat{x}, \hat{x}')\,\varphi + f_3(t, \hat{x}, \hat{x}')\,\varphi' \qquad (a \le t \le b)$$
$$\varphi(a) = \varphi(b) = 0$$

has only the trivial solution $\varphi=0$, then we have the similar result to Theorem. The successive use of this procedure gives the approximate solution $x_k(t)$ such that

(i) $$x_1(t) \lesseqgtr \hat{x}(t) \lesseqgtr x_2(t)$$

(ii) $$|\hat{x}(t)-x_k(t)| \le \varepsilon .$$

Now we consider the application of the stated method by the sample equations in [1]–[4].

2. Numerical Illustration

Example 1 ([1]).

$$\sigma x''-(2-t^2)\,x = -1 \qquad (\sigma = 10^{-8})$$
$$x(0) = 0.5, \qquad x(1) = 0 .$$

Exact solution: $\hat{x}(t) \fallingdotseq 1/(2-t^2)-\exp(-10^4(1-t))-\exp(-10^4(1+t))$.

Table 1

N	Remaining interval ($\varepsilon=10^{-4}$)	N	Remaining interval ($\varepsilon=10^{-6}$)
39	[0.825, 1]	39	[0.75, 1]
13	[0.9125, 1]	19	[0.875, 1]
13	[0.95625, 1]	19	[0.9375, 1]
13	[0.978125, 1]	19	[0.965625, 1]
13	[0.9890625, 1]	21	[0.9828125, 1]
13	[0.9953125, 1]	21	[0.99140625, 1]
11	[0.998046875, 1]	21	[0.997265625, 1]
9	[0.9990234375, 1]	13	[0.9984375, 1]
9	[0.9991210938, 1]	15	[0.9986328125, 1]
17	[0.9993164063, 1]	27	[0.998828125, 1]
27	[0.9995117188, 1]	47	[0.9989746094, 1]
39	[0.9997070313, 0999987793]	83	[0.9991455078, 1]
45	—	139	[0.9993041992, 1]
		227	[0.9994689941, 1]
		347	[0.9996520996, 0.999985559]
		441	—

(N is the number of interior points per subinterval.)

Example 2 ([3]).

$$x'' = 400\,(x+\cos^2 \pi t)+2\pi^2 \cos 2\pi t$$

$$x(0) = x(1) = 0\,.$$

Table 2

N	Remaining interval ($\varepsilon=10^{-4}$)
39	[0, 0.5]
41	[0, 0.25]
41	[0, 0.1375]
45	—

Example 3 ([4]).

$$x'' = k \sinh (kx) \qquad (k = 10)$$

$$x(0) = 0\,, \qquad x(1) = 1\,.$$

Table 3

N	Remaining interval ($\varepsilon=10^{-4}$)
399	[0.4375, 1]
449	[0.82, 1]
287	[0.93625, 1]
203	[0.9984375, 1]
9	[0.99953125, 1]
5	—

Example 4 ([2]).

$$x'' + (3\cot t + 2\tan t)\,x' + 0.7x = 0$$

$$x(30°) = 0\,, \quad x(60°) = 5\,.$$

Since this solution curve has a sharp spike approximately at 30.65°, we have computed $x_k(t)$ such that

$$|\hat{x}(t) - x_k(t)| \leq \varepsilon \hat{x}(t)\,.$$

Table 4

N	Remaining interval ($\varepsilon = 10^{-4}$)
79	[0, 0.5375]
85	[0, 0.01875], [0.01875, 0.24375]
5, 69	[0, 0.01875], [0.01875, 0.025]
11, 3	[0, 0.01875], [0.01875, 0.025]
23, 7	[0, 0.01875], [0.01875, 0.02265625]
47, 9	[0, 0.01875], [0.01875, 0.01921875]
95, 5	——

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