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A POWER OF THE BIVARIATE ANALOGUE OF THE TWO-SIDED t TEST

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Abstract

We consider the power function of the bivariate analogue of the two-sided t test and compare the power of this test with that of the ordinary test based on F distribution. The numerical results are given in Tables 1-4.

1. Introduction and summary

In the previous paper the authors [1] proposed a test criterion based on an analogy to that proposed by Kudô and Fujisawa [3], where a bivariate normal test with a two-sided alternative was discussed, and we furnished a table of the percentage points to facilitate the significance test. In this paper we are concerned with the power function of this test and also compare the power of this test with that of the ordinary test based on F distribution. For the sake of convenience we shall state the problem and the results given in detail in [1].

Suppose (X, Y) has a bivariate normal distribution with an unknown mean vector (θ_1, θ_2) and an unknown covariance matrix which is factored as a product of an unknown scalar σ^2 and a known matrix Λ, namely, $\Lambda=\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. And there is an available independent estimate S^2 of σ^2 whose distribution is proportional to a χ^2 distribution, namely, the distribution of nS^2/σ^2 is the χ^2 distribution with n degrees of feedom. We consider the problem of testing the null hypothesis H_0: $\theta_1=\theta_2=0$ against the alternative that the mean vector is either in the first or third quadrant, namely, H_1: ($\theta_1\geqq 0$ and $\theta_2\geqq 0$) or ($\theta_1\leqq 0$ and $\theta_2\leqq 0$), where at least one of the inequalities are strict in both cases.

Let us define

$$t_1=x/S, \quad t_2=y/S, \quad t_3=\sqrt{\frac{n}{\sigma^2}}S \qquad (-\infty<t_1,\ t_2<\infty,\ 0<t_3<\infty) \tag{1}$$

and

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$$\bar{F}^2 = \frac{t_1^2 - 2\rho t_1 t_2 + t_2^2}{1-\rho^2} \qquad (2)$$
$$= u_1^2 + t_2^2$$
$$= t_1^2 + u_2^2$$

where $$u_1 = \frac{t_1 - \rho t_2}{\sqrt{1-\rho^2}} \quad \text{and} \quad u_2 = \frac{t_2 - \rho t_1}{\sqrt{1-\rho^2}},$$

which give the rejection region

(i) $\bar{F}^2 \geqq \bar{F}_0^2$ if $t_1 \geqq 0,\ t_2 \geqq 0$

(ii) $u_1 \leqq -\bar{F}_0$ if $t_1 < 0,\ t_2 > 0,\ |t_1| \geqq |t_2|$

(iii) $u_2 \geqq \bar{F}_0$ if $t_1 < 0,\ t_2 > 0,\ |t_1| < |t_2|$

(iv) $\bar{F}^2 \geqq \bar{F}_0^2$ if $t_1 \leqq 0,\ t_2 \leqq 0$

(v) $u_2 \leqq -\bar{F}_0$ if $t_1 > 0,\ t_2 < 0,\ |t_1| \leqq |t_2|$

(vi) $u_1 \geqq \bar{F}_0$ if $t_1 > 0,\ t_2 < 0,\ |t_1| > |t_2|$

where $\bar{F}_0$ is a constant depending on ρ, n and the significance level α.

The constant $\bar{F}_0$ is determined, keeping the property of symmetry in mind, by the relation

$$\alpha = 2\{Pr(\bar{F}^2 \geqq \bar{F}_0, t_1 \geqq 0, t_2 \geqq 0) + Pr(-u_1 \geqq \bar{F}_0, t_1 < 0, t_2 > 0, |t_1| \geqq |t_2|)$$
$$+ Pr(u_2 \geqq \bar{F}_0, t_1 < 0, t_2 > 0, |t_1| < |t_2|)\}$$
$$= \frac{\cos^{-1}(-\rho)}{\pi}\left\{1 + I_{\alpha'}\left(\frac{n}{2}, 1\right)\right\} + I_{\alpha'}\left(\frac{n}{2}, \frac{1}{2}\right)$$
$$- \frac{2}{\pi}\int_0^{\cos^{-1}(-\rho)} I_{\alpha''}\left(\frac{n}{2}, 1\right) d\theta - 2\int_0^{\bar{F}_0}\int_0^{\bar{F}_0} g(u_1, u_2 : \rho)\, du_1\, du_2 \qquad (3)$$

where $I_x(\alpha, \beta)$ is an incomplete beta function ratio, namely,

$$I_x(\alpha, \beta) = \frac{1}{B(\alpha, \beta)}\int_0^x x^{\alpha-1}(1-x)^{\beta-1}\, dx,$$

and

$$g(u_1, u_2 : \rho) = \frac{1}{n\pi\sqrt{1-\rho^2}}\left\{1 + \frac{u_1^2 - 2\rho u_1 u_2 + u_2^2}{n(1-\rho^2)}\right\}^{-(n+2)/2} \cdot \frac{1}{B\left(\frac{n}{2}, 1\right)},$$

$$\alpha' = \frac{1}{1 + \frac{\bar{F}_0^2}{n}}, \qquad \alpha'' = \frac{1}{1 + \frac{\bar{F}_0^2}{n \sin^2\theta}}$$

In the previous paper [1] for $\alpha = 0.05$ and 0.01 we presented the values of $\bar{F}_0$ for $\rho = -0.95\ (0.05)\ 0.95$ and some values of n.

2. The power function

Let (X, Y) be distributed as a bivariate normal distribution with an unknown mean vector (θ_1, θ_2) and an unknown covariance matrix $\sigma^2 \Lambda$.
Therefore the density of (X, Y) is

$$f(x,y)=\frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2\sigma^2(1-\rho^2)}\{(x-\theta_1)^2-2\rho(x-\theta_1)(y-\theta_2)+(y-\theta_2)^2\}\right]$$

$$=\frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}}\exp\left[-\frac{\Delta^2}{2}\right]\exp\left[\frac{1}{\sigma^2(1-\rho^2)}\{x\theta_1-\rho(x\theta_2+y\theta_1)+y\theta_2\}\right]$$

$$\times\exp\left[-\frac{1}{2\sigma^2(1-\rho^2)}(x^2-2\rho xy+y^2)\right] \tag{4}$$

where

$$\Delta^2=\frac{\theta_1^2-2\rho\theta_1\theta_2+\theta_2^2}{\sigma^2(1-\rho^2)} \tag{5}$$

Suppose there is an available another statistic S^2, which is an unbiased estimate of σ^2 statistically independent of (X, Y), and the distribution of nS^2/σ^2 is the χ^2 distribution with n degrees of freedom. As the statistic (X, Y) and S^2 are mutually independent, the joint density of X, Y, S is

$$g(x,y,S)=\frac{2n^{n/2}}{\pi\sqrt{1-\rho^2}\,\Gamma\left(\frac{n}{2}\right)}\left(\frac{1}{2\sigma^2}\right)^{(n+2)/2}S^{n-1}\exp\left[-\frac{\Delta^2}{2}\right]$$

$$\times\exp\left[\frac{1}{\sigma^2(1-\rho^2)}\{x\theta_1-\rho(x\theta_2+y\theta_1)+y\theta_2\}\right]$$

$$\times\exp\left[-\frac{1}{2\sigma^2}\left\{nS^2+\frac{x^2-2\rho xy+y^2}{1-\rho^2}\right\}\right] \tag{6}$$

Making the transformation

$$t_1=x/S\,,\qquad t_2=y/S\,,\qquad t_3=\sqrt{\frac{n}{\sigma^2}}\,S\qquad(-\infty<t_1,\ t_2<\infty,\ 0<t_3<\infty) \tag{7}$$

whose Jacobian is

$$\frac{\partial(x,y,S)}{\partial(t_1,t_2,t_3)}=S^2\left(\frac{n}{\sigma^2}\right)^{-1/2}, \tag{8}$$

we have the joint density of t_1, t_2, t_3

$$g(t_1,t_2,t_3)=\frac{\sqrt{2}}{n\pi\Gamma\left(\frac{n}{2}\right)\sqrt{1-\rho^2}}\exp\left[\frac{\Delta^2}{2}\right]\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\left[\frac{\sqrt{2}}{\sqrt{n}\,\sigma(1-\rho^2)}\{t_1\theta_1\right.$$

$$\left.-\rho(t_1\theta_2+t_2\theta_1)+t_2\theta_2\}\right]^{\nu}\left(\frac{t_3^2}{2}\right)^{-(n+\nu+1)/2}\exp\left[-\frac{t_3^2}{2}\left\{1+\frac{t_1^2-2\rho t_1t_2+t_2^2}{n(1-\rho^2)}\right\}\right] \tag{9}$$

The marginal density of t_1, t_2 is obtained by integrating in t_3,

$$g(t_1, t_2) = \frac{1}{n\pi\Gamma\left(\frac{n}{2}\right)\sqrt{1-\rho^2}} \exp\left[-\frac{\Delta^2}{2}\right] \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \left[\frac{\sqrt{2}}{\sqrt{n}\,\sigma(1-\rho^2)} \{t_1\theta_1 - \rho(t_1\theta_2+t_2\theta_1)+t_2\theta_2\}\right]^{\nu} \left\{1+\frac{t_1^2-2\rho t_1t_2+t_2^2}{n(1-\rho^2)}\right\}^{-(n+\nu+2)/2} \Gamma\left(\frac{n+\nu+2}{2}\right)$$

$$= \frac{1}{n\pi\Gamma\left(\frac{n}{2}\right)\sqrt{1-\rho^2}} \exp\left[-\frac{\Delta^2}{2}\right] \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \left[\sqrt{\frac{2}{n}} \left\{\frac{t_1\mu_1-\rho(t_1\mu_2+t_2\mu_1)+t_2\mu_2}{1-\rho^2}\right\}\right]^{\nu}$$

$$\times \left\{1+\frac{t_1^2-2\rho t_1t_2+t_2^2}{n(1-\rho^2)}\right\}^{-(n+\nu+2)/2} \Gamma\left(\frac{n+\nu+2}{2}\right) \tag{10}$$

where $\theta_1/\sigma = \mu_1$, $\theta_2/\sigma = \mu_2$ and $\Delta^2 = (\mu_1^2-2\rho\mu_1\mu_2+\mu_2^2)/(1-\rho^2)$.

Let P (μ_1, μ_2) be a power function of the bivariate analogue of the two-sided t test. The probability of rejecting the null hypothesis, when the population mean vector is (μ_1, μ_2), is found to be

$$P(\mu_1, \mu_2) = 1-(P_1+P_2+P_3+P_4+P_5+P_6) \tag{11}$$

where

$$P_1 = Pr(\bar{F}^2<\bar{F}_0,\ t_1\geqq 0,\ t_2\geqq 0)\,,$$

$$P_2 = Pr(\bar{F}^2<\bar{F}_0,\ t_1\leqq 0,\ t_2\leqq 0)\,,$$

$$P_3 = Pr(u_1>-\bar{F}_0,\ t_1<0,\ t_2>0,\ |t_1|\geqq|t_2|)\,,$$

$$P_4 = Pr(u_1<\bar{F}_0,\ t_1>0,\ t_2<0,\ |t_1|>|t_2|)\,,$$

$$P_5 = Pr(u_2<\bar{F}_0,\ t_1<0,\ t_2>0,\ |t_1|\leqq|t_2|)$$

and

$$P_6 = Pr(u_2>-\bar{F}_0,\ t_1>0,\ t_2<0,\ |t_1|>|t_2|)\,.$$

In order to evaluate the sum of P_1 and P_2 we make the transformation

$$u_1 = \frac{t_1-\rho t_2}{\sqrt{1-\rho^2}}\,, \qquad t_2 = t_2 \tag{12}$$

in (10) and we have

$$P_1+P_2 = Pr(u_1^2+t_2^2<\bar{F}_0^2,\ \sqrt{1-\rho^2}\,u_1+\rho t_2\geqq 0,\ t_2\geqq 0) + Pr(u_1^2+t_2^2<\bar{F}_0^2,\ \sqrt{1-\rho^2}\,u_1+\rho t_2\leqq 0,\ t_2\leqq 0)$$

$$= \iint_{D_1} \{g_t(u_1, t_2)+g_t(-u_1, -t_2)\}\, du_1\, dt_2 \tag{13}$$

where

$$D_1:\ u_1^2+t_2^2<\bar{F}_0^2,\quad \sqrt{1-\rho^2}\,u_1+\rho t_2\geqq 0,\ t_2\geqq 0 \tag{14}$$

and

$$g_t(u_1,t_2)=\frac{1}{n\pi\Gamma\left(\frac{n}{2}\right)}\exp\left[-\frac{\Delta^2}{2}\right]\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\left[\sqrt{\frac{2}{1-\rho^2}}\left\{\frac{\mu_1}{\sqrt{n}}(\mu_1-\rho\mu_2)\right.\right.$$

$$\left.\left.+\frac{t_2}{\sqrt{n}}\sqrt{1-\rho^2}\,\mu_2\right\}\right]^{\nu}\left(1+\frac{u_1^2}{n}+\frac{t_2^2}{n}\right)^{-(n+\nu+2)/2}\Gamma\left(\frac{n+\nu+2}{2}\right). \tag{15}$$

This can be further calculated by the property of the gamma function,

$$\Gamma\left(\frac{n}{2}+\nu+1\right)=\frac{\Gamma\left(\frac{n}{2}\right)\Gamma(\nu+1)}{B\left(\frac{n}{2},\nu+1\right)}=\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{B\left(\frac{n}{2},\nu+1\right)B\left(\nu+\frac{1}{2},\frac{1}{2}\right)}, \tag{16}$$

and we have

$$P_1+P_2=\frac{2}{n\pi\Gamma\left(\frac{n}{2}\right)}\exp\left[-\frac{\Delta^2}{2}\right]\sum_{\nu=0}^{\infty}\frac{2^{\nu}}{(2\nu)!}\Gamma\left(\frac{n}{2}+\nu+1\right)$$

$$\times\iint_{D_1}\left\{\frac{u_1}{\sqrt{n}}\left(\frac{\mu_1-\rho\mu_2}{\sqrt{1-\rho^2}}\right)+\frac{t_2\mu_2}{\sqrt{n}}\right\}^{2\nu}\left(1+\frac{u_1^2}{n}+\frac{t_2^2}{n}\right)^{-(n+2\nu+2)/2}du_1\,dt_2$$

$$=\frac{2}{n\sqrt{\pi}}\exp\left[-\frac{\Delta^2}{2}\right]\sum_{\nu=0}^{\infty}\frac{2^{\nu}}{(2\nu)!}\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{B\left(\frac{n}{2},\nu+1\right)B\left(\nu+\frac{1}{2},\frac{1}{2}\right)}$$

$$\times\iint_{D_1}\left\{\frac{u_1}{\sqrt{n}}\left(\frac{\mu_1-\rho\mu_2}{\sqrt{1-\rho^2}}\right)+\frac{t_2\mu_2}{\sqrt{n}}\right\}^{2\nu}\left(1+\frac{u_1^2}{n}+\frac{t_2^2}{n}\right)^{-(n+2\nu+2)/2}du_1\,dt_2. \tag{17}$$

Using the same transformation (12) and the property (16), the sum of P_3 and P_4 can be also evaluated to be

$$P_3+P_4=Pr(u_1>-\bar{F}_0,\ \sqrt{1-\rho^2}\,u_1+(1+\rho)t_2\leqq 0,\ t_2>0)$$

$$+Pr(u_1<\bar{F}_0,\ \sqrt{1-\rho^2}\,u_1+(1+\rho)t_2>0,\ t_2<0)$$

$$=\iint_{D_2}\{g_t(-u_1,-t_2)+g_t(u_1,t_2)\}\,du_1\,dt_2$$

$$=\frac{2}{n\sqrt{\pi}}\exp\left[-\frac{\Delta^2}{2}\right]\sum_{\nu=0}^{\infty}\frac{2^{\nu}}{(2\nu)!}\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{B\left(\frac{n}{2},\nu+1\right)B\left(\nu+\frac{1}{2},\frac{1}{2}\right)}$$

$$\times\iint_{D_2}\left\{\frac{u_1}{\sqrt{n}}\left(\frac{\mu_1-\rho\mu_2}{\sqrt{1-\rho^2}}\right)+\frac{t_2\mu_2}{\sqrt{n}}\right\}^{2\nu}\left(1+\frac{u_1^2}{n}+\frac{t_2^2}{n}\right)^{-(n+2\nu+2)/2}du_1\,dt_2. \tag{18}$$

where

$$D_2:\ u_1<\bar{F}_0,\ \sqrt{1-\rho^2}\,u_1+(1+\rho)t_2>0,\ t_2<0. \tag{19}$$

And similarly we have

$$P_5+P_6=Pr(u_2<\bar{F}_0,\ t_1<0,\ \sqrt{1-\rho^2}\,u_2+(1+\rho)t_1\geqq 0)$$
$$+Pr(u_2>-\bar{F}_0, t_1>0,\ \sqrt{1-\rho^2}\,u_2+(1+\rho)\,t_1<0)$$
$$=\iint_{D_3}\{g_t(u_2,t_1)+g_t(-u_2,-t_1)\}\,du_2\,dt_1$$
$$=\frac{2}{n\sqrt{\pi}}\exp\left[-\frac{\Delta^2}{2}\right]\sum_{\nu=0}^{\infty}\frac{2^\nu}{(2\nu)!}\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{B\left(\frac{n}{2},\nu+1\right)B\left(\nu+\frac{1}{2},\frac{1}{2}\right)}$$
$$\times\iint_{D_3}\left\{\frac{u_2}{\sqrt{n}}\left(\frac{\mu_1-\rho\mu_2}{\sqrt{1-\rho^2}}\right)+\frac{t_1\mu_2}{\sqrt{n}}\right\}^{2\nu}\left(1+\frac{u_2^2}{n}+\frac{t_1^2}{n}\right)^{-(n+2\nu+2)/2}du_2\,dt_1 \tag{20}$$

where

$$D_3:\ u_2<\bar{F}_0,\quad t_1<0,\quad \sqrt{1-\rho^2}\,u_2+(1+\rho)\,t_1\geqq 0\,. \tag{21}$$

Therefore the power function is obtained by inserting (17), (18) and (20) in (10) and substituting the appropriate $100\alpha\%$ significance point for $\bar{F}_0$.

The power is computed for two configurations of the μ's. The first is $\mu_1=\mu_2=\mu$ for which $\Delta^2=2\mu^2/(1+\rho)$. In the second case we took $\mu_1=\mu$ and $\mu_2=0$ for which $\Delta^2=\mu^2/(1\text{-}\rho^2)$.

In the first case we can find that by transforming to polar coordinates $u_1/\sqrt{n}=r\cos\theta$, $t_2/\sqrt{n}=r\sin\theta$ and integrating with respect to r and θ, we obtain

$$P_1+P_2=\frac{2}{n\sqrt{\pi}}\exp\left[-\frac{\mu^2}{1+\rho}\right]\sum_{\nu=0}^{\infty}\frac{2^\nu}{(2\nu)!}\frac{\Gamma\left(\nu+\frac{1}{2}\right)n}{B\left(\frac{n}{2},\nu+1\right)B\left(\nu+\frac{1}{2},\frac{1}{2}\right)}\left(\frac{2}{1+\rho}\right)^\nu$$
$$\times\int_0^\alpha\left\{\cos\left(\theta-\frac{\alpha}{2}\right)\right\}^{2\nu}d\theta\int_0^{\bar{F}_0/\sqrt{n}}r^{2\nu+1}(1+r^2)^{-(n/2+\nu+1)}\,dr$$
$$=\frac{1}{\sqrt{\pi}}\exp\left[-\frac{\mu^2}{1+\rho}\right]\sum_{\nu=0}^{\infty}\frac{\left(\frac{4\mu^2}{1+\rho}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)}{(2\nu)!}$$
$$\times\left\{1-\frac{B_{\alpha'}\left(\frac{n}{2},\nu+1\right)}{B\left(\frac{n}{2},\nu+1\right)}\right\}\left\{1-\frac{B_\beta\left(\nu+\frac{1}{2},\frac{1}{2}\right)}{B\left(\nu+\frac{1}{2},\frac{1}{2}\right)}\right\} \tag{22}$$

where

$$\alpha' = \frac{1}{1+\frac{F_0^2}{n}}, \qquad \beta = \frac{1-\rho}{2}$$

and

$$P_3+P_4 = P_5+P_6$$

$$= \frac{2}{\sqrt{\pi}} \exp\left[-\frac{\mu^2}{1+\rho}\right] \sum_{\nu=0}^{\infty} \frac{(2\mu^2)^\nu}{(2\nu)!} \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{B\left(\frac{n}{2}, \nu+1\right) B\left(\nu+\frac{1}{2}, \frac{1}{2}\right)} I_\nu \tag{23}$$

where

$$I_\nu = \iint_{D_2'} \left\{u_1\left(\frac{1-\rho}{\sqrt{1-\rho^2}}\right)+t_2\right\}^{2\nu} (1+u_1^2+t_2^2)\, du_1\, dt_2,$$

$$D_2': \ u_1 \leqq F_0/\sqrt{n}, \quad \sqrt{1-\rho^2}\, u_1+(1+\rho)t_2>0, \ t_2<0.$$

Therefore the power function, which is denoted by $P(\mu_1=\mu_2=\mu)$, is found as follows:

$$P(\mu_1=\mu_2=\mu) = 1 - \frac{1}{\sqrt{\pi}} \exp\left[-\frac{\mu^2}{1+\rho}\right] \sum_{\nu=0}^{\infty} \frac{\left(\frac{4\mu^2}{1+\rho}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right)}{(2\nu)!}$$

$$\times \left\{1 - \frac{B_{\alpha'}\left(\frac{n}{2}, \nu+1\right)}{B\left(\frac{n}{2}, \nu+1\right)}\right\} \left\{1 - \frac{B_\beta\left(\nu+\frac{1}{2}, \frac{1}{2}\right)}{B\left(\nu+\frac{1}{2}, \frac{1}{2}\right)}\right\}$$

$$- \frac{4}{\sqrt{\pi}} \exp\left[-\frac{\mu^2}{1+\rho}\right] \sum_{\nu=0}^{\infty} \frac{(2\mu^2)^\nu}{(2\nu)!} \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{B\left(\frac{n}{2}, \nu+1\right) B\left(\nu+\frac{1}{2}, \frac{1}{2}\right)} I_\nu$$

$$= 1 - \exp\left[-\frac{\mu^2}{1+\rho}\right] \left[\left\{1 - \frac{B_{\alpha'}\left(\frac{n}{2}, 1\right)}{B\left(\frac{n}{2}, 1\right)}\right\} \left\{1 - \frac{B_\beta\left(\frac{1}{2}, \frac{1}{2}\right)}{B\left(\frac{1}{2}, \frac{1}{2}\right)}\right\}\right.$$

$$+ \frac{\mu^2}{1+\rho} \left\{1 - \frac{B_{\alpha'}\left(\frac{n}{2}, 2\right)}{B\left(\frac{n}{2}, 2\right)}\right\} \left\{1 - \frac{B_\beta\left(\frac{3}{2}, \frac{1}{2}\right)}{B\left(\frac{3}{2}, \frac{1}{2}\right)}\right\}$$

$$+ \frac{\mu^4}{2(1+\rho)^2} \left\{1 - \frac{B_{\alpha'}\left(\frac{n}{2}, 3\right)}{B\left(\frac{n}{2}, 3\right)}\right\} \left\{1 - \frac{B_\beta\left(\frac{5}{2}, \frac{1}{2}\right)}{B\left(\frac{5}{2}, \frac{1}{2}\right)}\right\}$$

$$+\frac{\mu^6}{6(1+\rho)^3}\left\{1-\frac{B_{\alpha'}\left(\frac{n}{2}, 4\right)}{B\left(\frac{n}{2}, 4\right)}\right\}\left\{1-\frac{B_\beta\left(\frac{7}{2}, \frac{1}{2}\right)}{B\left(\frac{7}{2}, \frac{1}{2}\right)}\right\}$$

$$+\frac{\mu^8}{24(1+\rho)^4}\left\{1-\frac{B_{\alpha'}\left(\frac{n}{2}, 5\right)}{B\left(\frac{n}{2}, 5\right)}\right\}\left\{1-\frac{B_\beta\left(\frac{9}{2}, \frac{1}{2}\right)}{B\left(\frac{9}{2}, \frac{1}{2}\right)}\right\}$$

$$+\frac{\mu^{10}}{120(1+\rho)^5}\left\{1-\frac{B_{\alpha'}\left(\frac{n}{2}, 6\right)}{B\left(\frac{n}{2}, 6\right)}\right\}\left\{1-\frac{B_\beta\left(\frac{11}{2}, \frac{1}{2}\right)}{B\left(\frac{11}{2}, \frac{1}{2}\right)}\right\}+\cdots$$

$$+\frac{4}{B\left(\frac{n}{2}, 1\right)B\left(\frac{1}{2}, \frac{1}{2}\right)}I_0+\frac{2\mu^2}{B\left(\frac{n}{2}, 2\right)B\left(\frac{3}{2}, \frac{1}{2}\right)}I_1$$

$$+\frac{\mu^4}{2B\left(\frac{n}{2}, 3\right)B\left(\frac{5}{2}, \frac{1}{2}\right)}I_2+\frac{\mu^6}{12B\left(\frac{n}{2}, 4\right)B\left(\frac{7}{2}, \frac{1}{2}\right)}I_3$$

$$\left.+\frac{\mu^8}{98B\left(\frac{n}{2}, 5\right)B\left(\frac{9}{2}, \frac{1}{2}\right)}I_4+\frac{\mu^{10}}{980B\left(\frac{n}{2}, 6\right)B\left(\frac{11}{2}, \frac{1}{2}\right)}I_5+\cdots\right] \tag{24}$$

where

$$\alpha'=\frac{1}{1+\frac{\bar{F}_0^2}{n}}, \quad \beta=\frac{1-\rho}{2}, \quad I_\nu=\iint_{D_2'}\left\{u_1\left(\frac{1-\rho}{\sqrt{1-\rho^2}}\right)+t_2\right\}^{2\nu}(1+u_1^2+t_2^2)\,du_1\,dt_2$$

$$(\nu=0, 1, 2, \cdots)$$

and

$$D_2': u_1\leqq\bar{F}_0/\sqrt{n}, \quad \sqrt{1-\rho^2}\,u_1+(1+\rho)\,t_2>0, \ t_2<0\,.$$

In the second case the power function, which is denoted by $P(\mu_1=\mu, \mu_2=0)$, can be obtained as follows:

$$P(\mu_1=\mu, \mu_2=0)=1-\frac{1}{2\sqrt{\pi}}\exp\left[-\frac{\mu^2}{2(1-\rho^2)}\right]\sum_{\nu=0}^{\infty}\frac{\left(\frac{2\mu^2}{1-\rho^2}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right)}{(2\nu)!}$$

$$\times\left\{1-\frac{B_{\alpha'}\left(\frac{n}{2}, \nu+1\right)}{B\left(\frac{n}{2}, \nu+1\right)}\right\}\left\{1-\frac{B_{\beta'}\left(\nu+\frac{1}{2}, \frac{1}{2}\right)}{B\left(\nu+\frac{1}{2}, \frac{1}{2}\right)}\right\}$$

$$-\frac{2}{\sqrt{\pi}}\exp\left[-\frac{\mu^2}{2(1-\rho^2)}\right]\sum_{\nu=0}^{\infty}\frac{\left(\frac{2\mu^2}{2-\rho^2}\right)^{\nu}+\left(\frac{2\rho^2\mu^2}{1-\rho^2}\right)^{\nu}}{(2\nu)!}$$

$$\times\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{B\left(\frac{n}{2},\nu+1\right)B\left(\nu+\frac{1}{2},\frac{1}{2}\right)}I'_{\nu}$$

$$=1-\frac{1}{2}\exp\left[-\frac{\mu^2}{2(1-\rho^2)}\right]\left[\left\{1-\frac{B_{\alpha'}\left(\frac{n}{2},1\right)}{B\left(\frac{n}{2},1\right)}\right\}\left\{1-\frac{B_{\beta'}\left(\frac{1}{2},\frac{1}{2}\right)}{B\left(\frac{1}{2},\frac{1}{2}\right)}\right\}\right.$$

$$+\frac{\mu^2}{2(1-\rho^2)}\left\{1-\frac{B_{\alpha'}\left(\frac{n}{2},2\right)}{B\left(\frac{n}{2},2\right)}\right\}\left\{1-\frac{B_{\beta'}\left(\frac{3}{2},\frac{1}{2}\right)}{B\left(\frac{3}{2},\frac{1}{2}\right)}\right\}$$

$$+\frac{\mu^4}{8(1-\rho^2)^2}\left\{1-\frac{B_{\alpha'}\left(\frac{n}{2},3\right)}{B\left(\frac{n}{2},3\right)}\right\}\left\{1-\frac{B_{\beta'}\left(\frac{5}{2},\frac{1}{2}\right)}{B\left(\frac{5}{2},\frac{1}{2}\right)}\right\}$$

$$+\frac{\mu^6}{48(1-\rho^2)^3}\left\{1-\frac{B_{\alpha'}\left(\frac{n}{2},4\right)}{B\left(\frac{n}{2},4\right)}\right\}\left\{1-\frac{B_{\beta'}\left(\frac{7}{2},\frac{1}{2}\right)}{B\left(\frac{7}{2},\frac{1}{2}\right)}\right\}$$

$$+\frac{\mu^8}{384(1-\rho^2)^4}\left\{1-\frac{B_{\alpha'}\left(\frac{n}{2},5\right)}{B\left(\frac{n}{2},5\right)}\right\}\left\{1-\frac{B_{\beta'}\left(\frac{9}{2},\frac{1}{2}\right)}{B\left(\frac{9}{2},\frac{1}{2}\right)}\right\}$$

$$\left.+\frac{\mu^{10}}{3840(1-\rho^2)^5}\left\{1-\frac{B_{\alpha'}\left(\frac{n}{2},6\right)}{B\left(\frac{n}{2},6\right)}\right\}\left\{1-\frac{B_{\beta'}\left(\frac{11}{2},\frac{1}{2}\right)}{B\left(\frac{11}{2},\frac{1}{2}\right)}\right\}+\cdots\right]$$

$$-2\exp\left[-\frac{\mu^2}{2(1-\rho^2)}\right]\left[\frac{1}{B\left(\frac{n}{2},1\right)B\left(\frac{1}{2},\frac{1}{2}\right)}I'_0\right.$$

$$+\frac{(1+\rho^2)\mu^2}{2(1-\rho^2)B\left(\frac{n}{2},2\right)B\left(\frac{3}{2},\frac{1}{2}\right)}I'_1+\frac{(1+\rho^4)\mu^4}{8(1-\rho^2)^2B\left(\frac{n}{2},3\right)B\left(\frac{5}{2},\frac{1}{2}\right)}I'_2$$

$$+\frac{(1+\rho^6)\mu^6}{48(1-\rho^2)^3B\left(\frac{n}{2},4\right)B\left(\frac{7}{2},\frac{1}{2}\right)}I'_3+\frac{(1+\rho^8)\mu^8}{384(1-\rho^2)^4B\left(\frac{n}{2},5\right)B\left(\frac{9}{2},\frac{1}{2}\right)}I'_4$$

$$+\frac{(1+\rho^{10})\mu^{10}}{3840(1-\rho^2)^5B\left(\frac{n}{2},6\right)B\left(\frac{11}{2},\frac{1}{2}\right)}I_5'+\cdots\Bigg] \tag{25}$$

where

$$\alpha'=\frac{1}{1+\frac{F_0^2}{n}}, \qquad \beta'=\rho^2, \qquad I_\nu'=\iint_{D_2'}u_1^{2\nu}(1+u_1^2+t_2^2)^{-(n/2+\nu+1)}\,du_1\,dt_2$$

$$(\nu=0,1,2,\cdots)$$

and

$$D_2'\colon\ u_1\leqq F_0/\sqrt{n}, \quad \sqrt{1-\rho^2}\,u_1+(1+\rho)t_2>0, \ t_2<0.$$

3. Tables and comparison of powers

We tabulate the powers of this test and their ratios to those of the ordinary test whose region of rejection at the $100\alpha\%$ significance level is

$$F=(x,y)\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}^{-1}\begin{pmatrix}x\\ y\end{pmatrix}\Bigg/S^2\geqq F_{2n}(\alpha) \tag{26}$$

where S^2 is an estimate of the common variance distributed in a χ^2 distribution with n degrees of freedom and statistically independent of the sample mean vector. The distribution of F is a noncentral F with 2 and n degrees of freedom and noncentrality

Table 1. The powers of a bivariate analogue of the two-sided t test

$\mu_1=\mu_2$	$\Delta=1$							
	Power				Ratio			
ρ \ n	5	10	15	20	5	10	15	20
−0.9	.1234	.1428	.1567	.1546	1.290	1.325	1.285	1.280
−0.8	.1197	.1396	.1475	.1518	1.251	1.260	1.259	1.257
−0.7	.1169	.1367	.1447	.1490	1.222	1.234	1.235	1.234
−0.6	.1145	.1342	.1421	.1464	1.197	1.212	1.212	1.212
−0.5	.1123	.1318	.1397	.1441	1.174	1.189	1.192	1.193
−0.4	.1103	.1294	.1375	.1416	1.152	1.168	1.173	1.172
−0.3	.1085	.1273	.1352	.1395	1 134	1.149	1.154	1.155
−0.2	.1069	.1252	.1330	.1372	1.118	1.130	1.135	1.136
−0.1	.1053	.1234	.1311	.1353	1.100	1.114	1.119	1.120
0.0	.1038	.1216	.1292	.1331	1.085	1.097	1.102	1.102
0.1	.1025	.1199	.1273	.1314	1.072	1.082	1.086	1.088
0.2	.1014	.1184	.1256	.1295	1.059	1.069	1.072	1.072
0.3	.1003	.1170	.1240	.1279	1.048	1.056	1.058	1.059
0.4	.0993	.1156	.1226	.1264	1.038	1.043	1.046	1.046
0.5	.0983	.1144	.1213	.1250	1.027	1.032	1.035	1.034
0.6	.0976	.1132	.1201	.1238	1.020	1.022	1.024	1.025
0.7	.0968	.1123	.1189	.1227	1.012	1.013	1.015	1.016
0.8	.0960	.1114	.1180	.1217	1.003	1.005	1.007	1.008
0.9	.0948	.1106	.1172	.1208	.991	.998	1.000	1.000

parameter $(\mu_1, \mu_2)\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}^{-1}(\mu_1, \mu_2)'$ and the probability of rejecting the null hypothesis, when the population mean vector is (μ_1, μ_2), is given as follows:

$$Pr\{F \geqq F_{2n}(\alpha)\} = \exp\left[-\frac{1}{2}(\mu_1, \mu_2)\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}^{-1}(\mu_1, \mu_2)'\right] \times \sum_{\nu=0}^{\infty} \frac{\left\{\frac{1}{2}(\mu_1, \mu_2)\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}^{-1}(\mu_1, \mu_2)'\right\}^{\nu} B_{\gamma}\left(\frac{n}{2}, \nu+1\right)}{\nu!\, B\left(\frac{n}{2}, \nu+1\right)} \tag{27}$$

where

$$\gamma = \frac{1}{1+\frac{2F_{2n}(\alpha)}{n}}.$$

The results are given in Table 1 and Table 2 at $\mu_1=\mu_2$ and $\mu_1 \neq 0$, $\mu_2=0$ respectively for the 5% significance level and similarly given in Table 3 and Table 4 at $\mu_1=\mu_2$ and $\mu_1 \neq 0$, $\mu_2=0$ respectively for the 1% significance level.

Persual of tables reveals that the power tends to decrease in the case where $\mu_1=\mu_2$ and to increase in the case where $\mu_1 \neq 0$, $\mu_2=0$ as ρ increases. It is more powerful than the ordinary test based on F distribution in the range $\rho \leqq 0.7$ when $\mu_1=\mu_2$ and in the range $\rho \geqq 0.3$ when $\mu_1 \neq 0$, $\mu_2=0$.

The authors are deeply indebted to Professor A. Kudô of Kyushu University for his helpful advices and encouragment.

and their ratios to those of the ordinary test based on F distribution

$\Delta=2$							
Power				Ratio			
5	10	15	20	5	10	15	20
.3456	.4221	.4506	.4648	1.399	1.325	1.295	1.277
.3303	.4098	.4395	.4549	1.337	1.286	1.263	1.250
.3194	.3998	.4303	.4460	1.293	1.255	1.236	1.225
.3104	.3912	.4219	.4379	1.256	1.228	1.212	1.203
.3024	.3832	.4143	.4307	1.224	1.203	1.190	1.183
.2953	.3756	.4072	.4234	1.196	1.179	1.170	1.163
.2890	.3689	.4004	.4169	1.170	1.158	1.150	1.145
.2834	.3624	.3938	.4103	1.147	1.137	1.131	1.127
.2781	.3566	.3880	.4046	1.126	1.119	1.115	1.112
.2733	.3510	.3824	.3986	1.106	1.102	1.099	1.095
.2690	.3459	.3769	.3956	1.089	1.086	1.083	1.081
.2651	.3413	.3720	.3884	1.073	1.071	1.069	1.067
.2616	.3369	.3674	.3839	1.059	1.057	1.056	1.055
.2584	.3328	.3634	.3796	1.046	1.044	1.044	1.043
.2554	.3292	.3596	.3756	1.034	1.033	1.033	1.032
.2529	.3259	.3562	.3723	1.024	1.023	1.023	1.023
.2507	.3232	.3531	.3693	1.015	1.014	1.014	1.015
.2484	.3207	.3505	.3666	1.005	1.006	1.007	1.007
.2455	.3185	.3483	.3643	.994	1.000	1.001	1.001

Table 2. The powers of a bivariate analogue of the two-sided t test

$\mu_1 \neq 0$, $\mu_2=0$	$\Delta=1$							
	Power				Ratio			
ρ \ n	5	10	15	20	5	10	15	20
−0.9	.0646	.0848	.0929	.0970	.675	.765	.793	.803
−0.8	.0624	.0821	.0899	.0942	.652	.741	.767	.779
−0.7	.0688	.0875	.0951	.0991	.719	.790	.811	.820
−0.6	.0780	.0959	.1031	.1070	.815	.866	.880	.886
−0.5	.0878	.1049	.1119	.1158	.917	.947	.954	.958
−0.4	.0972	.1135	.1205	.1240	1.015	1.025	1.028	1.026
−0.3	.1057	.1215	.1281	.1317	1.105	1.096	1.093	1.090
−0.2	.1131	.1283	.1346	.1381	1.182	1.158	1.149	1.143
−0.1	.1191	.1339	.1402	.1435	1.245	1.209	1.196	1.188
0.0	.1238	.1381	.1443	.1474	1.294	1.247	1.231	1.221
0.1	.1637	.1776	.1835	.1868	1.711	1.603	1.565	1.546
0.2	.2032	.2168	.2225	.2256	2.124	1.957	1.898	1.868
0.3	.2436	.2568	.2624	.2655	2.545	2.317	2.239	2.198
0.4	.2860	.2989	.3045	.3075	2.990	2.698	2.598	2.546
0.5	.3324	.3450	.3505	.3534	3.475	3.114	2.990	2.926
0.6	.3854	.3976	.4030	.4059	4.029	3.589	3.438	3.360
0.7	.4493	.4610	.4660	.4689	4.697	4.160	3.976	3.882
0.8	.5320	.5428	.5474	.5500	5.562	4.900	4.671	4.554
0.9	.6532	.6622	.6660	.6680	6.826	5.977	5.682	5.531

Table 3. The powers of a bivariate analogue of the two-sided t test

$\mu_1=\mu_2$	$\Delta=1$							
	Power				Ratio			
ρ \ n	5	10	15	20	5	10	15	20
−0.9	.0299	.0399	.0446	.0472	1.381	1.424	1.429	1.426
−0.8	.0288	.0387	.0434	.0460	1.333	1.382	1.390	1.390
−0.7	.0281	.0377	.0423	.0449	1.296	1.345	1.354	1.356
−0.6	.0273	.0367	.0413	.0438	1.263	1.310	1.322	1.324
−0.5	.0267	.0358	.0402	.0429	1.232	1.279	1.290	1.295
−0.4	.0261	.0350	.0394	.0419	1.204	1.248	1.261	1.267
−0.3	.0255	.0342	.0385	.0410	1.179	1.221	1.232	1.238
−0.2	.0250	.0335	.0377	.0402	1.155	1.194	1.207	1.213
−0.1	.0245	.0327	.0369	.0393	1.133	1.168	1.181	1.187
0.0	.0241	.0321	.0361	.0385	1.113	1.146	1.157	1.162
0.1	.0237	.0315	.0354	.0377	1.095	1.124	1.134	1.140
0.2	.0233	.0309	.0347	.0370	1.078	1.102	1.112	1.117
0.3	.0230	.0303	.0340	.0363	1.062	1.083	1.090	1.096
0.4	.0227	.0298	.0335	.0356	1.048	1.064	1.072	1.076
0.5	.0224	.0293	.0329	.0350	1.034	1.047	1.053	1.056
0.6	.0221	.0289	.0324	.0344	1.022	1.031	1.037	1.040
0.7	.0218	.0285	.0318	.0339	1.009	1.017	1.020	1.023
0.8	.0216	.0281	.0314	.0334	.997	1.004	1.006	1.009
0.9	.0208	.0275	.0306	.0326	.962	.980	.980	.985

and their ratios to those of the ordinary test based on F distribution

$\Delta=2$							
Power				Ratio			
5	10	15	20	5	10	15	20
.2407	.3238	.3550	.3706	.975	1.016	1.020	1.018
.2381	.3194	.3500	.3659	.964	1.002	1.006	1.005
.2465	.3250	.3550	.3705	.998	1.020	1.020	1.018
.2557	.3324	.3616	.3770	1.035	1.043	1.039	1.036
.2635	.3386	.3677	.3831	1.067	1.063	1.056	1.053
.2699	.3436	.3728	.3877	1.093	1.078	1.071	1.065
.2752	.3478	.3766	.3918	1.114	1.091	1.082	1.076
.2798	.3511	.3797	.3947	1.133	1.102	1.091	1.084
.2835	.3542	.3826	.3976	1.148	1.112	1.099	1.092
.2871	.3567	.3850	.3997	1.162	1.120	1.106	1.098
.2988	.3674	.3953	.4103	1.209	1.153	1.136	1.127
.3110	.3787	.4062	.4209	1.259	1.189	1.167	1.156
.3244	.3911	.4182	.4329	1.313	1.227	1.202	1.189
.3398	.4052	.4323	.4467	1.376	1.272	1.242	1.227
.3583	.4226	.4493	.4633	1.451	1.326	1.291	1.273
.3823	.4450	.4711	.4850	1.548	1.397	1.354	1.332
.4157	.4765	.5017	.5154	1.683	1.496	1.441	1.416
.4685	.5262	.5500	.5629	1.896	1.652	1.580	1.546
.5696	.6199	.6404	.6513	2.306	1.946	1.840	1.790

and their ratios to those of the ordinary test based on F distribution

$\Delta=2$							
Power				Ratio			
5	10	15	20	5	10	15	20
.1116	.1749	.2053	.2215	1.611	1.541	1.499	1.467
.1040	.1661	.1960	.2126	1.502	1.463	1.431	1.408
.0992	.1598	.1894	.2061	1.432	1.408	1.383	1.365
.0952	.1545	.1838	.2004	1.375	1.361	1.342	1.327
.0918	.1499	.1786	.1954	1.326	1.324	1.304	1.294
.0888	.1455	.1741	.1906	1.283	1.282	1.271	1.263
.0862	.1417	.1697	.1861	1.245	1.248	1.239	1.233
.0838	.1381	.1658	.1820	1.222	1.216	1.211	1.206
.0817	.1347	.1620	.1781	1.179	1.186	1.183	1.180
.0797	.1317	.1586	.1744	1.152	1.160	1.158	1.155
.0780	.1289	.1553	.1710	1.126	1.135	1.134	1.133
.0764	.1262	.1522	.1676	1.103	1.111	1.111	1.111
.0749	.1237	.1493	.1647	1.082	1.090	1.090	1.091
.0736	.1215	.1467	.1617	1.063	1.070	1.071	1.071
.0724	.1193	.1442	.1590	1.046	1.051	1.053	1.053
.0713	.1175	.1420	.1566	1.030	1.035	1.037	1.038
.0703	.1158	.1398	.1543	1.015	1.020	1.021	1.022
.0693	.1143	.1380	.1524	1.001	1.007	1.008	1.009
.0670	.1119	.1350	.1495	.968	.985	.986	.990

Table 4. The powers of a bivariate analogue of the two-sided t test

$\mu_1 \neq 0,\ \mu_2=0$	$\Delta=1$							
	Power				Ratio			
ρ \ n	5	10	15	20	5	10	15	20
−0.4	.0176	.0250	.0287	.0308	.814	.894	.918	.930
−0.3	.0286	.0357	.0391	.0411	1.321	1.273	1.253	1.243
−0.2	.0380	.0448	.0481	.0501	1.756	1.598	1.542	1.513
−0.1	.0458	.0523	.0555	.0574	2.116	1.866	1.778	1.734
0.0	.0519	.0582	.0613	.0631	2.397	2.076	1.963	1.906
0.1	.0946	.1007	.1037	.1055	4.373	3.593	3.226	3.187
0.2	.1368	.1426	.1456	.1473	6.321	5.092	4.664	4.448
0.3	.1796	.1853	.1881	.1899	8.299	6.614	6.028	5.734
0.4	.2246	.2301	.2329	.2346	10.381	8.217	7.463	7.087
0.5	.2738	.2791	.2818	.2834	12.652	9.960	9.033	8.562
0.6	.3300	.3352	.3378	.3394	15.249	11.963	10.824	10.250
0.7	.3980	.4029	.4054	.4069	18.392	14.382	12.990	12.291
0.8	.4867	.4913	.4936	.4950	22.492	17.538	15.818	14.952
0.9	.6181	.6219	.6238	.6250	28.563	22.202	19.988	18.879

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and their ratios to those of the ordinary test based on F distribution

$\Delta=2$							
Power				Ratio			
5	10	15	20	5	10	15	20
.0754	.1257	.1510	.1657	1.089	1.108	1.103	1.098
.0860	.1349	.1595	.1740	1.243	1.188	1.165	1.153
.0950	.1425	.1668	.1810	1.372	1.255	1.218	1.199
.1028	.1490	.1729	.1870	1.485	1.313	1.263	1.239
.1099	.1551	.1785	.1923	1.588	1.366	1.304	1.274
.1252	.1693	.1923	.2059	1.808	1.492	1.404	1.364
.1408	.1839	.2065	.2198	2.033	1.620	1.508	1.456
.1575	.1997	.2218	.2351	2.275	1.759	1.619	1.557
.1764	.2176	.2394	.2523	2.548	1.917	1.748	1.671
.1988	.2389	.2602	.2729	2.871	2.105	1.900	1.808
.2271	.2662	.2870	.2995	3.280	2.345	2.096	1.984
.2664	.3043	.3244	.3364	3.848	2.680	2.369	2.229
.3290	.3650	.3840	.3956	4.751	3.215	2.804	2.621
.4512	.4829	.4993	.5096	6.515	4.254	3.646	3.376