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Abstract

The purpose of the present paper is to give a general theory of generalized Berwald spaces.

§0. Introduction.

A Berwald space is an affinely connected space defined by Berwald [3, 4], which is a Finsler space such that the coefficients $G_j{}^i{}_k$ of the Berwald connection $B\Gamma$ [2] depend on position alone. If we obey the Cartan connection $C\Gamma$ [6], such a space is also the one in which the coefficients $\Gamma^*{}_j{}^i{}_k$ depend on position alone. Wagner [33] generalized the notion of Berwald space, and called a Finsler space as a generalized Berwald space if there is possible to introduce a generalized Cartan connection with torsion, in such a way that the coefficients ${}^*\Gamma_j{}^i{}_k$ depend on position alone. And in the two-dimensional case he characterized such a space in terms of the main scalar I (Berwald [4, 5]), and showed that a Finsler space with the so-called cubic metric is an example.

In his paper [7], Hashiguchi, one of the authors, investigated various axioms imposed on a Finsler connection, based on the modern theory of Finsler geometry by Matsumoto [20, 22], clarified a geometrical meaning of the generalized Cartan connection given by Wagner, and characterized Wagner's generalized Berwald space of general dimensions. Then, a generalized Cartan connection and so a generalized Berwald space were defined in broader sense than Wagner's, while Wagner's were called a Wagner connection and a Wagner space respectively.

On the other hand, Ichijyō, the other author, [13, 14] obtained the notion of $\{V, H\}$-manifold from the study about Finsler spaces modeled on a Minkowski space and showed that such a manifold is just a generalized Berwald space. This result is significant in the sense that global considerations are possible in generalized Berwald spaces.

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Recently, Matsumoto [23] generalized Okada's axioms [29] which determine the Berwald connection $B\Gamma$, and gave the notion of generalized Berwald connection. And he showed that a generalized Berwald space can be also defined in terms of a generalized Berwald connection. Since the notion of Berwald space was defined in terms of $B\Gamma$, Matsumoto's result is very satisfactory to the establishment of the notion of generalized Berwald space.

Generalized Berwald spaces thus defined might look peculiar, but the peculiarity is thought to be rather useful to characterize Finsler spaces with complicated character, and have been studied by the authors and the others (Aikou-Hashiguchi [1], Hashiguchi [9], Hashiguchi-Ichijyō [10, 11], Hashiguchi-Varga [12], Ichijyō [15, 16, 17], Matsumoto [24], Miron-Hashiguchi [27], Tamássy-Matsumoto [31], etc.), and have formed an interesting class among Finsler spaces, waiting for the further studies.

The purpose of the present paper is to give a general theory of generalized Berwald spaces. In §1, §3, §4, and §5 we state the respective definitions of generalized Berwald spaces by Wagner [33], Hashiguchi [7], Matsumoto [23] and Ichijyō [13, 14] comparatively, and in §6 we consider the geometrical significance from various standpoints. The definite definition of a generalized Berwald space is given in §3, and [7] is improved. In §5 $\{V, H\}$-manifolds are defined also for a vector space V with an eccentric norm.

The terminology and notations are referred to Matsumoto [20, 22]. As to Finsler connections, we sketch the materials necessary for our discussions, in §2.

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§1. Wagner's generalized Berwald spaces.

1.1. Let M and $T(M)$ be a differentiable manifold and the tangent bundle respectively. A coordinate system (x^i) in M induces a canonical coordinate system (x^i, y^i) in $T(M)$. And we put $\partial_k = \partial/\partial x^k$, $\dot\partial_k = \partial/\partial y^k$.

A positive-valued differentiable function $L(x, y)$ defined on a domain D of $T(M)-\{0\}$ is called a *Finsler metric* of M, if it satisfies the following conditions:

(i) L is (1) p-homogeneous: $L(x, \lambda y) = \lambda L(x, y)$ for $\lambda > 0$,

(ii) The matrix (g_{ij}) is regular: $g = \det(g_{ij}) \neq 0$, where $g_{ij} = 1/2\, \dot\partial_j \dot\partial_i L^2$.

An n-dimensional differentiable manifold M with a Finsler metric L is called a *Finsler space* and is denoted by $F^n = (M, L)$, if the length s of a curve $x^i(t)$ in M is measured by $s = \int L(x, dx/dt)dt$. Then L and g_{ij} are called the *Fundamental function* and the *fundamental tensor* of F^n respectively. And we put $y_i = g_{ir}y^r$, $l^i = y^i/L$, $l_i = g_{ir}l^r$ $(= \dot\partial_i L = y_i/L)$, and $(g^{ij}) = (g_{ij})^{-1}$.

A Finsler metric is traditionally defined in the more restrictive sense that $D = T(M) - \{0\}$ and (g_{ij}) be positive-definite. However, our definition may regard the following well-known metrics as Finsler metrics:

$$L(x,y) = (a_{ijk}(x)y^i y^j y^k)^{1/3}, \tag{1.1}$$

$$L(x,y) = (a_{ij}(x)y^i y^j)^{1/2} + b_i(x)y^i, \tag{1.2}$$

$$L(x,y) = a_{ij}(x)y^i y^j / b_i(x)y^i, \tag{1.3}$$

where $(a_{ij}(x)y^i y^j)^{1/2}$ is a Riemannian metric and $b_i(x)$ is a non-zero covariant vector field. The Finsler metrics (1.1), (1.2) and (1.3) are called the *cubic metric*, the *Randers metric* [30] and the *Kropina metric* [18, 19] respectively.

1.2. In the two-dimensional case, Cartan's torsion tensor $C_{ijk}=1/2\ \dot{\partial}_k g_{ij}$ is expressed as

$$LC_{ijk} = Im_i m_j m_k, \tag{1.4}$$

where m_i is the unit vector orthogonal to l^i: $m_1 = -l^2\sqrt{g}$, $m_2 = l^1\sqrt{g}$. The scalar I is called the *main scalar* of F^2. The differential equation $\dot{\partial}_i\theta = m_i/L$ is integrable, and the scalar θ is called the *Landsberg angle.*

1.3. Wagner [33] called a Finsler space F^2 as a *generalized Berwald space* if there is possible to introduce a generalized Cartan connection, with torsion $({}^*\Gamma_j{}^i{}_k - {}^*\Gamma_k{}^i{}_j \neq 0)$, in such a way that the coefficients ${}^*\Gamma_j{}^i{}_k$ depend on position alone: $\dot{\partial}_l {}^*\Gamma_j{}^i{}_k = 0$. ${}^*\Gamma_j{}^i{}_k$ were there given by

$$\begin{aligned} {}^*\Gamma_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k &+ s_r l^r (Im^i m_j m_k + l^i m_j m_k - m^i l_j m_k) \\ &+ s_r m^r (I^2 m^i m_j m_k + I(l^i m_j m_k - m^i l_j m_k - m^i m_j l_k) \\ &\qquad + m^i l_j l_k - l^i m_j l_k), \end{aligned} \tag{1.5}$$

where $\Gamma^*{}_j{}^i{}_k$ are the coefficients of the Cartan connection, $m^i = g^{ir} m_r$, and s_r is a covariant vector field. By considering the condition that $s_r (\neq 0)$ can be chosen in such a way that ${}^*\Gamma_j{}^i{}_k$ depend on position alone, he obtained

Theorem 1.1. (Wagner) *A necessary and sufficient condition that* F^2 $(\partial I/\partial\theta \neq 0)$ *is a generalized Berwald space is thst* $\partial I/\partial\theta$ *be a function of* I. *If* $\partial I/\partial\theta = 0$, *then* I *must be constsnt.*

Theorem 1.2. (Wagner) F^2 *with the cubic metric* (1.1) *is a generalized Berwald space and* $\partial I/\partial\theta = -3/2 - 3I^2$.

F^2 with a constant I is a Berwald space. If we consider a Berwald space as a special generalized Berwald space, the assumption $\partial I/\partial\theta \neq 0$ may be omitted. The detail of the cubic metric is referred to [25].

What is the generalized Cartan connection given by (1.5)? How can we characterize the generalized Berwald spaces of general dimensions? Are there other interesting examples of generalized Berwald spaces?

§2. Finsler connections.

2.1. Given an n-dimensional differentiable manifold M, we denote by $L(M)$ $(M, \pi, GL(n, R))$ and $T(M)$ $(M, \tau, V, GL(n, R))$ the linear frame bundle and the tangent bundle respectively. The standard fibre V is assumed that a base $\{e_a\}$ is fixed. The induced bundle $\tau^{-1}L(M)=\{(y, z) \in T(M)\times L(M) \mid \tau(y)=\pi(z)\}$ is called the *Finsler bundle* of M and denoted by $F(M)$ $(T(M), \pi_1, GL(n, R))$. The Lie algebra of the structural group $GL(n, R)$ of $L(M)$ and $F(M)$ is denoted by $\mathfrak{gl}(n, R)$ and the canonical base by $\{L_a{}^b\}$.

Since a point of $F(M)$ is a pair of a tangent vector y and a linear frame $z=(z_a)$ at a point x of the base manifold M, a coordinate system (x^i) in M induces a canonical coordinate system $(x^i, y^i, z_a{}^i)$ in $F(M)$ by $y=y^i(\partial/\partial x^i)_x$ and $z_a=z_a{}^i(\partial/\partial x^i)_x$.

2.2. The Finsler connection $F\Gamma$ in M is defined in three equivalent manners as a pair (Γ, N), as a pair (Γ^h, Γ^v) or as a triad (Γ_V, N, Γ^v), where Γ and Γ^h (resp. Γ^v) are a connection and a horizontal (resp. vertical) connection in $F(M)$, N is a non-linear connection in $T(M)$, and Γ_V is a V-connection in $L(M)$. In $F(M)$ the fundamental vector field $Z(A)$ $(A \in \mathfrak{gl}(n, R))$ and h- and v-basic vector fields $B^h(v)$, $B^v(v)$ $(v \in V)$ are defined, and these three fields span the tangent space of $F(M)$ at each point. They are expressed by

$$Z(A) = A_b{}^a z_a{}^k(\partial/\partial z_b{}^k)\,, \tag{2.1}$$

$$B^h(v) = v^a z_a{}^k(\partial/\partial x^k - N^i{}_k\partial/\partial y^i - z_b{}^j F_j{}^i{}_k\partial/\partial z_b{}^i)\,, \tag{2.2}$$

$$B^v(v) = v^a z_a{}^k(\partial/\partial y^k - z_b{}^j C_j{}^i{}_k\partial/\partial z_b{}^i)\,, \tag{2.3}$$

where $A=A_b{}^a L_a{}^b \in \mathfrak{gl}(n, R)$ and $v=v^a e_a \in V$. $F_j{}^i{}_k$, $N^i{}_k$, $C_j{}^i{}_k$ are called the *coefficients* of $F\Gamma$. The Finsler connection $F\Gamma$ having $F_j{}^i{}_k$, $N^i{}_k$, $C_j{}^i{}_k$ as the coefficients is denoted by $F\Gamma=(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$.

There is a tensor field D called the *deflection tensor*, which expresses a relation between Γ_V and N, and it is expressed as

$$D^i{}_k = y^j F_j{}^i{}_k - N^i{}_k\,. \tag{2.4}$$

Definition 2.1. A Finsler connection $F\Gamma=(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ is called *linear*, if $F_j{}^i{}_k$ depend on position alone.

Let $\Gamma=(\Gamma_j{}^i{}_k(x))$ be a linear connection of M, that is, a connection in $L(M)$. Then a linear Finsler connection $F(\Gamma)$ without deflection is obtained by $F(\Gamma)=(\Gamma_j{}^i{}_k, y^j\Gamma_j{}^i{}_k, C_j{}^i{}_k)$, which is called to be *associated to* Γ, where $C_j{}^i{}_k$ is freely chosen, for instance, $C_j{}^i{}_k=0$. Especially, in a Finsler space we specify $C_j{}^i{}_k$ to Cartan's torsion tensor $1/2\, g^{ir}\dot{\partial}_k g_{jr}$.

2.3. Let K be a Finsler tensor field. The h- and v-covariant derivatives of K are defined by $\Delta^h K(v)=B^h(v)K$ and $\Delta^v K(v)=B^v(v)K$ respectively. If K is assumed, for instance, to be of type (1.1), i.e., (2.5) $$K = z'_i{}^a z_b{}^j K_j{}^i e_a \otimes e^b ,$$

where $(z'_i{}^a)=(z_a{}^i)^{-1}$, and $\{e^b\}$ is the dual base of $\{e_a\}$, their components $K_j{}^i{}_{|k}$ and $K_j{}^i|_k$ are expressed as follows:

(2.6) $$K_j{}^i{}_{|k} = \delta_k K_j{}^i + K_j{}^m F_m{}^i{}_k - K_m{}^i F_j{}^m{}_k ,$$

(2.7) $$K_j{}^i|_k = \dot{\partial}_k K_j{}^i + K_j{}^m C_m{}^i{}_k - K_m{}^i C_j{}^m{}_k ,$$

where $\delta_k = \partial_k - N^m{}_k \dot{\partial}_m$.

2.4. If we consider the Lie brackets [,] of the basic vector fields, we have the following structure equations:

(2.8) $$[B^h(v), B^h(w)] = B^h(T(v, w)) + B^v(R^1(v, w)) + Z(R^2(v, w)) ,$$

(2.9) $$[B^h(v), B^v(w)] = B^h(C(v, w)) + B^v(P^1(v, w)) + Z(P^2(v, w)) ,$$

(2.10) $$[B^v(v), B^v(w)] = B^v(S^1(v, w)) + Z(S^2(v, w)) ,$$

from which we have five torsion tensors T, C, R^1, P^1, S^1 and three curvature tensors R^2, P^2, S^2. Their components are expressed as follows:

(2.11) $$T:\ T_j{}^i{}_k = \mathfrak{A}_{jk}\{F_j{}^i{}_k\} ; \quad S^1:\ S^i{}_{jk} = \mathfrak{A}_{jk}\{C_j{}^i{}_k\} ; \quad C:\ C_j{}^i{}_k ,$$

(2.12) $$R^1:\ R^i{}_{jk} = \mathfrak{A}_{jk}\{\delta_k N^i{}_j\} ; \ P^1:\ P^i{}_{jk} = \dot{\partial}_k N^i{}_j - F_k{}^i{}_j ,$$

(2.13) $$R^2:\ R_h{}^i{}_{jk} = \mathfrak{A}_{jk}\{\delta_k F_h{}^i{}_j + F_h{}^m{}_j F_m{}^i{}_k\} + C_h{}^i{}_m R^m{}_{jk} ,$$

(2.14) $$P^2:\ P_h{}^i{}_{jk} = \dot{\partial}_k F_h{}^i{}_j - C_h{}^i{}_{k|j} + C_h{}^i{}_m P^m{}_{jk} ,$$

(2.15) $$S^2:\ S_h{}^i{}_{jk} = \mathfrak{A}_{jk}\{\dot{\partial}_k C_h{}^i{}_j + C_h{}^m{}_j C_m{}^i{}_k\} ,$$

where $\mathfrak{A}_{jk}\{\dots\}$ denotes, for instance, $\mathfrak{A}_{jk}\{A_{jk}\} = A_{jk} - A_{kj}$.

For the later use we give

Definition 2.2. A Finsler connection $(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ is called a *C-zero connection* if $C_j{}^i{}_k=0$, and is called an *N-connection if* $P^i{}_{jk}=0$: $F_j{}^i{}_k = \dot{\partial}_j N^i{}_k$.

Let a Finsler connection $F\Gamma=(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ be given. A Finsler connection $(F_j{}^i{}_k, N^i{}_k, 0)$ is called the *C-zero connection of* $F\Gamma$, and a Finsler connection $(\dot{\partial}_j N^i{}_k, N^i{}_k, 0)$ is called the *N-connection of* $F\Gamma$.

§3. Generalized Cartan connections and generalized Berwald spaces.

3.1. Now we are concerned with a Finsler space $F^n=(M, L)$. We have

Proposition 3.1. *For a given alternate and* (0) *p-homogeneous Finsler tensor field* $T_j{}^i{}_k$, *there exists a unique Finsler connection* $C\Gamma(T)=(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ *satisfying the following four axioms*:

(C1) *It is metrical*: $g_{ij|k}=0$, $g_{ij}|_k=0$,

(C2) *The deflection tensor D vanishes*: $N^i{}_k = y^j F_j{}^i{}_k$,

(C3) *The torsion tensor T is the given* $T_j{}^i{}_k$: $F_j{}^i{}_k - F_k{}^i{}_j = T_j{}^i{}_k$, (C4) *The torsion tensor* S^1 *vanishes*: $C_j{}^i{}_k = C_k{}^i{}_j$.

The coefficients are given by

$$(3.1)\quad \begin{cases} F_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k - g^{ih}C_{jkm}(C_h{}^m{}_r A_0{}^r{}_0 - A_0{}^m{}_h) + C_j{}^i{}_m(C_k{}^m{}_r A_0{}^r{}_0 - A_0{}^m{}_k) \\ \qquad\qquad + C_k{}^i{}_m(C_j{}^m{}_r A_0{}^r{}_0 - A_0{}^m{}_j) + A_j{}^i{}_k , \\ N^i{}_k = G^i{}_k - C_k{}^i{}_r A_0{}^r{}_0 + A_0{}^i{}_k , \\ C_j{}^i{}_k = \dfrac{1}{2} g^{ir}\dot{\partial}_k g_{jr} , \end{cases}$$

where $A_j{}^i{}_k = (T_j{}^i{}_k + T_{jk}{}^i + T_{kj}{}^i)/2$, *and* $\Gamma^*{}_j{}^i{}_k$, $G^i{}_k$, $C_j{}^i{}_k$ *are the coefficients of the Cartan connection* $C\Gamma = C\Gamma(0)$, *and the subscript* 0 *means the contraction by* y^j: $A_0{}^m{}_h = y^j A_j{}^m{}_h$.

Definition 3.1. A Finsler connection $C\Gamma(T)$ given by Proposition 3.1 is called a *generalized Cartan connection.*

A Finsler space is called a *generalized Berwald space* if there is possible to introduce a *linear* generalized Cartan connection $C\Gamma(T)$.

3.2. As a typical generalized Cartan connection we have

Proposition 3.2. *For a given* (0) *p-homogeneous covariant Finsler vector field* s_k, *there exists a unique Finsler connection* $W\Gamma(s) = (F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ *satisfying* (C1), (C2), (C4) *and*

(C3*) *It is semi-symmetric with respect to the given* s_k:

$$F_j{}^i{}_k - F_k{}^i{}_j = \delta_j^i s_k - \delta_k^i s_j .$$

The coefficients are given by

$$(3.2)\quad \begin{cases} F_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k + L^2(S_j{}^i{}_{kl} + C_j{}^i{}_m C_k{}^m{}_l)s^l \\ \qquad\qquad + (y^i C_{jkl} - y_j C_k{}^i{}_l - y_k C_j{}^i{}_l)s^l + C_j{}^i{}_k s_0 + g_{jk}s^i - \delta_k^i s_j , \\ N^i{}_k = G^i{}_k - L^2 C_k{}^i{}_l s^l + y_k s^i - \delta_k^i s_0 , \\ C_j{}^i{}_k = \dfrac{1}{2} g^{ir}\dot{\partial}_k g_{jr} , \end{cases}$$

where $s^l = g^{lm}s_m$ *and* $S_j{}^i{}_{kl}$ *are the coefficients of* S^2 *of* $C\Gamma$.

Definition 3.2. A Finsler connection $W\Gamma(S)$ given by Proposition 3.2 is called a *Wagner connection.*

A Finsler space is called a *Wagner space* if there is possible to introduce a *linear* Wagner connection $W\Gamma(s)$.

In the two-dimensional case, the $F_j{}^i{}_k$ given in (3.2) become Wagner's ${}^*\Gamma_j{}^i{}_k$ given by (1.5). Thus we have noticed the geometrical meaning for the generalized Cartan connection given by Wagner.

3.3. In the paper [7] a generalized Cartan connection was defined as a Finsler connection satisfying the axioms (C1), (C4). Given a Finsler tensor field $D^i{}_k$ and an alternate Finsler tensor field $T_j{}^i{}_k$, there exists a unique Finsler connection satisfying (C1), (C3), (C4) and

(C2*) *The deflection tensor is the given* $D^i{}_k$.

However, only connections without deflection have been treated in almost the subsequent papers. Hence, we reform the definition of a generalized Cartan connection and so a generalized Berwald space, and adopt Definition 3.1. Then Theorem 3 of [7] is improved as follows.

Theorem 3.1. *A Finsler space is a generalized Berwald space if and only if there exists an alternate Finsler tensor field* $T_j{}^i{}_k(x)$ *such that* $C\Gamma(T)$ *satisfies the condition* $C_{ijk|l}=0$.

Especially, a Finsler space is a Wagner space if and only if there exists a covariant vector field $s_k(x)$ *such that* $W\Gamma(s)$ *satisfies the condition* $C_{ijk|l}=0$.

A Berwald space is characterized by the condition $C_{ijk|l}=0$ with respect to $C\Gamma$. Thus a generalized Berwald space and a Wagner space of general dimensions are characterized by the formally same condition as the one for a Berwald space.

3.4. As an example of a generalized Cartan connection with surviving deflection we have

Proposition 3.3. *For a given* (0) *p-homogeneous covariant Finsler vector field* s_k, *there exists a unique Finsler connection* $M\Gamma(s)=(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ *satisfying* (C1), (C3), (C4) *and*

(C2*) *The non-linear connection is the Cartan one*: $N^i{}_k=G^i{}_k$.

The coefficients are given by

$$F_j{}^i{}_k=\Gamma^*{}_j{}^i{}_k+g_{jk}s^i-\delta_k^i s_j\,,\quad N^i{}_k=G^i{}_k,\quad C_j{}^i{}_k=1/2\,g^{ir}\dot\partial_k g_{jr}\,. \tag{3.3}$$

Definition 3.3. A Finsler connection $M\Gamma(s)$ given by Proposition 3.3 is called a *Miron connection.*

Whereas the Wagner connection has the very complicated coefficients, the Miron connection is represented by the simple coefficients. Miron [26] treated the general theory of transformations of Finsler connections. The simplicity applied the theory gave us interesting invariants of the Miron connections ([9], [27]). On the other hand, complexity of the Wagner connection serves to characterize Finsler spaces with complicated characters.

§4. Generalized Berwald connections and generalized Berwald spaces.

4.1. In his recent paper [23], Matsumoto generalized the notion of Berwald connection as follows.

Proposition 4.1. (Matsumoto) *For a given alternate and* (0) *p-homogeneous Finsler tensor field* $T_j{}^i{}_k$ *satisfying the condition*

$$(4.1) \qquad y^r(\partial_k T_j{}^i{}_r - \partial_j T_k{}^i{}_r) = 0\,,$$

there exists a unique Finsler connection $B\Gamma(T)=(F_j{}^i{}_k, N^i{}_k, 0)$ *satisfying the following four axioms*:

(B1) $L_{|k}=0$,

(B2) *The deflection tensor* D *vanishes*: $N^i{}_k = y^j F_j{}^i{}_k$,

(B3) *The torsion tensor* P^1 *vanishes*: $F_j{}^i{}_k = \partial_j N^i{}_k$,

(B4) *The torsion tensor* T *is the given* $T_j{}^i{}_k$: $F_j{}^i{}_k - F_k{}^i{}_j = T_j{}^i{}_k$.

The coefficients are given by

$$(4.2) \qquad \begin{cases} F_j{}^i{}_k = G_j{}^i{}_k - (\partial_j \partial_k T^i{}_{00} + \partial_j T_k{}^i{}_0)/2\,, \\ N^i{}_k = G^i{}_k - (\partial_k T^i{}_{00} + T_k{}^i{}_0)/2\,, \end{cases}$$

where $G_j{}^i{}_k$, $G^i{}_k$ *are the coefficients of the Berwald connection* $B\Gamma = B\Gamma(0)$.

Definition 4.1. A Finsler connection $B\Gamma(T)$ given by Proposition 4.1 is called a *generalized Berwald connection.*

A Finsler space is called a *generalized Berwald space* if there is possible to introduce a *linear* generalized Berwald connection $B\Gamma(T)$.

4.2. Contrary to the case of $C\Gamma(T)$, $T_j{}^i{}_k$ in $B\Gamma(T)$ is not necessarily given arbitrarily. It must satisfy the condition (4.1). It is noted, however, that (4.1) holds good if $T_j{}^i{}_k$ depend on position alone, and we have

Theorem 4.1. (Matsumoto) *Let* $T_j{}^i{}_k$ *be an alternate and* (0) *p-homogeneous tensor field. If* $T_j{}^i{}_k$ *depend on position alone, then* $C\Gamma(T)=(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ *and* $B\Gamma(T)=(\bar{F}_j{}^i{}_k, \bar{N}^i{}_k, 0)$ *are defined, and* $B\Gamma(T)$ *is the N-connection of* $C\Gamma(T)$. *And* $C\Gamma(T)$ *is linear if and only if* $B\Gamma(T)$ *is linear. In this case* $B\Gamma(T)$ *is the C-zero Finsler connection of* $C\Gamma(T)$.

Thus Definition 4.1 is equivalent to Definition 3.1 for the definition of generalized Berwald space. A Wagner space is also defined in terms of a C-zero Wagner connection. Since the notion of Berwald space was originally defined in terms of $B\Gamma$, the above result is very satisfactory to the establishment of the notion of generalized Berwald space.

The discussions about the generalized Berwald connection need the homogeneity of $T_j{}^i{}_k$. So, in the definitions of $C\Gamma(T)$, $W\Gamma(s)$ and $M\Gamma(s)$ we imposed the homogeneity for $T_j{}^i{}_k$, s_k, too.

4.3. Given a linear connection $\Gamma=(\Gamma_j{}^i{}_k(x))$ of M, we have two linear Finsler connections $F(\Gamma)=(\Gamma_j{}^i{}_k, y^j\Gamma_j{}^i{}_k, C_j{}^i{}_k)$ and $\bar{F}(\Gamma)=(\Gamma_j{}^i{}_k, y^j\Gamma_j{}^i{}_k, 0)$ associated to Γ. Then we have

Proposition 4.2.. *The h-covariant derivative of a Finsler tensor field* K *with respect to* $F(\Gamma)$ *coincides with the one with respect to* $\bar{F}(\Gamma)$. *If the components of* K *depends on position alone, it coincides also with the one with respect to the original* Γ.

Since $F(\Gamma)$ and $\bar{F}(\Gamma)$ have the same torsion tensor $T_j{}^i{}_k(x) = \Gamma_j{}^i{}_k - \Gamma_k{}^i{}_j$, it follows from Theorem 4.1 that $F(\Gamma)$ is a generalized Cartan connection if and only if $\bar{F}(\Gamma)$ is a generalized Berwald connection, which is characterized by $L_{|i}=0$. Thus we have

Theorem 4.2. *Let a linear connection Γ of M be given. If it holds that $L_{|i}=0$ with respect to the Finsler connection $F(\Gamma)$ or $\bar{F}(\Gamma)$ associated with Γ, then $F(\Gamma)$ (resp. $\bar{F}(\Gamma)$) is a linear generalized Cartan (resp. Berwald) connection, and the space is a generalized Berwald space.*

Then, if Γ is symmetric (resp. semi-symmetric), then $F(\Gamma)$ is the Cartan connection (resp. a Wagner connection), and $\bar{F}(\Gamma)$ is the Berwald connection (resp. a C-zero Wagner connection), and the space is a Berwald space (resp. a Wagner space).

§5. $\{V, H\}$-manifolds.

5.1. Let V be an n-dimensional linear space with a fixed base $\{e_a\}$. A global coordinate system (v^a) is introduced on V by $v=v^a e_a \in V$, and the differentiability is defined for a function on V.

Definition 5.1. A positive-valued differentiable function $f(v)$ defined on V-$\{0\}$ is called a *Minkowski norm* of V, if it satisfies the following conditions:

(i) f is (1) p-homogeneous: $f(\lambda v)=\lambda f(v)$ for $\lambda>0$,

(ii) The matrix (g_{ab}) is positive-definite, where $g_{ab}=1/2\ \partial^2 f^2/\partial v^a \partial v^b$.

An n-dimensional linear space V with a Minkowski norm f is called an n-dimensional *Minkowski space*, and is denoted by (V, f).

Proposition 5.1. *In a Minkowski space (V, f), the set*

$$G=\{T \in GL(n, R) \mid f(Tv)=f(v) \text{ for any } v \in V\} \tag{5.1}$$

is a closed subgroup of $GL(n, R)$, and so becomes a Lie group.

5.2. The tagnent space $T_x(M)$ at any point x of a Finsler space (in the restrictive sense) (M, L) is a Minkowksi space, but the tangent spaces at two distinct points are not necessarily same. So an important class of Finsler spaces is given by the property that the tangent spaces at any points are linearly isomorphic to a single Minkowski space.

Proposition 5.2. *Let H be a Lie subgroup of a Lie group G defined in Proposition 5.1. Suppose that an n-dimensional differentiable manifold M admits the H-structure in the sense of G-structure. Let $\{U\}$ be a coordinate neighbourhood system and $z=(z_a)$ be a linear frame adapted to the H-structure. Then any tangent vector y at $x \in M$ is expressed as $y=y^i(\partial/\partial x^i)=v^a z_a$. The function L defined on $T(M)$-$\{0\}$ by*

$$L(x, y)=f(v) \qquad (v=v^a e_a,\ v^a=z'_i{}^a y^i) \tag{5.2}$$

does not depend on the choice of the local coordinate system and the adapted frame, and it gives globally a Finsler metric of M.

Definition 5.2. The Finsler metric given by Proposition 5.2 is called a $\{V, H\}$-*Finsler metric.*

A Finsler space (M, L) is called a $\{V, H\}$-*manifold* if L is a $\{V, H\}$-Finsler metric.

5.3. In a $\{V, H\}$-manifold (M, L), let Γ be a *G-connection relative to* the H-structure, then it holds $L_{|i}=0$ with respect to the Finlser connection $F(\Gamma)$ associated to Γ. Hence, by Theorem 4.2, a $\{V, H\}$-manifold is a generalized Berwald space. In the case that M is connected, the converse is also true. If a Finsler space F^n is a generalized Berwald space by a linear generalized Cartan connection $(\Gamma_j{}^i{}_k(x),\ y^j\Gamma_j{}^i{}_k, C_j{}^i{}_k)$, the F^n is a $\{V, H\}$-manifold, where H is the holonomy group of the linear connection $(\Gamma_j{}^i{}_k(x))$.

Theorem 5.1. *A $\{V, H\}$-manifold is a generalized Berwald space. Conversely, if M is connected, a generalized Berwald space (M, L) is a $\{V, H\}$-manifold whose $\{V, H\}$-Finsler metric coincides with L.*

A differentiable manifold M admitting an $\{e\}$-structure gives a simple example of a $\{V, H\}$-manifold.

Another example is given by a Minkowski space V with a Minkowski norm

$$f(v)=\left(\sum_{a=1}^{n}(v^a)^2\right)^{1/2}+kv^1\,, \tag{5.3}$$

where k is constant and $0<k<1$. Then G of (5.1) is $1\times O(n-1)$ and we have a Randers space.

Theorem 5.2. *Let M be an n-dimensional differentiable manifold. If M admits a $\{1\times O(n-1)\}$-structure, then M admits a Finsler metric such that*

$$L(x, y)=(a_{ij}(x)y^iy^j)^{1/2}+kb_i(x)y^i\,, \tag{5.4}$$

where $a_{ij}(x)$ is a Riemannian metric on M and $b_i(x)$ is a covariant vector field on M satisfying $a^{ij}b_ib_j=1$. Conversely, if M admits the above Finsler metric, then M is a $\{V, 1\times O(n-1)\}$-manifold.

5.4. The above stated notion of $\{V, H\}$-manifold [13, 14] followed from the consideration of a two-dimensional Finsler metric [5] given by

$$L(x^1, x^2, y^1, y^2)=(y^1+zy^2)^2/y^1 \qquad (z\in R)\,. \tag{5.5}$$

In order to treat such a non-restrictive Finsler metric, we can generalize the notion of $\{V, H\}$-manifold by defining a Minkowski norm f of V in a non-restrictive sense as a positie-valued differentiable function defined on an open set W of V-{0} satisfying the following conditions:

(i) If $v\in W$ then $\lambda v\in W$ for any $\lambda>0$, and $f(\lambda v)=\lambda f(v)$,

(ii) (g_{ab}) is regular,

(iii) There is a continuous function $\tilde{f}$ defined on a dense open set U of V containing W such that $f=\tilde{f}\,|\,W$.

Then the set

$$G=\{T\in GL(n,R)\,|\,\tilde{f}(Tv)=\tilde{f}(v) \text{ for any } v,\ Tv\in U\} \tag{5.6}$$

becomes a Lie group, too. Let H be a Lie subgroup of G. If an n-dimensional differentiable manifold M admits the H-structure, we can define a generalized Berwald space (M, L) in the way shown in Proposition 5.2 by $L(x, y)=f(v)$ $(v=v^a e_a$ for $y=v^a z_a)$.

For example, the Finsler metric (5.5) follows from the Minkowski norm given by

$$f(v)=(v^1+zv^2)^2/v^1 \qquad (z\in R), \tag{5.7}$$

where $W=U=\{v\in V\,|\,v^1\neq 0\}$, $f=\tilde{f}$ and G is given by

$$G=\left\{\begin{bmatrix} a & za(1-a) \\ 0 & a^2 \end{bmatrix}\,\middle|\, a\neq 0,\ a\in R\right\}. \tag{5.8}$$

Other interesting examples are obtained from the Minkowski norms of V given by the arithmetic, geometric and harmonic means of the components v^a of $v\in V$.

§6. The geometrical significance of a generalized Berwald space.

6.1. An interesting example [10] of a generalized Berwald space is obtained from an (α, β)-metric $L(\alpha, \beta)$, which is by definition [21] a (1) p-homogeneous function of $\alpha(x, y)=(a_{ij}(x)y^i y^j)^{1/2}$ and $\beta(x, y)=b_i(x)y^i$, where α is a Riemannian metric and b_i is a covariant vector field.

A Finsler space $F^n=(M, L(\alpha, \beta))$ has two metrics. The one is the Finsler metric itself, and the other is the Riemannian metric α. A linear connection $\Gamma=(\Gamma_j{}^i{}_k)$ of M is called to be metrical if it is metrical with respect to the latter: $\nabla_k a_{ij}=0$, where ∇_k denotes the covariant differentiation with respect to Γ. Let b_i be parallel with respect to a metrical linear connection Γ: $\nabla_k b_i=0$. With respect to the associated Finsler connection $F(\Gamma)$ we have from Proposition 4.2 that $a_{ij|k}=\nabla_k a_{ij}=0$, $b_{i|k}=\nabla_k b_i=0$, which implies $L_{|i}=0$. Thus we have from Theorem 4.2

Theorem 6.1. *If there exists in $F^n=(M, L(\alpha, \beta))$ a metrical linear connection Γ such that b_i is parallel with respect to Γ, the associated Finsler connection $F(\Gamma)$ is a linear generalized Cartan connection, and F^n becomes a generalized Berwald space.*

Especially, if Γ is semi-symmetric, $F(\Gamma)$ is a linear Wagner connection, and F^n becomes a Wagner space. If b_i is parallel with respect to the Riemannian connection determined by α, the F^n is a Berwald space.

The interest of geometry is in the classification theory. A generalized Berwald space offers a criterion of classification to get interesting models of Finsler spaces. And, it is noted that this theorem rose naturally from considerations of $\{V, H\}$-manifolds.

6.2. As is shown in [11], a generalized Berwald space plays an important role in the conformal theory of Finsler metrics [8].

Proposition 6.1. *Let a generalized Cartan connection $F\Gamma=(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ be* *given in a Finsler space* $F^n=(M, L)$. *If for a conformal change* $\bar{L}=e^{\sigma(x)}L$ *we put*

(6.1) $$\bar{F}_j{}^i{}_k = F_j{}^i{}_k+\delta_j^i\sigma_k\,, \quad \bar{N}^i{}_k = N^i{}_k+y^i\sigma_k\,, \quad \bar{C}_j{}^i{}_k = C_j{}^i{}_k\,,$$

where $\sigma_k=\partial_k\sigma$, *then* $\bar{F}\Gamma=(\bar{F}_j{}^i{}_k, \bar{N}^i{}_k, \bar{C}_j{}^i{}_k)$ *is a generalized Cartan connection of the Finsler space* $\bar{F}^n=(M, \bar{L})$.

The torsion tensor $T_j{}^i{}_k$ is changed as $\bar{T}_j{}^i{}_k=T_j{}^i{}_k+\delta_j^i\sigma_k-\delta_k^i\sigma_j$, but the other torsions C, R^1, P^1, S^1 and all curvatures R^2, P^2, S^2 are invariant for $\bar{F}\Gamma$ and $F\Gamma$.

It is noted that the invariabilities of R^1 and R^2 are due to the fact that $F\Gamma$ is p-homogeneous and σ_k is gradient. Since σ_k depend on position alone, we have

Theorem 6.2. *A generalized Berwald space (esp. a Wagner space) remains to be a generalized Berwald space (esp. a Wagner space) by any conformal change of Finsler metrics.*

Definition 6.1. A Wagner connection $W\Gamma$ (σ) is called a *σ-Wagner connection* if σ_k is a gradient vector field $\sigma_k=\partial_k\sigma$ of a fuction $\sigma(x)$. A Finsler space F^n is called a *σ-Wagener space*, if F^n becomes a Wagner space by a σ-Wagner connection.

If $F^n=(M, L)$ is a Berwald space, then $\bar{F}^n=(M, e^\sigma L)$ becomes a σ-Wagner space by $W\Gamma(\sigma)$. Conversely, if $F^n=(M, L)$ is a σ-Wagner space by $W\Gamma(\sigma)$, then $\bar{F}^n=(M, e^{-\sigma}L)$ is a Berwald space. Thus we have

Theorem 6.3. *A Finsler space* F^n *is conformal to a Berwald space, if and only if* F^n *becomes a σ-Wagner space.*

Since R^2 is invariant by (6.1), we have

Theorem 6.4. *A Finsler space* F^n *is conformal to a locally Minkowski space, if and only if* F^n *becomes a σ-Wagner space by a σ-Wagner connection whose curvature* R^2 *vanishes.*

A *locally Minkowski space* is by the original definition a Finsler space such that there exists a coordinate system (x^i) in which g_{ij} are functions of y^i alone, and is characterized as a Berwald space whose curvature R^2 vanishes. Tamássy-Matsumoto [31] proved directly Theorem 6.4 by the original definition.

The above theorems show that if we know a result about a Berwald space (resp. a locally Minkowski space), we can directly obtain a result about a space conformal to a Berwald space (resp. to a locally Minkowski space) in terms of a σ-Wagner space. For example, Hashiguchi-Varge [12] generalized a result (Numata [29] and Varga [32]) about a Berwald space of scalar curvature.

6.3. Could a fixed Finsler space (esp. Berwald space) become various generalized Berwald spaces or Wagner spaces ? In order to solve this difficult problem partially, Aikou-Hashiguchi [1] consider whether the paths in generalized Berwald spaces can coincide with the geodesics, and obtained

Theorem 6.5. *Let* F^n *be a generalized Berwald space by a generalized Cartan connection* $C\Gamma(T)=(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$. *Then the paths with respect to* $C\Gamma(T)$ *coincide with the geodesics of* F^n, *if and only if* F^n *is a Berwald space and* $C\Gamma(T)$ *is given by*

(6.2) $$F_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k + T_j{}^i{}_k/2\,, \quad N^i{}_k = G^i{}_k + T_0{}^i{}_k/2\,,$$

and $T_j{}^i{}_k$ *satisfy*

(6.3) $$Q_{ijs}{}^r T_r{}^s{}_k = 0\,,$$

where $Q_{ijs}{}^r = 2C_{ijs}y^r + g_{is}\delta_j^r + g_{js}\delta_i^r$ [16].

Theorem 6.6. *A Berwald space cannot become a non-trivial Wagner space in such a way that the paths coincide with the geodesics of* F^n.

Are there $T_j{}^i{}_k(x) \neq 0$ satisfying (6.3)? We have from (6.3)

(6.4) $$T_i{}^s{}_s = 0\,.$$

So, in the two-dimensional case, (6.4) implies $T_j{}^i{}_k = 0$.

6.4. In his appearing paper [24] Matsumoto finds all two-dimensional Wagner spaces as follows. Putting $z = y^2/y^1$ for a positive y^1, we have a function λ of x^1, x^2 and z by

(6.5) $$\lambda(x^1, x^2;\, z) = L(x^1, x^2, 1, y^2/y^1)\,,$$

which is called the *associated fundamental function* of F^n. Then I^2 and $\partial I/\partial\theta$ are expressed as follows:

(6.6) $$I^2 = 9(\lambda')^2/4\lambda\lambda'' + 3\lambda'\lambda'''/2(\lambda'')^2 + \lambda(\lambda''')^2/4(\lambda'')^3\,,$$

(6.7) $$2(\partial I/\partial\theta) = 3 - 3(\lambda')^2/2\lambda\lambda'' - \lambda'\lambda'''/(\lambda'')^2 - 3\lambda(\lambda''')^2/2(\lambda'')^3 + \lambda\lambda''''/(\lambda'')^2\,,$$

where $\lambda' = \partial\lambda/\partial z$ etc. Thus from Theorem 1.1 we have

Theorem 6.7. (Matsumoto) *The associated fundamental function* λ *of a two-dimensional Wagner space with* $\partial I/\partial\theta \neq 0$ *is given by an ordinary differential equation of fourth order.*

By specifying the above differential equations to be solved, various intersting examples of two-dimensional Wagner spaces have been obtained. For example, the differential equation obtained from $\partial I/\partial\theta = 3/2 - I^2/3$ gives all the Kropina metrics (1.3) as the solutions. Thus, every two-dimensional Finsler space with a Kropina metric is a Wagner space.

This research is significant in the sense that various fundamental functions with interesting character spring out concretely.

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