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QUADRATIC SPLINE APPROXIMATION FOR BOUNDARY VALUE PROBLEM

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Abstract

We consider an application of quadratic spline function to a numerical solution of two-point boundary value problem. We derive asymptotic expansion of the error which is of use for a posteriori improvement of the quadratic spline approximation and a mesh selection strategy (chopping procedure).

1. Introduction

We shall consider the numerical solution of the following two-point boundary value problem:

$$\begin{aligned} &x''(t)=f(t, x(t), x'(t)), \quad 0\le t\le 1 \\ &a_0x(0)-b_0x'(0)=c_0 \\ &a_1x(1)+b_1x'(1)=c_1 \end{aligned} \tag{1.1}$$

where $f(t, x, y)$ is defined and sufficiently smooth in a region D of (t, x, y)-space intercepted by two hyperplanes $t=0$ and $t=1$.

Here we rewrite (1.1) in the following form:

$$\begin{cases} x'(t)=y(t), & 0\le t\le 1 \\ y'(t)=f(t, x(t), y(t)), & 0\le t\le 1 \\ a_0x(0)-b_0y(0)=c_0 & \\ a_1x(1)+b_1y(1)=c_1. & \end{cases} \tag{1.2}$$

By making use of the B-spline $Q_{m+1}(t)$:

$$Q_{m+1}(t)=(1/m!)\sum_{i=0}^{m+1}(-1)^i\binom{m+1}{i}(t-i)_+^m \tag{1.3}$$

where

$$(t-i)_+^m=\begin{cases}(t-i)^m, & t>i \\ 0, & t\le i,\end{cases}$$

we consider spline functions of the form $x_h(t)=\sum_{i=-2}^{n-1}\alpha_iQ_3(t/h-i)$ and $y_h(t)=\sum_{i=-1}^{n-1}\beta_iQ_2(t/h-i)$ $(nh=1)$ with undetermined coefficients $(\alpha_{-2}, \alpha_{-1}, \dots, \alpha_{n-1}, \beta_{-1}, \beta_0, \dots, \beta_{n-1})$. The above x_h and y_h will be approximate solutions to the problem (1.2) if they satisfy

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$$(1.4)\qquad \begin{cases} x_h'(t)=y_h(t), & 0\le t\le 1\\ y_h'(t)=P[f(t,\ x_h(t),\ y_h(t))], & 0\le t\le 1\\ a_0x_h(0)-b_0y_h(0)=c_0 \\ a_1x_h(1)+b_1y_h(1)=c_1. \end{cases}$$

Here P is an operator defined by

$$(1.5)\qquad (Pg)(t)=\sum_{i=0}^{n-1} g_{i+1/2}\chi_i(t)$$

where $g_{i+1/2}=g((i+1/2)h)$ and $\chi_i(t)$ is the characteristic function on $[t_i,\ t_{i+1}]$ $(t_i=ih)$. For simplicity, let us denote $i=p,\ p+r,\ \dots,\ q\ (=p+(m-1)r)$ by $i=p(r)q$. Since $Q_3'(t)\equiv Q_2(t)-Q_2(t-1)$, from (1.4) we have a system of determining equations with respect to $(\alpha_i,\ \beta_i)$:

$$(1.6)\qquad \begin{cases} (1/h)(\alpha_i-\alpha_{i-1})=\beta_i, & i=-1(1)n-1\\ (1/h)(\beta_i-\beta_{i-1})=f(t_{i+1/2},\ (1/8)(\alpha_i+6\alpha_{i-1} & \\ \qquad\qquad +\alpha_{i-2}),\ (1/2)(\beta_i+\beta_{i-1})), & i=0(1)n-1\\ (1/2)a_0(\alpha_{-1}+\alpha_{-2})-b_0\beta_{-1}=c_0 & \\ (1/2)a_1(\alpha_{n-1}+\alpha_{n-2})+b_1\beta_{n-1}=c_1 & \end{cases}$$

where $t_{i+1/2}=(i+1/2)h$.

In practical computation, by eliminating β_i, $i=-1(1)\ n-1$, we have the system of nonlinear equations with only α_i, $i=-2(1)n-1$:

$$(1.7)\qquad \begin{cases} (1/h^2)(\alpha_i-2\alpha_{i-1}+\alpha_{i-2})=f(t_{i+1/2},\ (1/8)(\alpha_i & \\ \qquad\qquad +6\alpha_{i-1}+\alpha_{i-2}),\ 1/(2h)(\alpha_i-\alpha_{i-2})), & i=0(1)n-1\\ (1/2)a_0(\alpha_{-1}+\alpha_{-2})-(b_0/h)(\alpha_{-1}-\alpha_{-2})=c_0 & \\ (1/2)a_1(\alpha_{n-1}+\alpha_{n-2})+(b_1/h)(\alpha_{n-1}-\alpha_{n-2})=c_1. & \end{cases}$$

In the present paper we shall assume that the problem (1.2) has an isolated solution $(\hat{x},\ \hat{y})$ satisfying the internality condition

$$(1.8)\qquad U=\{(t,\ x,\ y)|\,|x-\hat{x}(t)|+|y-\hat{y}(t)|\le\delta,\ t\in[0,\ 1]\}\subset D \quad \text{for some } \delta>0.$$

The solution $(\hat{x},\ \hat{y})$ is isolated if and only if

(1.9) $G=A_0\Phi(0)+A_1\Phi(1)=A_0+A_1\Phi(1)$ is nonsingular where

$$(1.10)\qquad A_0=\begin{bmatrix} a_0 & -b_0\\ 0 & 0\end{bmatrix},\ A_1=\begin{bmatrix} 0 & 0\\ a_1 & b_1\end{bmatrix},$$

and $\Phi(t)$ is the solution of the first variation equation to (1.2) subject to the initial condition $\Phi(0)=E$ (E the unit matrix):

$$(1.11)\qquad \Phi'(t)=\begin{bmatrix} 0 & 1\\ \sigma_2(t) & \sigma_3(t)\end{bmatrix}\Phi(t)$$

$$(\sigma_2(t)=f_x(t,\ \hat{x}(t),\ \hat{y}(t)),\ \sigma_3(t)=f_y(t,\ \hat{x}(t),\ \hat{y}(t))).$$

In Section 2, we shall prove the following

THEOREM 1. *In a sufficiently small neighbourhood of the isolated solution* $(\hat{x},\ \hat{y})$ *of* (1.2), *there exists an approximate solution* $(\bar{x}_h,\ \bar{y}_h)$ *of* (1.4) *such that*

$$\|\hat{x}-\bar{x}_h\|=\max_{0\leq t\leq 1}|\hat{x}(t)-\bar{x}_h(t)|=O(h^2)$$
$$\|\hat{y}-\bar{y}_h\|=O(h^2). \tag{1.12}$$

In Section 3, by making use of this Theorem 1 we shall prove the asymptotic expansions of errors: $\hat{x}-\bar{x}_h$ and $\hat{y}-\bar{y}_h$.

THEOREM 2. *Under the same assumption of Theorem 1, we have*

$$\begin{aligned}\hat{x}(t)-\bar{x}_h(t)&=-(h^2/24)\theta(t)+O(h^3)\\ \hat{y}(t)-\bar{y}_h(t)&=-(h^2/24)\lambda(t)+(1/2)\hat{x}^{(3)}(t)\\ &\quad\times(t-t_i)(t-t_{i+1})+O(h^3)\\ &\quad t_i\leq t\leq t_{i+1}\end{aligned} \tag{1.13}$$

where (θ, λ) *is the solution of the following linear problem :*

$$\begin{aligned}\begin{bmatrix}\theta'\\ \lambda'\end{bmatrix}&=\begin{bmatrix}0 & 1\\ \sigma_2 & \sigma_3\end{bmatrix}\begin{bmatrix}\theta\\ \lambda\end{bmatrix}+\begin{bmatrix}2\hat{x}^{(3)}\\ -\hat{x}^{(4)}+3\sigma_3\hat{x}^{(3)}\end{bmatrix}\\ A_0\begin{bmatrix}\theta(0)\\ \lambda(0)\end{bmatrix}&+A_1\begin{bmatrix}\theta(1)\\ \lambda(1)\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}.\end{aligned} \tag{1.14}$$

In Section 4, we shall notice that (θ, λ) is approximately determined with very little additional computation by using the already obtained approximation $(\bar{x}_h, \bar{y}_h)$. Hence we may consider the application of the above stated asymptotic expansion (1.13) to a posteriori improvement of the approximate solution and a mesh selection strategy (chopping procedure).

2. Proof of Theorem 1

In what follows, we shall denote the vector and matrix maximum norms by $\|\cdot\|$. In this section, by using Newton-Kantorovitch's theorem we shall prove the existence and convergence of a solution $(\alpha_{-2}, \alpha_{-1}, \ldots, \alpha_{n-1}, \beta_{-1}, \ldots, \beta_{n-1})$ of (1.6):

$$\begin{aligned}F_i(\alpha, \beta)&=(1/h)(\alpha_i-\alpha_{i-1})-\beta_i, && i=-2(1)n-1\\ G_{-1}(\alpha, \beta)&=(1/2)a_0(\alpha_{-1}+\alpha_{-2})-b_0\beta_{-1}-c_0\\ G_i(\alpha, \beta)&=(1/h)(\beta_i-\beta_{i-1})-f(t_{i+1/2}, (1/2)(\alpha_i && i=0(1)n-1\\ &\quad+\alpha_{i-1}), (1/2)(\beta_i+\beta_{i-1})),\\ G_n(\alpha, \beta)&=(1/2)a_1(\alpha_{n-1}+\alpha_{n-2})+b_1\beta_{n-1}-c_1.\end{aligned} \tag{2.1}$$

For simplicity, let us denote

$$(F(\alpha, \beta), G(\alpha, \beta))=(F_{-2}(\alpha, \beta), \ldots, F_{n-1}(\alpha, \beta), G_{-1}(\alpha, \beta), \ldots, G_n(\alpha, \beta)).$$

Corresponding to $(\hat{x}, \hat{y})$, one can determine piecewise quadratic and linear functions $\hat{x}_h$ and $\hat{y}_h$ of the form:

$$\hat{x}_h(t)=\sum_{i=-2}^{n-1}\hat{\alpha}_i Q_3(t/h-i)$$
$$\hat{y}_h(t)=\sum_{i=-1}^{n-1}\hat{\beta}_i Q_2(t/h-i)\quad(\equiv\hat{x}'_h(t))$$

so that

$$(2.3)\qquad \begin{aligned} \hat{x}_h(t_{i+1/2}) &= \hat{x}(t_{i+1/2}), & i&=0(1)n-1 \\ \hat{x}_h(t_i) &= \hat{x}(t_i), & i&=0,\ n. \end{aligned}$$

Since $\hat{x}(t)$ is sufficiently smooth due to the assumption that $f(t, x, y)$ is sufficiently smooth, it is valid that

$$(2.4)\qquad \begin{aligned} &\|\hat{x}^{(k)}-\hat{x}_h^{(k)}\| = O(h^2), \quad k=0,\ 1 \\ &\hat{x}''(t_{i+1/2})-\hat{x}_h''(t_{i+1/2}) = O(h^2) \qquad ([4]). \end{aligned}$$

Hence we have the estimate of $\|F(\hat{\alpha}, \hat{\beta}),\ G(\hat{\alpha}, \hat{\beta}))\|$ of the form

$$(2.5)\qquad \|F(\hat{\alpha}, \hat{\beta}),\ G(\hat{\alpha}, \hat{\beta}))\| = O(h^2).$$

Next, in order to investigate of the property of the Jacobian matrix $J(\hat{\alpha}, \hat{\beta})$ of $(F(\alpha, \beta),\ G(\alpha, \beta))$ with respect to (α, β), we consider a linear system:

$$(2.6)\qquad J(\hat{\alpha}, \hat{\beta})(\xi_1,\ \xi_2) = (\gamma_1,\ \gamma_2)$$

where $\xi_1 = (u_{-2},\ u_{-1},\ \ldots,\ u_{n-1})$, $\xi_2 = (v_{-1},\ v_0,\ \ldots,\ v_{n-1})$, $\gamma_1 = (c_{-1},\ c_0,\ \ldots,\ c_{n-1})$ and $\gamma_2 = (d_{-1},\ d_0,\ \ldots,\ d_n)$.

Corresponding to ξ_1 and ξ_2, we consider quadratic and linear functions $y_1(t)$ and $y_2(t)$ defined by

$$(2.7)\qquad \begin{aligned} y_1(t) &= \sum_{i=-2}^{n-1} u_i Q_3(t/h-i) \\ y_2(t) &= \sum_{i=-1}^{n-1} v_i Q_2(t/h-i), \end{aligned}$$

and in a similar way, corresponding to γ_1 and γ_2, we consider linear and step functions $\phi_1(t)$ and $\phi_2(t)$ defined by

$$(2.8)\qquad \begin{aligned} \phi_1(t) &= \sum_{i=-1}^{n-1} c_i Q_2(t/h-i) \\ \phi_2(t) &= \sum_{i=0}^{n-1} d_i \chi_i(t). \end{aligned}$$

Hence, corresponding to (2.6), we have

$$(2.9)\qquad \begin{cases} y_1'(t) = y_2(t) + \phi_1(t), & 0 \le t \le 1 \\ a_0 y_1(0) - b_0 y_2(0) = d_{-1} & \\ y_2'(t_{i+1/2}) = \hat{\sigma}_2(t_{i+1/2}) y_1(t_{i+1/2}) + \hat{\sigma}_3(t_{i+1/2}) y_2(t_{i+1/2}) + \phi_2(t_{i+1/2}), & \\ \qquad\qquad i=0(1)n-1 & \\ a_1 y_1(1) + b_1 y_2(1) = d_n & \end{cases}$$

$$(\ \hat{\sigma}_2(t) = f_x(t,\ \hat{x}_h(t),\ \hat{y}_h(t)),\ \hat{\sigma}_3(t) = f_y(t,\ \hat{x}_h(t),\ \hat{y}_h(t))\ ).$$

Since $y_2'(t)$ and $\phi_2(t)$ are both step functions, from above we have

$$(2.10)\qquad \begin{cases} y_1'(t) = y_2(t) + \phi_1(t) \\ y_2'(t) = P[\hat{\sigma}_2(t) y_1(t) + \hat{\sigma}_3(t) y_2(t)] + \phi_2(t) \\ a_0 y_1(0) - b_0 y_2(0) = d_{-1} \\ a_1 y_1(1) + b_1 y_2(1) = d_n. \end{cases}$$

Here we rewrite the second equation of (2.10) as follows:

$$(2.11)\qquad y_2'(t) = \sigma_2(t) y_1(t) + \sigma_3(t) y_2(t) + r(t) + \phi_2(t)$$

where $r(t) = -(I-P)(\hat{\sigma}_2 y_1 + \hat{\sigma}_3 y_2) + (\hat{\sigma}_2 - \sigma_2) y_1 + (\hat{\sigma}_3 - \sigma_3) y_2$.

Since $\|(I-P)g\|_{[t_i, t_{i+1}]} = O(h)\|g'\|$ for any $g \in C[0,\ 1]$,

$$\|r\|_{[t_i, t_{i+1}]} = O(h)[\|y_1\| + \|y_1'\| + \|y_2\| + \|y_2'\|_{[t_i, t_{i+1}]}]$$

$$= O(h)[\|y_1\|+\|y_2\|+\|\phi_1\|+\|\phi_2\|+\|y_2'\|_{[t_i,t_{i+1}]}].$$

By (2.11), we have

(2.12) $$\|r\|_{[t_i,t_{i+1}]}=O(h)[\|y_1\|+\|y_2\|+\|\phi_1\|+\|\phi_2\|] \quad i=0(1)n-1 \quad \text{for} \quad h\leq h_0$$

provided that h_0 is sufficiently small.

By (2.10) and (2.11), we have

(2.13) $$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \Phi G^{-1}\begin{bmatrix} d_{-1} \\ d_n \end{bmatrix} + \int_0^1 H(\cdot, s)\begin{bmatrix} \phi_1 \\ r+\phi_2 \end{bmatrix} ds$$

where

(2.14) $$H(t, s)=\begin{cases} \Phi(t)[E-G^{-1}A_1\Phi(1)]\Phi^{-1}(s), & s\leq t \\ -\Phi(t)G^{-1}A_1\Phi(1)\Phi^{-1}(s), & t<s \end{cases} \quad ([5], [11]).$$

From above, we have the inequality of the form

(2.15) $$\|y_1\|, \|y_2\|\leq C\{\|d_{-1}, d_n)\|+\|r\|+\|\phi_1\|+\|\phi_2\|\}$$

where C is a generic constant independent of h, and $\|r\|=\max_i \|r\|_{[t_i,t_{i+1}]}$.

By the use of (2.12) and (2.15), we obtain the inequality of the form

(2.16) $$\|y_1\|, \|y_2\|\leq C\{\|(d_{-1}, d_n)\|+\|\phi_1\|+\|\phi_2\|\} \quad \text{for} \quad h\leq h_1 \ (\leq h_0)$$

provided that h_1 is sufficiently small.

By a simple calculation, we have

(2.17) $$\|y_1\|\geq C\|\xi_1\|, \ \|y_2\|\geq C\|\xi_2\|$$
$$\|\phi_1\|\leq C\|\gamma_1\|, \ \|\phi_2\|\leq C\|\gamma_2\| \quad ([11]).$$

Therefore we finally have the inequality of the form

(2.18) $$\|(\xi_1, \xi_2)\|\leq C\|(\gamma_1, \gamma_2)\| \quad \text{for any} \quad h\leq h_1.$$

By (2.6), inequality (2.18) implies the nonsingularity of $J(\hat{\alpha}, \hat{\beta})$ and in addition the inequality

(2.19) $$\|J^{-1}(\hat{\alpha}, \hat{\beta})\|\leq C \quad \text{for} \quad h\leq h_1.$$

By (2.4), let us note that there exists a constant h_2 such that

(2.20) $$\|\hat{x}-\hat{x}_h\|+\|\hat{y}-\hat{y}_h\|\leq \delta_0<\delta \quad \text{for any} \quad h\leq h_2$$

Now let us define the set $\Omega_h(\hat{\alpha}, \hat{\beta})$ which is a neighbourhood of $(\hat{\alpha}, \hat{\beta})$:

(2.21) $$\Omega_h(\hat{\alpha}, \hat{\beta})=\{(\alpha, \beta) \mid \|\alpha-\hat{\alpha}\|+\|\beta-\hat{\beta}\|\leq \delta-\delta_0\}.$$

Then for

(2.22) $$x_h(t)=\sum_{i=-2}^{n-1} \alpha_i Q_3(t/h-i)$$
$$y_h(t)=\sum_{i=-1}^{n-1} \beta_i Q_2(t/h-i)$$

with $(\alpha, \beta)\in\Omega_h(\hat{\alpha}, \hat{\beta})$, we have

$$\|\hat{x}-x_h\|+\|\hat{y}-y_h\|\leq\|\hat{x}-\hat{x}_h\|+\|\hat{y}-\hat{y}_h\|+\|\hat{x}_h-x_h\|+\|\hat{y}_h-y_h\|\leq\delta-\delta_0+\delta_0=\delta$$

consequently $(t, x_h(t), y_h(t))\in U$ for any $h\leq h_2$.

Here we have used the following results:

$$0 \leq Q_{m+1}(t) \leq 1$$

$$\sum_{i=-\infty}^{\infty} Q_{m+1}(t-i)=1$$

i.e., $$\|\hat{x}_h - x_h\| \leq \|\hat{\alpha}-\alpha\|, \ \|\hat{y}_h - y_h\| \leq \|\hat{\beta}-\beta\|.$$

This means that $(F(\alpha, \beta), G(\alpha, \beta))$ is defined on the region $\Omega_h(\hat{\alpha}, \hat{\beta})$ for any $h \leq h_2$. Hence by means of the mean-value theorem we have

$$\|J(\alpha_1, \beta_1)-J(\alpha_2, \beta_2)\| \leq C\|(\alpha_1-\alpha_2, \beta_1-\beta_2)\| \quad \text{for } (\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \Omega_h(\hat{\alpha}, \hat{\beta}). \tag{2.23}$$

The expressions (2.5), (2.19) and (2.23) show that the conditions of Newton-Kantorovitch's theorem are fulfilled for sufficiently small $h(\leq \min(h_1, h_2))$. Thus we see that the determining equation $(F(\alpha, \beta), G(\alpha, \beta))=0$ has one and only one solution $(\alpha, \beta)=(\bar{\alpha}, \bar{\beta})$ in a neighbourhood of $(\hat{\alpha}, \hat{\beta})$ ([8]).

3. Proof of Theorem 2

Let $e_1=\hat{x}-\bar{x}_h$ and $e_2=\hat{y}-\bar{y}_h$, then in virtue of Theorem 1 we have

$$\begin{cases} e_1'=e_2 \\ e_2'=\sigma_2 e_1+\sigma_3 e_2+(I-P)\hat{x}''+r+O(h^4) \\ a_0 e_1(0)-b_0 e_2(0)=0 \\ a_1 e_1(1)+b_1 e_2(1)=0 \end{cases} \tag{3.1}$$

where

$$r=-(I-P)(\sigma_2 e_1+\sigma_3 e_2).$$

Hence we have

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \int_0^1 H(\cdot, s)\begin{bmatrix} 0 \\ (I-P)\hat{x}''(s) \end{bmatrix} ds + \int_0^1 H(\cdot, s)\begin{bmatrix} 0 \\ r(s) \end{bmatrix} ds + O(h^4). \tag{3.2}$$

Now we shall prove the asymptotic expansion at mesh point $t=t_j$:

$$\int_0^1 H(t, s)\begin{bmatrix} 0 \\ (I-P)\hat{x}''(s) \end{bmatrix} ds = -(h^2/24)\left\{\int_0^1 H(t, s)\begin{bmatrix} 0 \\ \hat{x}^{(4)}(s) \end{bmatrix} ds - 2Z(t)\begin{bmatrix} 0 \\ \hat{x}^{(4)}(1) \end{bmatrix} + 2W(t)\begin{bmatrix} 0 \\ \hat{x}^{(4)}(0) \end{bmatrix} + \begin{bmatrix} 0 \\ \hat{x}^{(3)}(t) \end{bmatrix}\right\} + O(h^4), \tag{3.3}$$

where

$$Z(t)=-\Phi(t)G^{-1}A_1 \quad \text{and} \quad W(t)=\Phi(t)[E-G^{-1}A_1\Phi(1)].$$

Let $K(t, s)=H_{i,2}(t, s)$, $i=1, 2$ of the $(i, 2)$-component of $H(t, s)$. Since $K(t, s)$ is sufficiently smooth except $t=s$, by Taylor series expansion we have for $j=1(1)n-1$

$$\begin{aligned} \int_0^1 K(t_j, s)(I-P)\hat{x}''(s)ds &= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \{K(t_j, t_{i+1/2}) \\ &\quad + K'(t_j, t_{i+1/2})(s-t_{i+1/2})+ \ldots\}\{\hat{x}_{i+1/2}^{(3)}(s-t_{i+1/2}) \\ &\quad + (1/2)\hat{x}_{i+1/2}^{(4)}(s-t_{i+1/2})^2+ \ldots\}ds \\ &= \sum_{i=0}^{n-1} \{(h^3/24)\hat{x}_{i+1/2}^{(4)}K(t_j, t_{i+1/2})+(h^3/12)\hat{x}_{i+1/2}^{(3)}K'(t_j, t_{i+1/2})\}+O(h^4) \\ &= (h^2/24)\int_0^1 K(t_j, s)\hat{x}^{(4)}(s)ds+(h^2/12)\int_0^1 K'(t_j, s)\hat{x}^{(3)}(s)ds+O(h^4)=(*), \end{aligned} \tag{3.4}$$

(3.5)
$$(*) = -(h^2/24)\int_0^1 K(t_j, s)\hat{x}^{(4)}(s)ds + (h^2/12)[K(t_j, 1) \times \hat{x}^{(3)}(1) - \{K(t_j, t_j+) - K(t_j, t_j-)\}\hat{x}^{(3)}(t_j) - K(t_j, 0)\hat{x}^{(3)}(0)] + O(h^4)$$

where $K'(t, s)$ means the partial derivative of $K(t, s)$ with respect to the second variable s.

Since

$$H_{i2}(t_j, t_j+) - H_{i2}(t_j, t_j-) = -\delta_{i2}$$
$$H(t_j, 1) = -\Phi(t_j)G^{-1}A_1 = Z(t_j)$$
$$H(t_j, 0) = \Phi(t_j)[E - G^{-1}A_1\Phi(1)] = W(t_j),$$

we have the desired result for $t = t_j$, $j = 1(1)n-1$.

For $t=1$, by (3.4) we have

(3.6)
$$\int_0^1 K(1, s)(I-P)\hat{x}''(s)ds = -(h^2/24)\int_0^1 K(1, s)\hat{x}^{(4)}(s)ds + (h^2/12)\{K(1, 1-)\hat{x}^{(3)}(1) - K(1, 0)\hat{x}^{(3)}(0)\} + O(h^4)$$

where

$$H(1, 1-) = \Phi(1)[E - G^{-1}A_1\Phi(1)]\Phi^{-1}(1) = E + Z(1)$$
$$H(1, 0) = \Phi(1)[E - G^{-1}A_1\Phi(1)]\Phi^{-1}(0) = W(1)$$

from which follows the desired result for $t=1$. Similarly we have the desired asymptotic expansion for $t=0$. This completes the proof of equation (3.3) at any mesh point.

Next we shall consider the second term of (3.2). Let $\tau = \sigma_2 e_1 + \sigma_3 e_2$, then for $K(t, s) = H_{i2}(t, s)$

(3.6)
$$\int_0^1 K(t_j, s)r(s)ds = -\int_0^1 K(t_j, s)(I-P)\tau(s)ds$$
$$= -\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \{K(t_j, t_{i+1/2}) + K'(t_j, t_{i+1/2})(s - t_{i+1/2}) + \ldots\}\{\tau'_{i+1/2}(s - t_{i+1/2}) + (1/2)\tau''_{i+1/2}(s - t_{i+1/2})^2 + \ldots\}ds$$

Since $\bar{x}_h$ is piecewise quadratic, in virtue of Theorem 1 we have

(3.7)
$$\tau'_{i+1/2} = \sigma_3(t_{i+1/2})e'_2(t_{i+1/2}) + O(h^2) = O(h^2)$$
$$\tau''_{i+1/2} = \sigma_3(t_{i+1/2})e''_2(t_{i+1/2}) + O(h^2) = \sigma_3(t_{i+1/2})\hat{x}^{(3)}_{i+1/2} + O(h^2)$$

where

$$e'_2(t_{i+1/2}) = x''_{i+1/2} - f(t_{i+1/2}, \bar{x}_h(t_{i+1/2}), \bar{y}_h(t_{i+1/2}))$$
$$= f(t_{i+1/2}, x_{i+1/2}, y_{i+1/2}) - f(t_{i+1/2}, \bar{x}_h(t_{i+1/2}), \bar{y}_h(t_{i+1/2}))$$
$$= O(h^2).$$

Combining (3.6) and (3.7) yields

(3.8)
$$\int_0^1 K(t_j, s)r(s)ds = -(h^3/24)\sum_{i=0}^{n-1} K(t_j, t_{i+1/2})\sigma_3(t_{i+1/2})\hat{x}^{(3)}_{i+1/2} + O(h^4)$$
$$= -(h^2/24)\int_0^1 K(t_j, s)\sigma_3(s)\hat{x}^{(3)}(s)ds + O(h^4).$$

Thus we have

(3.9)
$$\int_0^1 H(t_j, s)\begin{bmatrix} 0 \\ r(s) \end{bmatrix} ds = -(h^2/24)\int_0^1 H(t_j, s)\begin{bmatrix} 0 \\ \sigma_3(s)\hat{x}^{(3)} \end{bmatrix} ds + O(h^4).$$

By (3.3) and (3.9), we have

$$\begin{bmatrix} e_1(t_j) \\ e_2(t_j) \end{bmatrix} = -(h^2/24)\{\int_0^1 H(t_j, s)\begin{bmatrix} 0 \\ \hat{x}^{(4)}(s)+\sigma_3\hat{x}^{(3)}(s) \end{bmatrix} ds$$
$$-2Z(t_i)\begin{bmatrix} 0 \\ \hat{x}^{(4)}(1) \end{bmatrix} + 2W(t_j)\begin{bmatrix} 0 \\ \hat{x}^{(4)}(0) \end{bmatrix} - 2\begin{bmatrix} 0 \\ \hat{x}^{(3)}(t_j) \end{bmatrix}\}$$
$$+O(h^4) = -(h^2/24)\begin{bmatrix} \theta(t_j) \\ \lambda(t_j) \end{bmatrix} + O(h^4). \tag{3.10}$$

By a simple calculation, (θ, λ) is shown to be the solution of (1.14). This completes the proof of Theorem 2 at any mesh point.

By Taylor series expansion, we have

$$\begin{aligned} \hat{x}_{i+1/2} &= (1/2)(\hat{x}_{i+1}+\hat{x}_i)-(h/8)(\hat{x}'_{i+1}-\hat{x}'_i)+O(h^4) \\ \hat{x}'_{i+1/2} &= (1/2)(\hat{x}'_{i+1}+\hat{x}'_i)-(h^2/8)\hat{x}^{(3)}_{i+1/2}+O(h^4) \end{aligned} \tag{3.11}$$

from which follow

$$\begin{aligned} e_1(t_{i+1/2}) &= -(h^2/24)\theta(t_{i+1/2})+O(h^4) \\ e_2(t_{i+1/2}) &= -(h^2/24)\lambda(t_{i+1/2})-(h^2/8)\hat{x}^{(3)}_{i+1/2}+O(h^4). \end{aligned} \tag{3.12}$$

Since $\bar{x}_h(t)$ and $\bar{y}_h(t)$ are piecewise quadratic and linear,

$$\begin{aligned} \bar{x}_h(t) &= (1/h^2)\{2\bar{x}_h(t_i)(t-t_{i+1/2})(t-t_{i+1})-4\bar{x}_h(t_{i+1/2}) \\ &\quad (t-t_i)(t-t_{i+1})-2\bar{x}_h(t_{i+1})(t-t_i)(t-t_{i+1/2})\} \\ \bar{y}_h(t) &= (1/h)\{\bar{y}_h(t_i)(t_{i+1}-t)+\bar{y}_h(t_{i+1})(t-t_i)\}. \end{aligned} \tag{3.13}$$

Hence, in virtue of (3.10) at mesh point and (3.12) at mid point, we have the desired asymptotic expansions at any point $t \in [0, 1]$. This completes the proof of Theorem 2.

4. Computation of Principal Part of the Errors

By the definition of $H(t, s)$, we have

$$\begin{cases} \theta'' = \sigma_2\theta+\sigma_3\theta'+\hat{x}^{(4)}+\sigma_3\hat{x}^{(3)} \\ a_0\theta(0)-b_0\theta'(0) = -2b_0\hat{x}_0^{(3)} \\ a_1\theta(1)+b_1\theta'(1) = 2b_1\hat{x}_n^{(3)}. \end{cases} \tag{4.1}$$

Here we notice the following asymptotic expansions for other numerical methods to two point boundary value problems.

PEMARK 1 ([5]). *If we apply the box scheme to the problem* (1.2), *under the same assumption of Theorem 1 there exists a solution* $(\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_n, \tilde{y}_0, \ldots, \tilde{y}_n)$ *of the box scheme so that*

$$\hat{x}(t)-\sum_{i=0}^{n}\tilde{x}_iQ_2(t/h-i+1) = -(h^2/24)\nu(t)+O(h^3) \tag{4.2}$$

where

$$\begin{cases} \nu'' = \sigma_2\nu+\sigma_3\nu'+\hat{x}^{(4)}+\sigma_3\hat{x}^{(3)}+3\sigma_2\hat{x}'' \\ a_0\nu(0)-b_0\nu'(0) = -2b_0\hat{x}_0^{(3)} \\ a_1\nu(1)+b_1\nu'(1) = 2b_1\hat{x}_n^{(3)}. \end{cases} \tag{4.3}$$

By making use of the B-spline $Q_4(t)$, we consider a cubic spline of the form

$$x_h(t)=\sum_{i=-3}^{n-1} a_i Q_4(t/h-i).$$

Then the above x_h is an approximate solution to (1.1) if it satisfies

$$\begin{cases} x_h''=P_j[f(t,\ x_h,\ x_h')], & 0\le t\le 1 \\ a_0x_h(0)-b_0x_h'(0)=c_0 \\ a_1x_h(1)+b_1x_h'(1)=c_1, \end{cases}$$

where P_j, $j=1, 2$, are operators defined by

(i) $(P_1g)(t)=\sum_{i=0}^{n} g_iQ_2(t/h-i+1)$

(ii) $(P_2g)(t)=\sum_{n-1}^{n} \gamma_iQ_2(t/h-i+1)$

$$\begin{cases} \Delta^r\gamma_0=0, & r\ge 4 \\ (1/6)(\gamma_{i+1}+4\gamma_i+\gamma_{i-1})=g, & i=1(1)n-1 \\ \nabla^r\gamma_n=0 \end{cases}$$

(Δ and ∇ are forward and backward difference operators, respectively).
By using the same argument in Sections 2 and 3, we have

REMARK 2. *Under the same assumption of Theorem 1, there exist approximate solutions $\bar{x}_{h,i}$, $i=1, 2$ of (4.4) according to P_i, $i=1, 2$, respectively :*

(4.5)
$$\hat{x}(t)-\bar{x}_{h,i}(t)=(-1)^i(h^2/12)\rho(t)+O(h^3)$$

where

(4.6)
$$\begin{cases} \rho''=\sigma_2\rho+\sigma_3\rho'+\hat{x}^{(4)} \\ a_0\rho(0)-b_0\rho'(0)=0 \\ a_1\rho(1)+b_1\rho'(1)=0. \end{cases}$$

Now, in order to compute the principal part of the error θ, we consider an approximate problem to (4.1):

(4.7)
$$\begin{cases} \theta_h''=P[\sigma_2\theta_h+\sigma_3\theta_h+g] \\ a_0\theta_h(0)-b_0\theta_h'(0)=2\mu_0 \\ a_1\theta_h(1)+b_1\theta_h'(1)=2\mu_1. \end{cases}$$

Since the problem (4.1) has the (isolated) solution $\theta(t)$, by using again the same argument in Section 2 there exists a solution $\hat{\theta}_h(t)$ of (4.7) for sufficiently small h so that

(4.8)
$$\|\theta^{(k)}-\bar{\theta}_h^{(k)}\|\le C\max[|\mu_0+b_0\hat{x}_0^{(3)}|,\ |\mu_1-b_1\hat{x}_n^{(3)}|,$$
$$\max_{0\le i\le n-1}|g_{i+1/2}-\hat{x}_{i+1/2}^{(4)}-\sigma_3(t_{i+1/2})\hat{x}_{i+1/2}^{(3)}|],\quad k=0,\ 1.$$

Thus, in order to define a computable approximate problem for $\theta(t)$, we shall require $O(h^2)$ approximations to $\hat{x}_0^{(3)}$, $\hat{x}_n^{(3)}$, $\hat{x}_{i+1/2}^{(4)}$, $\hat{x}_{i+1/2}^{(3)}$, $i=0(1)n-1$.
Here, by the definition of P and (3.12) we have

(4.9)
$$\hat{x}_{i+1/2}''-\bar{x}_h''(t_{i+1/2})=-(h^2/24)(\sigma_2\theta+\sigma_3\lambda+3\sigma_3\hat{x}^{(3)})|_{t=t_{i+1/2}}$$
$$+O(h^4),\quad i=0(1)n-1.$$

From above, we have

(i) $\hat{x}_0^{(3)}=(1/h)\{-2\bar{x}_h''(t_{1/2})+3\bar{x}_h''(t_{3/2})-\bar{x}_h''(t_{5/2})\}+O(h^2),$

$$\hat{x}_n^{(3)} = \ldots,$$

(ii) $\hat{x}_{1/2}^{(4)} = (1/h^2)\{2\bar{x}_h''(t_{1/2}) - 5\bar{x}_h'')t_{3/2}) + 4\bar{x}_h''(t_{5/2}) - \bar{x}_h''(t_{7/2})\} + O(h^2),$

$$\hat{x}_{n-1/2}^{(3)} = \ldots,$$

(4.10) (iii) $\hat{x}_{i+1/2}^{(4)} = (1/h^2)\{\bar{x}_h''(t_{i+3/2}) - 2\bar{x}_h''(t_{i+1/2}) + \bar{x}_h''(t_{i-1/2})\} + O(h^2), \quad i=1(1)n-2,$

(iv) $\hat{x}_{1/2}^{(3)} = (1/2h)\{-3\bar{x}_h''(t_{1/2}) + 4\bar{x}_h''(t_{5/2}) - \bar{x}_h''(t_{7/2})\} + O(h^2),$

$$\hat{x}_{n-1/2}^{(3)} = \ldots,$$

(v) $\hat{x}_{i+1/2}^{(3)} = (1/2h)\{\bar{x}_h''(t_{i+3/2}) - \bar{x}_h''(t_{i-1/2})\} + O(h^2), \quad i=1(1)n-2.$

Let $\bar{\sigma}_2(t) = f_x(t, \bar{x}_h(t), \bar{x}_h'(t))$ and $\bar{\sigma}_3(t) = f_y(t, \bar{x}_h(t), \bar{x}_h'(t))$ which have been already computed in the process of the Newton method for determining $\bar{x}_h$, then in virtue of Theorem 1

$$\|\sigma_2 - \bar{\sigma}_2\|,\ \|\sigma_3 - \bar{\sigma}_3\| = O(h^2). \tag{4.11}$$

By combining (4.10) and (4.11), we have a posteriori computable approximate problem to (4.1):

$$\begin{cases} \theta_h'' = P[\bar{\sigma}_2\theta_h + \bar{\sigma}_3\theta_h' + \bar{g}_h] \\ a_0\theta_h(0) - b_0\theta_h'(0) = 2\bar{\mu}_0 \\ a_1\theta_h(1) + b_1\theta_h'(1) = 2\bar{\mu}_1 \end{cases} \tag{4.12}$$

where $\bar{g}_h(t_{i+1/2})$, $i=0(1)n-1$, $2\bar{\mu}_0$, $2\bar{\mu}_1$ are approximations to $\hat{x}_{i+1/2}^{(4)} + \sigma_3(t_{i+1/2})\hat{x}_{i+1/2}^{(3)}$, $i=0(1)n-1$, $-2b_0\hat{x}_0^{(3)}$ and $2b_1\hat{x}_n^{(3)}$ obtained by (4.10). Since the problem (4.7) has the solution $\bar{\theta}_h$, in virtue of (4.11), the problem (4.12) has a solution $\tilde{\theta}_h$ so that

$$\|\theta^{(k)} - \tilde{\theta}_h^{(k)}\| = O(h^2), \quad k=0,\ 1 \tag{4.13}$$

for sufficiently small h.

By using $\tilde{\theta}_h$, we have a posteriori improved approximations:

$$\begin{aligned} &\hat{x}_i - \{\bar{x}_h(t_i) - (h^2/24)\tilde{\theta}_h(t_i)\} = O(h^4), && i=0(1)n \\ &\hat{x}_i' - \{\bar{y}_h(t_i) - (h^2/24)\tilde{\theta}_h'(t_i) + (h^2/12)\tilde{x}_i^{(3)}\} = O(h^4), && i=0(1)n \end{aligned} \tag{4.14}$$

where $\tilde{x}_i^{(3)} = (1/h)\{\bar{x}_h''(t_{i+1/2}) - \bar{x}_h''(t_{i-1/2})\}$, $i=1(1)n-1$ and $\tilde{x}_0^{(3)}$, $\tilde{x}_n^{(3)}$ by 4.10 (i).

Here we remark that the coefficient matrix of (4.12) for determining $\tilde{\theta}_h$ is exactly the same one of the Newton method at the final stage by which we determine the approximate solution $\bar{x}_h(t)$. That is., we may compute $\tilde{\theta}_h^{(k)}(t)$, $k=0$, 1 with very little additional effort.

5. Mesh Selection Strategy (Chopping Procedure)

In this section, we shall consider chopping procedure applied to two-point boundary value problem by Russell and Christansen ([9]). Our procedure uses only uniform meshes at each step, which can be automatically refined in order to reduce the (estimated) error below a requested tolerance. It behaves quiet adequately for various problems whose solutions have sharp gradients. If $\tilde{\theta}_h(t)$ satisfies the inequality:

$$\begin{aligned} &(h^2/24)|\tilde{\theta}_h(t)| \le \varepsilon \quad (\varepsilon \text{ a desired tolerance}) \\ &\text{for}\quad t \in [0,\ a],\ [b,\ 1] \quad (a,\ b \text{ mesh points}), \end{aligned} \tag{5.1}$$

we chop off intervals [0, a] and [b, 1] and consider the following new approximate problem on the remaining interval [a, b] with $h := h/2$

(5.2) $$\begin{cases} x_h'' = P[t, x_h(t), x_h'(t)], \quad a \le t \le b \\ x_h(a) = \bar{x}_h(a) - (h^2/24)\tilde{\theta}_h(a) \\ x_h(b) = \bar{x}_h(b) - (h^2/24)\tilde{\theta}_h(b). \end{cases}$$

By (4.14), we probably have

(5.3) $$\begin{aligned} &|\hat{x}(a) - \{\bar{x}_h(a) - (h^2/24)\tilde{\theta}_h(a)\}| \ll |\hat{x}(a) - \bar{x}_h(a)| \le \varepsilon \\ &|\hat{x}(b) - \{\bar{x}_h(b) - (h^2/24)\tilde{\theta}_h(b)\}| \ll |\hat{x}(b) - \bar{x}_h(b)| \le \varepsilon \end{aligned}$$

from which the boundary conditions of (5.2) are considered to be suitable ones. While inequalities of (5.3) are not assured theoretically for any h, no numerical difficulties are encountered (see Examples 1 and 2 in Section 6).

If $\tilde{\theta}_h(t)$ satisfies the inequality:

(5.4) $$(h^2/24)|\tilde{\theta}_h(t)| \le \varepsilon \quad \text{for} \quad t \in [a, b] \ (a, b \text{ mesh points}),$$

then we consider the following two problems on [0, a] and [b, 1] with $h := h/2$;

(5.5) $$\begin{cases} x_h'' = P[f(t, x_h(t), x_h'(t))], \quad 0 \le t \le a \\ a_0 x_h(0) - b_0 x_h'(0) = c_0 \\ x_h(a) = x_h(a) - (h^2/24)\tilde{\theta}_h(a); \end{cases}$$

(5.6) $$\begin{cases} x_h'' = P[f(t, x_h(t), x_h'(t))], \quad b \le t \le 1 \\ x_h(b) = x_h(b) - (h^2/24)\tilde{\theta}_h(b) \\ a_1 x_h(1) + b_1 x_h'(1) = c_1. \end{cases}$$

Continuating these processes would yield the approximate solution $\bar{x}_h$ such that

(5.7) $$\|\hat{x} - \{\bar{x}_h - (h^2/24)\tilde{\theta}_h\}\| \ll \|\hat{x} - \bar{x}_h\| \le \varepsilon$$

for sufficiently small h.

6. Numerical Illustration

In this section we shall consider the application of the above stated asymptotic expansion to a posteriori improvement of spline approximations of solutions of two point boundary value problems and mesh selection strategy. Numerical results conform the theoretical accuracies established in previous sections. The rates of decrease of the errors $O(h^\alpha)$, where α are computed from the results from $h=1/16$ to $1/32$, are given in parentheses in each Tables.

As our examples, we choose

Problem 1.

$$x'' = (1/2)\exp(-t)(x^2 + x'^2)$$
$$x(0) = 1, \ x(1) = e.$$

Problem 2.

The same equation in Problem 1 subject to the boundary conditions:

$$x(0) - x'(0) = 0$$
$$x(1) + x'(1) = 2e.$$

The exact solutions of the above problems are exp (t).

Table 6.1 The observed maximum errors in function values.

h	1/16	1/32	(α)
Prob. 1	0.614−4* → 0.674−7	0.154−4 → 0.435−8	2.0 → 4.0
Prob. 2	0.246−3 → 0.657−6	0.616−4 → 0.464−7	2.0 → 3.8

* We denote 0.614×10^{-4} by 0.614-4.

Table 6.2 The observed maximum errors in derivatives.

h	1/16	1/32	(α)
Prob. 1	0.588−3 → 0.241−5	0.147−3 → 0.161−6	2.0 → 3.9
Prob. 2	0.284−3 → 0.232−5	0.710−4 → 0.154−6	2.0 → 3.9

In the following Tables, the left and right hand sides of (...)→(...) mean $\max_{0\leq i\leq n}|\hat{x}_i^{(k)}-\bar{x}_h^{(k)}(t_i)|$, $k=0$, 1 and $\max_{0\leq i\leq n}|\hat{x}_i-\{\bar{x}_h(t_i)-(h^2/24)\tilde{\theta}_h(t_i)\}|$, $\max_{0\leq i\leq n}|\hat{x}_i'-[\hat{x}_h(t_i)-(h^2/24)\tilde{\theta}_h'(t_i)+(h^2/12)(1/h)\{\bar{x}_h''(t_{i+1/2})-\hat{x}_h''(t_{i-1/2})\}]|$, respectively.

The above stated method is also applicable to the numerical solution of the nonlinear boundary value problem having the singularity at $t=0$:

$$x''+(\kappa/t)x'+f(t,x)=0,\quad 0<t\leq1$$
$$x'(0)=0\quad\text{and}\quad x(1)=c_1$$

with $\kappa=0$, 1, 2, respectively.

While Theorems 1-2 are not assured for the above problem, no numerical difficulties are encountered.

Problem 3. We treat the nonlinear problem:

$$x''+(2/t)x'+x^5=0,\ c_1=\sqrt{3}/2.$$

The unique solution is $1/\sqrt{1+t^2/3}$.

Problem 4. Consider another nonlinear problem:

$$x''+(1/t)x'+\exp(x)=0,\quad 0<t\leq1$$
$$c_1=0.$$

The solutions are $x(t)=2\times\ln[(B+1)/(Bt^2+1)]$, where $B=3\pm2\sqrt{3}$. In the following

Table 6.3 The observed maximum errors in function values.

h	1/16	1/32	(α)
Prob. 3	0.834−5 → 0.422−7	0.209−5 → 0.248−8	2.0 → 4.1
Prob. 4	0.589−4 → 0.196−6	0.147−4 → 0.134−7	2.0 → 3.9

Table 6.4 The observed maximum errors in derivatives.

h	1/16	1/32	(α)
Prob. 3	0.159−3 → 0.868−6	0.398−4 → 0.503−7	2.0 → 4.1
Prob. 4	0.202−3 → 0.593−6	0.505−4 → 0.292−7	2.0 → 4.3

Table, we only list up numerical results for the smaller solution.

Now we consider the application of chopping procedure to the following problems in which we take a desired tolerance $\varepsilon=10^{-4}$ and $h=1/32$ as starting mesh sizes.

Problem 5.

$$10^{-4}x''+(1-1/2\,t)x'-(1/2)x=0, \quad x(0)=0 \quad \text{and} \quad x(1)=1.$$

The exact solution is approximately $1/(2-t)$ on $(0,1]$ and has a boundary layer of thickness 10^{-4} at $t=0$.

Problems 6 and 7. We consider Troesch's equation:

Table 6.5 Remaining subintervals.

	Prob. 5	Prob. 6	Prob. 7
	$[0, \frac{32}{32}]$	$[1-\frac{19}{32}, 1]$	$[1-\frac{12}{32}, 1]$
	$[0, \frac{64}{64}]$	$[1-\frac{31}{64}, 1]$	$[1-\frac{21}{64}, 1]$
	$[0, \frac{128}{128}]$	$[1-\frac{48}{128}, 1]$	$[1-\frac{36}{128}, 1]$
	$[0, \frac{97}{256}]$	$[1-\frac{69}{256}, 1]$	$[1-\frac{64}{256}, 1]$
	$[0, \frac{47}{512}]$	$[1-\frac{84}{512}, 1]$	$[1-\frac{105}{512}, 1]$
	$[0, \frac{21}{1024}]$	$[1-\frac{72}{1024}, 1]$	$[1-\frac{164}{1024}, 1]$
	$[0, \frac{11}{2048}]$	$[1-\frac{39}{2048}, 1]$	$[1-\frac{234}{2048}, 1]$
	$[0, \frac{5}{4096}]$	$[1-\frac{8}{4096}, 1]$	$[1-\frac{296}{4096}, 1]$
	$[0, \frac{7}{8192}]$		$[1-\frac{607}{8192}, 1]$
	$[0, \frac{11}{16384}]$		$[1-\frac{388}{16384}, 1]$
	$[0, \frac{17}{32768}]$		$[1-\frac{534}{32768}, 1]$
	$[0, \frac{21}{65536}]$		$[1-\frac{877}{65536}, 1]$
			$[1-\frac{1610}{131072}, 1]$
			$[1-\frac{3220}{262144}, 1]$
			$[1-\frac{6373}{524288}, 1]$
h	2^{-16}	2^{-12}	2^{-19}
N	256	168	496

* h is the smallest mesh size and N is the maximum number of partitions of the remaining subintervals.

Here we notice that we have to solve at least one time a linear system of order $N+2$.

$$x'' = k \sinh kx, \quad x(0) = 0 \quad \text{and} \quad x(1) = 1.$$

The solution has a singularity for $t < 1$ for values of $x'(0)$ slightly greater than its real value. This creats problems for any shooting technique. The results for $k = 10$ and 20, using identically zero starting guess, are shown in Table 6.5. We have obtained 147..... and 21875...... as approximate values to $\hat{x}'(1)$ for $k = 10$ and 20, respectively.

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