SEMI-SYMMETRIC MIRON CONNECTIONS IN DUAL FINSLER SPACES

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SEMI-SYMMETRIC MIRON CONNECTIONS IN DUAL FINSLER SPACES

Dedicated to Professor Dr. Radu Miron on the occasion of his sixtieth birthday

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Abstract

The purpose of the present paper is to give a dual theory, corresponding to the theory of semi-symmetric metrical Finsler connections in Finsler spaces given in R. Miron-M. Hashiguchi [6].

Introduction

In his recent papen [5], R. Miron studied a dual space (M, H) of a Finsler space (M, L), and established a beautiful dual theory of Finsler geomtry. All properties of such a space (M, H) are based on the existence of one canonical d-connection $M^*\Gamma$. Since these concepts were obviously made explicit by him for the first time, we hope to call (M, H) a *Miron space* and $M^*\Gamma$ a *Miron connection.*

In the present paper we shall define semi-symmetric metrical d-connections called *semi-symmetric Miron connections* in a Miron space, and study the group of transformations of these connections and its invariants. As the results of these considerations, we have two important d-tensor fields $L_j{}^i{}_{kl}$ and $M_j{}^{ikl}$ which are invariants of semi-symmetric Miron connections (Theorem 5.2), and consider some of their properties.

The terminology and notaions follow those in Miron [5], with some modifications (e. g., *Cartan space* → *Miron space*, $N_{ir} \rightarrow N_{ri}$, $H\Gamma \rightarrow F^*\Gamma$, $H^i_{jk} \rightarrow F_j{}^i{}_k$, $R_{jkr} \rightarrow R_{rjk}$, etc.). For convenience' sake, in the first two preliminary sections we shall sketch the materials necessary for our discussions from the theory of Miron [5].

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1. Distinguished geomerical objects on the cotangent bundle

Let M be an n-dimensional C^∞-manifold and (T^*M, π^*, M) its cotangent bundle.

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Since a point of T^*M is a covector (x, p) at a point x of the base manifold M, a coordinate system $x=(x^i)$ in M induces a canonical coordinate system $(x, p)=(x^i, p_i)$ in T^*M by $p=p_i(dx^i)$.

Let V be the vertical distribution on T^*M given by $(x, p)\in T^*M \rightarrow V_{(x,p)} =\{X\in T_{(x,p)}T^*M \mid \pi^{*'}X=0\}$. A *non-linear connection* N on T^*M is a distribution of class C^∞ given by $(x, p)\in T^*M \rightarrow N_{(x,p)}\subset T_{(x,p)}T^*M$ such that $T_{(x,p)}T^*M=N_{(x,p)}\oplus V_{(x,p)}$. N is characterized by

$$\delta_i=\partial_i+N_{ri}\dot{\partial}^r, \tag{1.1}$$

where $\partial_i=\partial/\partial x^i$, $\dot{\partial}^i=\partial/\partial p_i$. $\{\delta_i, \dot{\partial}^i\}$ is a local basis adapted to the distributions N and V, and its dual basis is $\{dx^i, \delta p_i\}$, where

$$\delta p_i=dp_i-N_{ir}dx^r. \tag{1.2}$$

A *distinguished tensor field*, a *d-tensor field* for short, of type (r, s) on M is defined by its components, satisfying the classical law of transformations with respect to a transformation of canonical coordinate systems in T^*M:

$$\tilde{x}^i=\tilde{x}^i(x^1, \cdots, x^n),\ \tilde{p}_i=(\partial x^r/\partial\tilde{x}^i)p_r;\ \det(\partial\tilde{x}^i/\partial x^r)\neq 0. \tag{1.3}$$

A *d-connection* on M is a triad $F^*\Gamma=(N, F, C)$, where $N=(N_{jk})$ is a non-linear connection, $F=(F_j{}^i{}_k)$ is a special d-object and $C=(C_i{}^{jk})$ is a d-tensor field of type (2, 1). The transformation formulas of N_{jk}, $F_j{}^i{}_k$, $C_i{}^{jk}$ with respect to (1.3) are as follows:

$$\begin{aligned}\tilde{N}_{jk}&=(\partial x^s/\partial\tilde{x}^j)(\partial x^t/\partial\tilde{x}^k)N_{st}+p_r(\partial^2x^r/\partial\tilde{x}^j\partial\tilde{x}^k),\\ \tilde{F}_j{}^i{}_k&=(\partial\tilde{x}^i/\partial x^r)(\partial x^s/\partial\tilde{x}^j)(\partial x^t/\partial\tilde{x}^k)F_s{}^r{}_t+(\partial\tilde{x}^i/\partial x^r)(\partial^2x^r/\partial\tilde{x}^j\partial\tilde{x}^k),\\ \tilde{C}_i{}^{jk}&=(\partial x^r/\partial\tilde{x}^i)(\partial\tilde{x}^j/\partial x^s)(\partial\tilde{x}^k/\partial x^t)C_r{}^{st}.\end{aligned} \tag{1.4}$$

Given a d-connection, the h- and v-*covariant derivatives* are defined for d-tensor field, e. g., K^i_j, by

$$K^i_{j|k}=\delta_kK^i_j+K^r_jF_r{}^i{}_k-K^i_rF_j{}^r{}_k,\quad K^i_j|^k=\dot{\partial}^kK^i_j+K^r_jC_r{}^{ik}-K^i_rC_j{}^{rk}. \tag{1.5}$$

The five *torsion* and three *curvature tensor fields* of a d-connection are given by

$$T_j{}^i{}_k=\mathfrak{A}_{jk}\{F_j{}^i{}_k\},\quad C_i{}^{jk}, \tag{1.6}$$

$$R_{ijk}=-\mathfrak{A}_{jk}\{\delta_kN_{ij}\},\quad P_{ij}{}^k=-(\dot{\partial}^kN_{ij}-F_i{}^k{}_j),\quad S_i{}^{jk}=-\mathfrak{A}_{jk}\{C_i{}^{jk}\}, \tag{1.7}$$

$$R_j{}^i{}_{kl}=\mathfrak{A}_{kl}\{\delta_lF_j{}^i{}_k+F_j{}^r{}_kF_r{}^i{}_l\}+C_j{}^{ir}R_{rkl}, \tag{1.8}$$

$$P_j{}^i{}_k{}^l=\dot{\partial}^lF_j{}^i{}_k-C_j{}^{il}{}_{|k}+C_j{}^{ir}P_{rk}{}^l, \tag{1.9}$$

$$S_j{}^{ikl}=\mathfrak{A}_{kl}\{\dot{\partial}^lC_j{}^{ik}+C_j{}^{rk}C_r{}^{il}\}, \tag{1.10}$$

where $\mathfrak{A}_{jk}\{\cdots\}$ denotes the alternate summation.

2. Miron spaces and metrical connections

We shall consider a C^∞-function $H(x, p)$ defined in $T^*_0M=\{(x, p)\in T^*M \mid p\neq 0\}$, and we put

(2.1) $$g^{ij}=(\dot\partial^i\dot\partial^j H)/2.$$

The pair (M, H) is called a *Miron space*, if H satisfies the following conditions:

(i) H is $(2)p$-homogeneous: $H(x, \lambda p)=\lambda^2 H(x, p)$ for $\lambda>0$,

(ii) g^{ij} is non-degenerate: $\det(g^{ij})\neq 0$.

The H and g^{ij} are called the *fundamental function* and the *fundamental metric tensor field* of (M, H) respectively. We put $p^i=g^{ir}p_r$ and $(g_{ij})=(g^{ij})^{-1}$.

It is noted that a Miron space differs with a so-called Cartan space [2], where p_i is not a covector but a covector density.

In a Miron space (M, H) there exists a canonical d-connection determined by H only. A d-connection is called *metrical* if $g^{ij}{}_{|k}=0$, $g^{ij}|^k=0$. Let $\gamma_j{}^i{}_k$ be the Christoffel symbols formed with respect to g_{ij}.

Theorem 2.1. *In a Miron space there exists a unique metrical d-connection $F^*\Gamma$ satisfying $D_{jk}=-(p_r F_j{}^r{}_k-N_{jk})$, $T_j{}^i{}_k=0$, $S_i{}^{jk}=0$. If we denote this $F^*\Gamma$ by $M^*\Gamma=(\overset{m}{N}, \overset{m}{F}, \overset{m}{C})$, then it is given by*

(2.2) $$\overset{m}{N}_{jk}=p_r\gamma_j{}^r{}_k-p_r p^t\gamma_s{}^r{}_t(\dot\partial^s g_{jk})/2,$$

(2.3) $$\overset{m}{F}_j{}^i{}_k=g^{ir}(\delta_k g_{jr}+\delta_j g_{kr}-\delta_r g_{jk})/2,$$

(2.4) $$\overset{m}{C}_i{}^{jk}=-g_{ir}(\dot\partial^k g^{jr})/2.$$

The above d-connection $M^*\Gamma$ was first obtained by Miron [5] as a dual notion of the well-known Cartan connection of a Finsler space (cf. [3, 4]). So we shall call $M^*\Gamma$ the *Miron connection* and also $\overset{m}{N}$ the *Miron non-linear connection*.

When we discuss d-connections with a fixed non-linear connection N, a d-connection is denoted by $F^*\Gamma(N)=(F, C)$. The set of all metrical d-connections $F^*\Gamma(\overset{m}{N})$ was given in the following form by Miron [5], and was first used in [1], together with $M^*\Gamma$.

Theorem 2.2. *In a Miron space there exists a unique metical d-connection $F^*\Gamma(\overset{m}{N})=(F, C)$ having the torsion tensor fields $T_j{}^i{}_k$, $S_i{}^{jk}$ given a priori. It is given by*

(2.5) $$F_j{}^i{}_k=\overset{m}{F}_j{}^i{}_k+g^{ir}(g_{rs}T_j{}^s{}_k-g_{js}T_r{}^s{}_k+g_{ks}T_j{}^s{}_r)/2,$$

(2.6) $$C_i{}^{jk}=\overset{m}{C}_i{}^{jk}-g_{ir}(g^{rs}S_s{}^{jk}-g^{js}S_s{}^{rk}+g^{ks}S_s{}^{jr})/2.$$

3. Semi-symmetric Miron connections and their group of transformations

We denote the set of all d-covector fields and the set of all d-vector fields in M by $X^*(M)$ and $X(M)$ respectively.

A d-connection on M is called *semi-symmetric if* the torsion tensor fields $T_j{}^i{}_k$, $S_i{}^{jk}$ have the form

(3.1) $$T_j{}^i{}_k=\sigma_j\delta^i_k-\sigma_k\delta^i_j,\quad S_i{}^{jk}=-(\tau^j\delta^k_i-\tau^k\delta^j_i),$$

where $\sigma_j \in X^*(M)$, $\tau^j \in X(M)$.

The Miron connection $M^*\Gamma$ on a $\underset{m}{\text{Miron}}$ space is considered as a special semi-symmetric $\underset{m}{\text{metrical}}$ d-connection $F^*\Gamma(N)$. We call a semi-symmetric metrical d-connection $F^*\Gamma(N)$ a *semi-symmetric Miron connection*. From Theorem 2.2 and (3.1) we have

Theorem 3.1. *In a Miron space the set of all semi-symmetic Miron connections* (N, F, C) *is given by*

(3.2) $$N_{jk} = \overset{m}{N}_{jk},\ F_j{}^i{}_k = \overset{m}{F}_j{}^i{}_k + \sigma_j \delta^i_k - g_{jk}\sigma^i,\ C_i{}^{jk} = \overset{m}{C}_i{}^{jk} + \tau^j \delta^k_i - g^{jk}\tau_i,$$

where $\sigma_j \in X^*(M)$, $\tau^j \in X(M)$ *are arbitrary, and* $\sigma^i = g^{ir}\sigma_r$, $\tau_i = g_{ir}\tau^r$.

Hence two semi-symmetric Miron connections (N, F, C), $(\bar{N}, \bar{F}, \bar{C})$ are related in the form

(3.3) $$\bar{N}_{jk} = N_{jk},\ \bar{F}_j{}^i{}_k = F_j{}^i{}_k + s_j \delta^i_k - g_{jk}s^i,\ \bar{C}_i{}^{jk} = C_i{}^{jk} + t^j \delta^k_i - g^{jk}t_i,$$

where $s_j = \bar{\sigma}_j - \sigma_j$, $t^j = \bar{\tau}^j - \tau^j$, and $s^i = g^{ir}s_r$, $t_i = g_{ir}t^r$.

Conversely, given $s_j \in X^*(M)$, $t^j \in X(M)$, the above (3.3) is thought to be a transformation of d-connections preserving the non-linear connection. Then it transforms a semi-symmetic Miron connection to a semi-symmetric Miron one. We shall denote this transformation by $\mathfrak{t}(s_j, t^j)$. Let $\overset{s}{T}$ be the set of all such transformations. In $\overset{s}{T}$ a product is defined by the mapping product $\mathfrak{t}(\bar{s}_j, \bar{t}^j) \circ \mathfrak{t}(s_j, t^j) = \mathfrak{t}(s_j + \bar{s}_j, t^j + \bar{t}^j)$, and we have

Theorem 3.2. *The set* $\overset{s}{T}$ *of all transformations* $\mathfrak{t}(s_j, t^j)$ *given by* (3. 3), *together with the mapping product, is an Abelian group. This group acts on the set of all* $\underset{m}{d}$*-connections effectively, and it acts on the set of all semi-symmetric Miron connections* $F^*\Gamma(N)$ *transitively. The group* $\overset{s}{T}$ *is the direct product of its subgroups* $\overset{s}{T}_C = \{\mathfrak{t}(s_j, 0) \in \overset{s}{T}\}$ *and* $\overset{s}{T}_F = \{\mathfrak{t}(0, t^j) \in \overset{s}{T}\}$: $\overset{s}{T} = \overset{s}{T}_C \times \overset{s}{T}_F$.

4. Various transformation formulas

In order to find invariants of the group $\overset{s}{T}$, we shall first treat a transformation of general d-connections, preserving the non-linear connection:

(4.1) $$\bar{N}_{jk} = N_{jk},\ \bar{F}_j{}^i{}_k = F_j{}^i{}_k - B_j{}^i{}_k,\ \bar{C}_i{}^{jk} = C_i{}^{jk} - D_i{}^{jk}.$$

Proposition 4.1. *By a transformation* (4.1) *of* d*-connections the torsion and curvature tensor fields are transformed as follows*:

(4.2) $$\bar{T}_j{}^i{}_k = T_j{}^i{}_k - \mathfrak{A}_{jk}\{B_j{}^i{}_k\},\ \bar{C}_i{}^{jk} = C_i{}^{jk} - D_i{}^{jk},$$

(4.3) $$\bar{R}_{ijk} = R_{ijk},\ \bar{P}_{ij}{}^k = P_{ij}{}^k - B_i{}^k{}_j,\ \bar{S}_i{}^{jk} = S_i{}^{jk} + \mathfrak{A}_{jk}\{D_i{}^{jk}\},$$

(4.4) $$\bar{R}_j{}^i{}_{kl} = R_j{}^i{}_{kl} - B_j{}^i{}_r T_k{}^r{}_l - D_j{}^{ir} R_{rkl} - \mathfrak{A}_{kl}\{B_j{}^i{}_{k|l} - B_j{}^r{}_k B_r{}^i{}_l\},$$

(4.5) $$\bar{P}_j{}^i{}_k{}^l = P_j{}^i{}_k{}^l - B_j{}^i{}_r C_k{}^{rl} - D_j{}^{ir} P_{rk}{}^l - (B_j{}^i{}_k|^l - B_j{}^r{}_k D_r{}^{il}) + (D_j{}^{il}{}_{|k} - D_j{}^{rl} B_r{}^i{}_k),$$

(4.6) $$\bar{S}_j{}^{ikl}=S_j{}^{ikl}-D_j{}^{ir}S_r{}^{kl}-\mathfrak{A}_{kl}\{D_j{}^{ik}|^{l}-D_j{}^{rk}D_r{}^{il}\}.$$

If we eliminate R_{rkl} from (4.4), we have

Proposition 4.2. *The d-tensor field defined by*

(4.7) $$K_j{}^i{}_{kl}=R_j{}^i{}_{kl}-C_j{}^{ir}R_{rkl}$$

is transformed by the transformation (4.1) *as follows*:

(4.8) $$\bar{K}_j{}^i{}_{kl}=K_j{}^i{}_{kl}-B_j{}^i{}_rT_k{}^r{}_l-\mathfrak{A}_{kl}\{B_j{}^i{}_{k|l}-B_j{}^r{}_kB_r{}^i{}_l\}.$$

In a Miron space, let us consider the d-tensor fields

(4.9) $$\Omega^{hi}_{jk}=(\delta^h_j\delta^i_k-g^{hi}g_{jk})/2,\ \Omega^{*hi}_{jk}=(\delta^h_j\delta^i_k+g^{hi}g_{jk})/2,$$

which have the well-known properties and play the same role as the operators Λ_1, Λ_2 given by M. Obata [7] (cf. [6]).

Now, we shall treat the transformation (3.3) of semi-symmetric Miron connections. Then $B_j{}^i{}_k$, $D_i{}^{jk}$ in (4.1) are expressed as $B_j{}^i{}_k=-2\Omega^{ri}_{jk}s_r$, $D_i{}^{jk}=-2\Omega^{jk}_{ri}t^r$. From Proposition 4.1 we have

Proposition 4.3. *The d-tensor fields* $T_j{}^i{}_k$, $C_i{}^{jk}$, $S_{ij}{}^k$, $P_i{}^{jk}$, $K_j{}^i{}_{kl}$, $S_j{}^i{}_{kl}$ *of a semi-symmetric Miron connection are transfomed by the transformation* (3.3) *as follows*:

(4.10) $$\bar{T}_j{}^i{}_k=T_j{}^i{}_k+2\mathfrak{A}_{jk}\{\Omega^{ri}_{jk}s_r\},\ \bar{C}_i{}^{jk}=C_i{}^{jk}+2\Omega^{jk}_{ri}t^r,$$

(4.11) $$\bar{S}_i{}^{jk}=S_i{}^{jk}-2\mathfrak{A}_{jk}\{\Omega^{jk}_{ri}t^r\},\ \bar{P}_{ij}{}^k=P_{ij}{}^k+2\Omega^{rk}_{ij}s_r,$$

(4.12) $$\bar{K}_j{}^i{}_{kl}=K_j{}^i{}_{kl}+2\mathfrak{A}_{kl}\{\Omega^{ri}_{jk}s_{rl}\},$$

(4.13) $$\bar{S}_j{}^{ikl}=S_j{}^{ikl}+2\mathfrak{A}_{kl}\{\Omega^{ik}_{rj}t^{rl}\},$$

where

(4.14) $$s_{rl}=s_{r|l}-s_rs_l-s_r\sigma_l+(s/2)g_{rl};\ s=g^{rs}s_rs_s,$$

(4.15) $$t^{rl}=t^r|^l-t^rt^l+t^r\tau^l+(t/2)g^{rl};\ t=g_{rs}t^rt^s.$$

5. Invariants of semi-symmetric Miron connections

We have the following important invariants from Proposition 4.3 by the well-known elimination method [4], [6].

Theorem 5.1. *The following d-tensor fields of semi-symmetric Miron connections are invariants of the group* $\underset{S}{T}$:

(5.1) $$T_j{}^i{}_k-\mathfrak{A}_{jk}\{T_j\delta^i_k\}/(n-1),\ C_i{}^{jk}-2\Omega^{jk}_{ri}C^r/(n-1),\ \tilde{C}^k,$$

(5.2) $$R_{ijk},\ S_i{}^{jk}-\mathfrak{A}_{jk}\{S^j\delta^k_i\}/(n-1),\ P_{ij}{}^k-2\Omega^{rk}_{ij}P_r/(n-1),\ \tilde{P}_j,$$

where $T_j=T_j{}^i{}_i$, $C^j=C_i{}^{ji}$, $S^j=S_i{}^{ji}$, $P_i=P_{ij}{}^j$, and $\tilde{C}^k=C_i{}^{ik}$, $\tilde{P}_j=P_{ij}{}^i$.

Theorem 5.2. *For $n>2$ the following d-tensor fields of semi-symmetric Miron connections are invariants of the group $\overset{s}{T}$:*

$$L_j{}^i{}_{kl}=K_j{}^i{}_{kl}+2\mathfrak{A}_{kl}\{\Omega^{ri}_{jk}(K_{rl}-g_{rl}K/2(n-1))\}/(n-2), \tag{5.3}$$

$$M_j{}^{ikl}=S_j{}^{ikl}+2\mathfrak{A}_{kl}\{\Omega^{ik}_{rj}(S^{rl}-g^{rl}S/2(n-1))\}/(n-2), \tag{5.4}$$

where $K_{jk}=K_j{}^i{}_{ki}$, $S^{ik}=S_j{}^{ikj}$, $K=g^{jk}K_{jk}$ and $S=g_{ik}S^{ik}$.

We shall show some properties of the invariants $L_j{}^i{}_{kl}$, $M_j{}^{ikl}$.

Theorem 5.3. *In a Miron space the invariants $L_j{}^i{}_{kl}$, $M_j{}^{ikl}$ of semi-symmetric Miron connections $F^*\overset{m}{\Gamma}(N)$ have the following properties:*

$$\Omega^{*si}_{jr}L_s{}^r{}_{kl}=-\overset{m}{C}_j{}^{is}R_{skl};\qquad \Omega^{*si}_{jr}M_s{}^{rkl}=0, \tag{5.5}$$

$$L_i{}^i{}_{kl}=-\tilde{C}^sR_{skl},\ L_j{}^i{}_{ki}=0;\qquad M_i{}^{ikl}=0,\ M_j{}^{ikj}=0, \tag{5.6}$$

$$\mathfrak{S}_{jkl}\{L_j{}^i{}_{kl}\}=-\tilde{C}^s\mathfrak{S}_{jkl}\{\delta^i_jR_{skl}\}/(n-2);\qquad \mathfrak{S}_{ikl}\{M_j{}^{ikl}\}=0, \tag{5.7}$$

where $\tilde{C}^s=S^s+C^s$, and $\mathfrak{S}_{jkl}\{\cdots\}$ denotes the cyclic summation.

For the proof it is sufficient to study the properties of $\overset{m}{L}_j{}^i{}_{kl}$, $\overset{m}{M}_j{}^{ikl}$ formed with respect to the Miron connection $M^*\Gamma$.

As is shown in [5], from the metrical property of $M^*\Gamma$ we have $\Omega^{*si}_{jr}\overset{m}{R}_s{}^r{}_{kl}=0$, $\Omega^{*si}_{jr}\overset{m}{S}_s{}^{rkl}=0$, from which we have

$$\Omega^{*si}_{jr}\overset{m}{K}_s{}^r{}_{kl}=-\overset{m}{C}_j{}^{is}\overset{m}{R}_{skl},\quad \overset{m}{K}_i{}^i{}_{kl}=-\overset{m}{C}_i{}^{is}\overset{m}{R}_{skl},\quad \overset{m}{S}_i{}^{ikl}=0. \tag{5.8}$$

On the other hand, from the Bianchi identities $\mathfrak{S}_{jkl}\{\overset{m}{K}_j{}^i{}_{kl}\}=0$, $\mathfrak{S}_{ikl}\{\overset{m}{S}_j{}^{ikl}\}=0$, we have

$$\mathfrak{A}_{kl}\{\overset{m}{K}_{kl}\}=\overset{m}{C}_i{}^{is}\overset{m}{R}_{skl},\quad \mathfrak{A}_{kl}\{\overset{m}{S}{}^{kl}\}=0. \tag{5.9}$$

Thus the proof follows in the same way as in [6, p. 34].

We shall finally give an example of a Miron space satisfying $L_j{}^i{}_{kl}=0$, $M_j{}^{ikl}=0$.

Theorem 5.4. *If the Miron connection $M^*\Gamma$ has the properties of h- and v-isotropy:*

$$\overset{m}{K}_j{}^i{}_{kl}=h(\delta^i_lg_{jk}-\delta^i_kg_{jl}),\quad \overset{m}{S}_j{}^{ikl}=v(\delta^l_jg^{ik}-\delta^k_jg^{il}), \tag{5.10}$$

where $h(x,p)$, $v(x,p)$, are arbitrary functions in T^*_0M, then for any semi-symmetric Miron connection $F^*\overset{m}{\Gamma}(N)$ we have

$$L_j{}^i{}_{kl}=0,\ M_j{}^{ikl}=0. \tag{5.11}$$

Indeed, $\overset{m}{L}_j{}^i{}_{kl}=0$, $\overset{m}{M}_j{}^{ikl}=0$ follow from the respective conditions of (5.10).

It would be an interesting problem to characterize the Miron spaces satisfying $\overset{m}{L}_j{}^i{}_{kl}=0$, $\overset{m}{M}_j{}^{ikl}=0$. On the other hand, we can also treat semi-symmtric metrical d-connection such that deflection tensor fields D_{jk} vanish, and consider some special Miron spaces.

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