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Abstract

In the present paper, we discuss the condition that any Finsler connection in a Finsler space be linear, and especially give the characterization of Cartan type.

Introduction

Let (M, L) be a Finsler space, where M is a differentiable manifold and $L(x, y)$ $(y^i = \dot{x}^i)$ is a Finsler metric function on M. The fundamental tensor field g_{ij} is given by $g_{ij} = (\dot{\partial}_i \dot{\partial}_j L^2)/2$, where $\dot{\partial}_i = \partial/\partial y^i$. We shall express a Finsler connection $F\Gamma$ in terms of its coefficients as $F\Gamma = (F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$. Various distinguished tensor fields are defined as follows: $U_{ijk} = (g_{ij|k})/2$, $D^i{}_k = y^j F_j{}^i{}_k - N^i{}_k$, $P^i{}_{jk} = \dot{\partial}_k N^i{}_j - F_k{}^i{}_j$, $T_j{}^i{}_k = F_j{}^i{}_k - F_k{}^i{}_j$ and $P_j{}^i{}_{kl} = \dot{\partial}_l F_j{}^i{}_k - C_j{}^i{}_{l|k} + C_j{}^i{}_m P^m{}_{kl}$, where a short bar denotes the h-covariant differentiation.

A Finsler connection $F\Gamma$ is called *linear* if the coefficients $F_j{}^i{}_k$ depend on position alone: $\dot{\partial}_l F_j{}^i{}_k = 0$, since then $(F_j{}^i{}_k)$ defines a linear connection on M.

A Finsler space is called a *Berwald space* if the Berwald connection is linear. A Berwald space is also defined as a Finsler space whose Cartan connection is linear, and is characterized by the well-known condition $C_{ijk\|l} = 0$, where $C_{ijk} = (\dot{\partial}_k g_{ij})/2$. Suggested by Wagner [9], the author generalized the notion of the Cartan connection, and gave a characterization of linear *generalized Cartan connections* (cf. Hashiguchi [2, Theorem 1], Hashiguchi-Ichijyō [3, Theorem 3]). Recently, Prasad, Shukla and Singh treated *h-recurrent Finsler connections*, and obtained the condition that such a Finsler connection be linear (cf. [7, Theorem 1.1], [8, Theorem 3.1]). On the other hand, in order to study conformal changes of a Finsler metric L on M, Ichijyō has introduced the notion of (L, N)*-connection,* where N is a fixed non-linear connection, and given the condition that such a Finsler connection be linear (cf. [4, Theorem 3]).

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The above Finsler connections are of special type among general Finsler connections. In their recent paper [1], Aikou and the author expressed any Finsler connection in terms of some distinguished tensor fields. From this standpoint, in the present paper we shall generally discuss the condition that any Finsler connection in a Finsler space be linear, and give various characterizations in terms of such tensor fields. In the first section we give characterizations in terms of U_{ijk}, $P^i{}_{jk}$, $P_j{}^i{}_{kl}$ (Theorem 1.1) and in terms of U_{ijk}, $P^i{}_{jk}$, $T_j{}^i{}_k$ (Theorem 1.2), and in the second section the characterization of Cartan type, i.e., in terms of U_{ijk}, $D^i{}_k$, $T_j{}^i{}_k$ (Theorem 2.1). Theorem 1.2 shows that Theorem 3 of [4] is due to $U_{ijk}=T_j{}^i{}_k=0$, and Theorem 2.1 gives other expressions for Theorem 1.1 of [7] and Theorem 3 of [3] (Theorem 2.2 and Theorem 2.3).

The motive of the present paper is influenced by the above literature. The author wishes to express here his sincere gratitude to Professor Dr. M. Matsumoto who drew the author's interest to the subject.

The terminology and notations are referred to Matsumoto's monograph [5].

1. Linear Finsler connections

Given a Finsler connection $F\Gamma=(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$, we can associate the Finsler connection $'F\Gamma=(F_j{}^i{}_k, N^i{}_k, 0)$ with $F\Gamma$. Since the hv-curvature tensor field $'P^2$ of $'F\Gamma$ is given by

(1.1) $$'P_j{}^i{}_{kl}=\dot{\partial}_l F_j{}^i{}_k,$$

we have

Proposition 1.1. *A Finsler connection $F\Gamma$ is linear if and only if the associated Finsler connection $'F\Gamma$ has the vanishing hv-curvature tensor field $'P^2$.*

Now, we shall consider the problem on a Finsler space (M, L). Since the v-connection is unessential, we assume that C_{ijk}, $C_j{}^i{}_k$ are always

(1.2) $$C_{ijk}=(\dot{\partial}_k g_{ij})/2, \quad C_j{}^i{}_k=g^{im}C_{jmk},$$

where $(g^{ij})=(g_{ij})^{-1}$. It is noted that contrary to the notation in [1] we put

(1.3) $$U_{ijk}=(g_{ij|k})/2.$$

Throughout the present paper the following Proposition and Lemma are useful.

Proposition 1.2. (1) *$F\Gamma$ is v-metrical*: $g_{ij}|_k = 0$, *where a long bar denotes the v-covariant differentiation.*

(2) *$F\Gamma$ satisfies the C_1-condition*: $y^j C_j{}^i{}_k = 0$, *and we have*

$$y^i{}_{|k} = D^i{}_k, \quad y^i C_{ijl|k} = -C_{ijl} D^i{}_k, \quad y^j C_j{}^i{}_{k|l} = -C_j{}^i{}_k D^j{}_l.$$

(3) *U_{ijk} is symmetric in i, j*: $U_{ijk} = U_{jik}$, *and we have*

$$g_{jm} C_i{}^m{}_{l|k} = C_{ijl|k} - 2 U_{jmk} C_i{}^m{}_l.$$

Lemma. *In a Finsler space (M, L), a Finsler connection $F\Gamma = (F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ satisfies the following formulas*:

$$C_{ijl|k} = C_{ijm} P^m{}_{kl} + \dot\partial_l U_{ijk} + (g_{mj} \dot\partial_l F_i{}^m{}_k + g_{im} \dot\partial_l F_j{}^m{}_k)/2, \tag{1.4}$$

$$P_0{}^i{}_{kl} = P^i{}_{kl} + \dot\partial_l D^i{}_k + C_m{}^i{}_l D^m{}_k, \tag{1.5}$$

$$P_{ijkl} = g_{jm} \dot\partial_l F_i{}^m{}_k + 2 U_{jmk} C_i{}^m{}_l - (C_{ijl|k} - C_{ijm} P^m{}_{kl}), \tag{1.6}$$

$$P_{ijkl} = \mathfrak{A}_{ij} \{ C_{jkl|i} - C_{jkm} P^m{}_{il} - \dot\partial_l U_{jki} \} + (2 U_{jmk} C_i{}^m{}_l - \dot\partial_l U_{ijk}) + \Lambda_{ijkl}, \tag{1.7}$$

where we put $P_{ijkl} = g_{jm} P_i{}^m{}_{kl}$, *and*

$$\Lambda_{ijkl} = (g_{jm} \dot\partial_l T_i{}^m{}_k + g_{im} \dot\partial_l T_k{}^m{}_j - g_{km} \dot\partial_l T_j{}^m{}_i)/2, \tag{1.8}$$

and $\mathfrak{A}_{ij}\{\cdots\}$ *denotes the alternating summation* : $\mathfrak{A}_{ij}\{F_i{}^m{}_j\} = F_i{}^m{}_j - F_j{}^m{}_i$, *and the subscript* 0 *the contraction by* y^j : $P_0{}^i{}_{kl} = y^j P_j{}^i{}_{kl}$.

Proof. (1) Let $'F\Gamma = (F_j{}^i{}_k, N^i{}_k, 0)$ be the Finsler connection associated with $F\Gamma$. Under the process from $F\Gamma$ to $'F\Gamma$, the $(v)hv$-torsion tensor field P^1 and the h-covariant differentiation are unchanged. The v-covariant differentiation is nothing but the partial differentiation by y^i. Applying to g_{ij} one of the Ricci identities with respect to $'F\Gamma$, we have from (1.1)

$$\dot\partial_l (g_{ij|k}) - (\dot\partial_l g_{ij})_{|k} = -(g_{mj} \dot\partial_l F_i{}^m{}_k + g_{im} \dot\partial_l F_j{}^m{}_k + (\dot\partial_m g_{ij}) P^m{}_{kl}). \tag{1.9}$$

(1.4) follows from (1.2), (1.3).

(2) (1.5) and (1.6) directly follow from

$$P_j{}^i{}_{kl} = \dot\partial_l F_j{}^i{}_k - C_j{}^i{}_{l|k} + C_j{}^i{}_m P^m{}_{kl}. \tag{1.10}$$

(3) Applying to g_{ij} one of the Ricci identities with respect to $F\Gamma$, we have

(1.11) $$g_{ij|k}|_l - g_{ij}|_{l|k} = -(g_{mj}P_i{}^m{}_{kl} + g_{im}P_j{}^m{}_{kl} + g_{ij|m}C_k{}^m{}_l),$$

which is reduced to

(1.12) $$P_{ijkl} + P_{jikl} = 2(U_{mjk}C_i{}^m{}_l + U_{imk}C_j{}^m{}_l - \dot{\partial}_l U_{ijk}).$$

Substituting the expression for $g_{jm}\dot{\partial}_l F_i{}^m{}_k$ from (1.6) in $g_{jm}\dot{\partial}_l T_i{}^m{}_k = \mathfrak{A}_{ik}\{g_{jm}\dot{\partial}_l F_i{}^m{}_k\}$, and calculating the right-hand side of (1.8) by the Christoffel process, we have (1.7) from (1.12). Q.E.D.

From (1.4) and (1.6) we have the following characterization of a linear Finsler connection in terms of U_{ijk}, $P^i{}_{jk}$, $P_i{}^j{}_{kl}$.

Theorem 1.1. *In a Finsler space, a Finsler connection $F\Gamma = (F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ is linear : $\dot{\partial}_l F_j{}^i{}_k = 0$, if and only if $F\Gamma$ satisfies the following conditions :*

(1.13) $$C_{ij|l|k} = C_{ijm}P^m{}_{kl} + \dot{\partial}_l U_{ijk},$$

(1.14) $$P_{ijkl} = 2U_{jmk}C_i{}^m{}_l - \dot{\partial}_l U_{ijk}.$$

Paying attention to (1.7), the condition (1.14) is equivalent to $\Lambda_{ijkl}=0$ under the condition (1.13). Taking the alternating part in i, k of Λ_{ijkl}, the condition $\Lambda_{ijkl} = 0$ is also equivalent to

(1.15) $$\dot{\partial}_l T_j{}^i{}_k = 0.$$

Thus we have the following characterization of a linear Finsler connection in terms of U_{ijk}, $P^i{}_{jk}$, $T_j{}^i{}_k$.

Theorem 1.2. *In a Finsler space, a Finsler connection $F\Gamma$ is linear if and only if $F\Gamma$ satisfies the conditions* (1.13) *and* (1.15).

Especially, in the case that $F\Gamma$ is metrical: $U_{ijk} = 0$, and has the vanishing (h) h-torsion tensor field: $T_j{}^i{}_k = 0$, then $F\Gamma$ is linear if and only if $F\Gamma$ satisfies the condition

(1.16) $$C_{ij|l|k} = C_{ijm}P^m{}_{kl}.$$

Given a non-linear connection N in a Finsler space (M, L), Ichijyō [4] introduced a Finsler connection called the (L, N)-*connection* and the tensor field $Q_{ijkl}=C_{ijl|k}-C_{ijm}P^m{}_{kl}$, and showed that $Q_{ijkl}=0$ is the condition that it be linear. In fact, the (L, N)-connection is characterized by $U_{ijk}=T_j{}^i{}_k=0$.

2. The characterization of Cartan type

The Cartan connection of a Finsler space is characterized as the Finsler connection with $C_j{}^i{}_k$ given by (1.2) and satisfying $U_{ijk}=D^i{}_k=T_j{}^i{}_k=0$. So we hope to characterize a linear Finsler connection in terms of U_{ijk}, $D^i{}_k$, $T_j{}^i{}_k$. In the condition (1.13) of Theorem 1.2, we can replace $P^i{}_{kl}$ by

$$P^i{}_{kl}=-\dot{\partial}_l D^i{}_k, \tag{2.1}$$

and we have

Theorem 2.1. *In a Finsler space, a Finsler connection* $F\Gamma$ *is linear if and only if* $F\Gamma$ *satisfies the conditions* (1.15) *and*

$$C_{ijl|k}=-C_{ijm}\dot{\partial}_l D^m{}_k+\dot{\partial}_l U_{ijk}. \tag{2.2}$$

Proof. Let $F\Gamma$ be linear. Substituting $N^i{}_k=y^jF_j{}^i{}_k-D^i{}_k$ in $P^i{}_{kl}=\dot{\partial}_l N^i{}_k-F_l{}^i{}_k$, we have (2.1) from $\dot{\partial}_l F_j{}^i{}_k=0$, and so (1.13) becomes (2.2).

Conversely, assume that $F\Gamma$ satisfies (1.15) and (2.2). Contracting (2.2) by y^i, we have

$$C_{ijl}D^i{}_k=-y^i\dot{\partial}_l U_{ijk}. \tag{2.3}$$

Since (1.15) yields $\Lambda_{ijkl}=0$, the condition (2.2) reduces (1.7) to

$$P_{ijkl} = -\mathfrak{A}_{ij}\{C_{jkm}(P^m{}_{il}+\dot{\partial}_l D^m{}_i)\}+(2U_{jmk}C_i{}^m{}_l-\dot{\partial}_l U_{ijk}). \tag{2.4}$$

Contracting (2.4) by y^i, we have from (1.5), (2.3)

$$P^i{}_{kl}+\dot{\partial}_l D^i{}_k=-C_k{}^i{}_m(P^m{}_{0l}+y^r\dot{\partial}_l D^m{}_r), \tag{2.5}$$

from which we have (2.1), and so (1.13). Q.E.D.

As is shown in the above proof, we have

Proposition 2.1. *In a Finsler space, if a Finsler connection $F\Gamma$ is linear, $F\Gamma$ satisfies* (2.1), (2.3), *and*

$$y^i y^j \dot{\partial}_l U_{ijk}=0. \tag{2.6}$$

(2.3) is the consequence of (2.2), and so of $\dot{\partial}_l F_j{}^i{}_k=0$. (2.6) follows from (2.3).

Now, we shall discuss some special cases. In a Finsler space (M, L), a Finsler connection $F\Gamma$ is called *h-recurrent* if there exists a covariant Finsler vector field $a_k(x, y)$ satisfying

$$g_{ij|k}=a_k g_{ij}, \tag{2.7}$$

that is, $2U_{ijk}=a_k g_{ij}$ (cf. [7, 8], Miron-Hashiguchi [6]).

If an h-recurrent Finsler connection $F\Gamma$ is linear, we have from (2.6)

$$\dot{\partial}_l a_k=0, \tag{2.8}$$

that is, a_k depend on position alone. Then we have

$$\dot{\partial}_l U_{ijk}=a_k C_{ijl}. \tag{2.9}$$

Thus we have

Theorem 2.2. *In a Finsler space* (M, L), *an h-recurrent Finsler connection $F\Gamma$ is linear if and only if $F\Gamma$ satisfies the conditions* (1.15), (2.8) *and*

$$C_{ijl|k}=-C_{ijm}\dot{\partial}_l D^m{}_k+a_k C_{ijl}. \tag{2.10}$$

Paying attention to (2.3), (2.9), it is shown that the condition (2.10) in Theorerm 2.2 is equivalent to the following two conditions in Theorem 1.1 of [7]:

$$C_{ijl}D^i{}_k=0, \tag{2.11}$$

$$C_{ijl|k}=C_{ijm}|_l D^m{}_k+a_k C_{ijl}. \tag{2.12}$$

It is noted that these conditions are expressed as the single condition (2.10) by using the partial derivative instead of the v-covariant derivative, and the additional condition $\lambda_{ijkl}=0$ is reduced to the simple conditions (1.15), (2.8).

Especially, for a metrical Finsler connection we have

Theorem 2.3. *In a Finsler space* (M, L), *a metrical Finsler connection* $F\Gamma$ *is linear if and only if* $F\Gamma$ *satisfies the conditions* (1.15) *and*

$$C_{ijl}|_k = -C_{ijm}\dot{\partial}_l D^m{}_k. \tag{2.10'}$$

The condition (2.10′) in Theorem 2.3 is equivalent to the two conditions (2.11) and

$$C_{ijl|k} = C_{ijm}|_l D^m{}_k, \tag{2.12'}$$

which were given in Theorem 3 of [3].

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