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著者	KAWAI Toru
journal or publication title	鹿児島大学理学部紀要. 数学・物理学・化学
volume	21
page range	39-43
別言語のタイトル	連続関数のフーリエ変換の超準表現
URL	<a href="http://hdl.handle.net/10232/00010054">http://hdl.handle.net/10232/00010054</a>

# A NONSTANDARD REPRESENTATION OF FOURIER TRANSFORMS OF CONTINUOUS FUNCTIONS

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(Received September 10, 1988)

## Abstract

We give a nonstandard representation of Fourier transforms of continuous functions on compact abelian groups. Applying the representation, we prove the Bochner-Weil theorem on compact abelian groups.

## Introduction

Let  $G$  be a compact abelian group with dual  $\Gamma$ . We denote  $C(G)$  the space of all continuous functions on  $G$  and define the Fourier transform  $\hat{f}$  of  $f \in C(G)$  by

$$\hat{f}(\gamma) = \int_G f(x) (-x, \gamma) dx \quad (\gamma \in \Gamma).$$

In 1, we represent  $G$  and  $\Gamma$  by  $*$ -finite abelian groups and give a nonstandard representation of functions in  $C(G)$  and their Fourier transforms. In [2, 6.5 Theorem], Luxemburg has given a nonstandard proof of the Bochner's theorem. Using our representation, we give a nonstandard proof of the Bochner-Weil theorem on compact abelian groups in 2. It is simpler than the proof in [2].

Throughout the present paper, we adopt a nonstandard set theory NST in [1] to express nonstandard analysis by axiomatic method. For example, every standard infinite set has nonstandard elements.

## 1. A nonstandard representation

**Theorem 1.** *Let  $G$  be a standard compact abelian group with dual  $\Gamma$ . Then there*

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exist an internal closed subgroup  $H$  of  $G$  and a  $*$ -finite subgroup  $D$  of  $G/H$  satisfying the following (i) - (vi) :

- (i) Let  $\Lambda$  be the annihilator of  $H$ . Then  ${}^\circ\Gamma \subset \Lambda$ , where  ${}^\circ\Gamma$  is the set of all standard elements of  $\Gamma$ .
- (ii) Let  $\Delta$  be the dual of  $D$  and  $\Theta : \Lambda \rightarrow \Delta$  be the natural mapping. Then the restriction of  $\Theta$  to  ${}^\circ\Gamma$  is one-to-one.
- (iii) Let  $\xi : G \rightarrow G/H$  be the natural mapping. Then for every  $x \in G$  there is  $y \in G$  such that  $\xi(y) \in D$  and  $x \approx y$ .
- (iv) Suppose that  $f \in C(G)$  is standard. Define the internal function  $\Phi(f) \in C(G/H)$  by

$$\Phi(f)(\xi(x)) = \int_H f(x+h) dh \quad (\xi(x) \in G/H)$$

and let  $\Psi(f)$  be the restriction of  $\Phi(f)$  to  $D$ . If  $x \in G$  and  $\xi(x) \in D$ , then  $f(x) \approx \Psi(f)(\xi(x))$ .

- (v) If  $f \in C(G)$  is standard, then

$$\int_G f(x) dx \approx \frac{1}{|D|} \sum_{t \in D} \Psi(f)(t),$$

where  $|D|$  is the cardinal of  $D$ .

- (vi) If  $f \in C(G)$  and  $\gamma \in \Gamma$  are standard, then

$$\hat{f}(\gamma) \approx \widehat{\Psi(f)}(\Theta(\gamma)).$$

Proof. Let  $\{F_j\}$  be any finite set of standard finite subsets of  $\Gamma$ . Then there is a standard finitely generated subgroup  $\Lambda$  of  $\Gamma$  such that  $F_j \subset \Lambda$  for each  $j$ . By the saturation principle, there exists an internal finitely generated subgroup  $\Lambda$  of  $\Gamma$  such that  $F \subset \Lambda$  for all standard finite subset  $F$  of  $\Gamma$ . Let  $H$  be the annihilator of  $\Lambda$ . Then  $H \subset U$  for every neighborhood  $U$  of  $0$  in  $G$ . Since the internal group  $G/H$  is topologically isomorphic to the direct sum of a  $*$ -finite dimensional torus group and a  $*$ -finite abelian group, it follows from the transfer principle that for any finite set of neighborhoods  $V_j$  of  $0$  in  $G/H$ ,  $g_j \in C(G/H)$ ,  $\gamma_j \in \Lambda$ , and standard  $\varepsilon_j > 0$  ( $j=1, 2, \dots, n$ ), there exists a  $*$ -finite subgroup  $D$  of  $G/H$  such that

$$\forall t \in G/H \exists w \in D \quad t-w \in U_j; \quad \dots(1)$$

$$\gamma_j \in A(\Lambda, D) \rightarrow \gamma_j = 0; \quad \dots(2)$$

$$\left| \int_{G/H} g_j(t) dt - \frac{1}{|D|} \sum_{t \in D} g_j(t) \right| < \varepsilon_j; \quad \dots(3)$$

for each  $j$  ( $j=1, 2, \dots, n$ ), where  $A(\Lambda, D)$  is the annihilator of  $D$  in  $\Lambda$ . Put

$$M = \{ \xi[U] : U \text{ is a standard neighborhood of } 0 \text{ in } G \}$$

and

$$K = \{ \Phi(f) : f \text{ is a standard function in } C(G) \}.$$

Since  $M$  and  $K$  are  $S$ -size, it follows that for any finite sets of  $V_j \in M$ ,  $g_j \in K$ , standard  $\gamma_j \in \Gamma$ , and standard  $\varepsilon_j > 0$  ( $j=1, 2, \dots, n$ ), there exists a  $*$ -finite subgroup  $D$  of  $G/H$  satisfying (1)-(3) for each  $j=1, 2, \dots, n$ . The saturation principle implies that there exists a  $*$ -finite subgroup  $D$  of  $G/H$  such that for any  $V \in M$ ,  $g \in K$ , standard  $\gamma \in \Gamma$ , and standard  $\varepsilon > 0$ ,

$$\forall t \in G/H \exists w \in D \ t-w \in V ; \quad \dots(4)$$

$$\gamma \in A(\Lambda, D) \rightarrow \gamma = 0 ; \quad \dots(5)$$

$$\left| \int_{G/H} g(t) dt - \frac{1}{|D|} \sum_{t \in D} g(t) \right| < \varepsilon \quad \dots(6)$$

Let  $\Delta = \Lambda / A(\Lambda, D)$ . Then  $\Delta$  is the dual of  $D$ . From (5), we have  $\Gamma \cap A(\Lambda, D) = \{0\}$  and so (ii) follows. By (4) and the saturation principle, there is  $w \in D$  such that for any standard neighborhood  $U$  of  $0$  in  $G$ ,  $t-w \in \xi[U]$ . Thus we obtain (iii). From the continuity of  $g$ , we obtain (iv). Let  $f$  be a standard function in  $C(G)$ . From (6) and the fact that

$$\int_G f(x) dx = \int_{G/H} \Phi(f)(t) dt,$$

we obtain (v). Suppose that  $f \in C(G)$  and  $\gamma \in \Gamma$  are standard. Then it follows from (v) that

$$\begin{aligned} \hat{f}(\gamma) &= \int_G f(x) \overline{(x, \gamma)} dx \approx \frac{1}{|D|} \sum_{t \in D} \Psi(f\bar{\gamma})(t) \\ &= \frac{1}{|D|} \sum_{t \in D} \Psi(f)(t) \overline{(t, \Theta(\gamma))} = \widehat{\Psi}(f)(\Theta(\gamma)). \end{aligned}$$

This completes the proof of the theorem.

## 2. An application to positive-definite functions

Applying Theorem 1, we obtain a nonstandard proof of the Bochner Weil theorem.

**Theorem 2 (Bochner-Weil).** *Let  $G$  be a compact abelian group with dual  $\Gamma$  and  $f$  be a continuous positive-definite function on  $G$ . Then  $\hat{f}(\gamma) \geq 0$  for all  $\gamma \in \Gamma$  and  $\sum_{\gamma \in \Gamma} \hat{f}(\gamma)$  is finite.*

Proof. First let  $D$  be a standard finite abelian group with dual  $\Delta$  and  $f$  be a standard positive definite function on  $D$ . It follows from the definition of  $\hat{f}$  that

$$\hat{f}(\delta) = \frac{1}{|D|} \sum_{x \in D} f(x) (-x, \delta) = \frac{1}{|D|} \sum_{x \in D} f(x-y) (-x+y, \delta) \quad (\delta \in \Delta, y \in D).$$

Summing over  $y \in D$ , we have

$$|D| \hat{f}(\delta) = \frac{1}{|D|} \sum_{x \in D} \sum_{y \in D} f(x-y) (-x, \delta) \overline{(-y, \xi)} \geq 0 \quad (\delta \in \Delta).$$

Also we have  $\sum_{\delta \in \Delta} \hat{f}(\delta) = f(0)$ .

Now suppose that  $G$  is a standard compact abelian group and  $f$  is a standard continuous positive-definite function on  $G$ . Then there exist an internal closed subgroup  $H$  of  $G$  and a  $*$ -finite subgroup  $\mathfrak{D}$  of  $G/H$  as in Theorem 1. If  $g$  is an internal function in  $C(G/H)$ , then we have

$$\begin{aligned} & \int_{G/H} \int_{G/H} g(u) \overline{g(v)} \Phi(f)(u-v) \, du \, dv \\ &= \int_G \int_G g(\xi(x)) \overline{g(\xi(y))} f(x-y) \, dx \, dy \geq 0. \end{aligned}$$

This shows that  $\Phi(f)$  is internally positive-definite on  $G/H$ , and so is  $\Psi(f)$  on  $D$ . By the transfer principle, we have  $\widehat{\Psi(f)}(\delta) \geq 0$  for all  $\delta \in \Delta$  and  $\sum_{\delta \in \Delta} \widehat{\Psi(f)}(\delta) = \Psi(f)(0)$ . If  $\gamma \in \Gamma$  is standard, then it follows from (vi) in Theorem 1 that

$$\hat{f}(\gamma) \approx \widehat{\Psi(f)}(\theta(\gamma))$$

Since the right hand side is non-negative, it follows that  $\hat{f}(\gamma) \geq 0$  for all standard  $\gamma \in \Gamma$ . Let  $\{\gamma_1, \dots, \gamma_m\}$  be any standard finite subset of  $\Gamma$ . Since  $\theta(\gamma_1), \dots, \theta(\gamma_m)$  are distinct by (ii) in Theorem 1, it follows that

$$\sum_{j=1}^m \widehat{\Psi(f)}(\theta(\gamma_j)) \leq \sum_{\delta \in \Delta} \widehat{\Psi(f)}(\delta) = \Psi(f)(0).$$

Taking the standard parts, we have

$$\sum_{j=1}^m \hat{f}(\gamma_j) \leq f(0)$$

This implies that  $\sum_{\gamma \in \Gamma} \hat{f}(\gamma)$  is finite. The proof of the theorem is complete.

### References

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