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On a Finsler-Geometrical Expression of the Gaussian Curvature of a Hypersurface in an Euclidean Space

Masao HASHIGUCHI[1)]

Abstract

The present paper is a revised note of the lecture presented by the author at "The XXVIth Symposium on Finsler Geometry" held at Kushiro during October 5-8, 1991. Let a hypersurface S in an euclidean space R^n be implicitly defined by a differentiable function f in R^n. Then the Gaussian curvature of S is expressed, in terms of f itself, in a Finsler-geometrically striking form, so this result is applicable to Finsler geometry. We discuss the Gaussian curvature of the indicatrix of a Finsler space (R^n, L), especially the effects by some changes of the Finsler metric L in R^n.

Key words: Gaussian curvature, Indicatrix, Finsler space, Randers change, Kropina change.

1. Introduction

In a three-dimensional euclidean space R^3, let a surface S be implicitly defined by a differentiable function f in R^3 as $f(x)=0$, where $x=(x^1, x^2, x^3)$ is a rectangular coordinate system of R^3. We put $f_i=\partial f/\partial x^i$, $f_{ij}=\partial^2 f/\partial x^i \partial x^j$. Around a point $x\in S$ such that $f_3(x)\neq 0$ the surface S is graphically expressed by a differentiable function g as $x^3=g(x^1, x^2)$, and the Gaussian curvature K of S is given by $K=(p_{11}p_{22}-p_{12}^2)/(1+p_1^2+p_2^2)^2$, where $p_i=\partial g/\partial x^i$, $p_{ij}=\partial^2 g/\partial x^i \partial x^j$. If we directly calculate from

$$f_3 p_i=-f_i,\ f_3^3 p_{ij}=-f_{ij}f_3^2+f_{i3}f_jf_3+f_{j3}f_if_3-f_{33}f_if_j,$$

we have

$$(1.1)\qquad K=-\begin{vmatrix} f_{11} & f_{12} & f_{13} & f_1 \\ f_{21} & f_{22} & f_{23} & f_2 \\ f_{31} & f_{32} & f_{33} & f_3 \\ f_1 & f_2 & f_3 & 0 \end{vmatrix} /(f_1^2+f_2^2+f_3^2)^2.$$

Especially, in the case where a treated function f is a quadratic polynomial of the coordinates:

1) Department of Mathematics, Faculty of Science, Kagoshima University, 1-21-35 Korimoto, Kagoshima 890, Japan. This research was partially supported by Grant-in-Aid for Scientific Research (No. 03640080), Ministry of Education, Science and Culture.

$$(1.2) \qquad 2f(x)=a_{ij}x^i x^j+2b_i x^i+c \qquad (a_{ij}=a_{ji}),$$

the formula (1.1) is reduced to

$$(1.3) \qquad K=-\begin{vmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \\ b_1 & b_2 & b_3 & c \end{vmatrix} /(f_1^2+f_2^2+f_3^2)^2,$$

where $f_i(x)=a_{ij}x^j+b_i$. We use the summation convention in proper case. It is noted that in this formula the value of K depends only on the magnitude of the gradient of f reciprocally.

Generally, in an n-dimensional euclidean space R^n we shall consider a hypersurface S defined by a differentiable function f in R^n as

$$(1.4) \qquad S=\{x\in R^n \mid f(x)=0,\ (\nabla f)(x)\neq 0\},$$

where $x=(x^1,\cdots,x^n)$ is a rectangular coordinate system of R^n, and ∇f denotes the gradient of f.

Throughout the present paper, we put $\partial_i=\partial/\partial x^i$, and denote a vector with components $v_1,\cdots,v_n$ by an $n\times 1$ matrix ${}^t(v_1,\cdots,v_n)$ and also by (v_i) briefly. A letter ${}^tA$ denotes the transpose of a matrix A. The inner product $\sum_i u_i v_i$ of vectors $\boldsymbol{u}=(u_i)$ and $\boldsymbol{v}=(v_i)$ is denoted by $\boldsymbol{u}\cdot\boldsymbol{v}$, and the length $(\boldsymbol{v}\cdot\boldsymbol{v})^{1/2}$ of a vector $\boldsymbol{v}$ by $|\boldsymbol{v}|$. Then we have

$$(1.5) \qquad \nabla f={}^t(f_1,\cdots,f_n),\ |\nabla f|=(\sum_i f_i^2)^{1/2}\ (f_i=\partial_i f).$$

The notion of Gaussian curvature is generally defined for a hypersurface S in R^n, and in the case where S is implicitly given by (1.4) we can get the same expression as (1.1) (Theorem 2.1). This is derived, for example, from Theorem 5 of Thorpe [5, Chap. 12, p 89], but in the previous paper [3] we showed a self-contained proof, based on Lemma 2.1 concerning with the determinant of a linear transformation of a hypersubspace of a vector space R^n. We sketch this proof in Section 2, where an orientation N of S is fixed by $N=-\nabla f/|\nabla f|$ and the proof of Lemma 2.1 is improved.

This result is applied to Finsler geometry. We denote by $y=(y^1,\cdots,y^n)$ the canonical coordinate system of the tangent space R^n_x at each point $x\in R^n$, and put $\dot\partial_i=\partial/\partial y^i$. Let (R^n, L) be a Finsler space, where L is the fundamental function defined in R^n. Each tangent space R^n_x is regarded as an n-dimensional euclidean space with the rectangular coordinate system y.

A hypersurface $I_x=\{y\in R^n_x \mid L(x,y)=1\}$ in R^n_x is called the ***indicatrix*** at x. In Section 3 we shall express the Gaussian curvature of I_x in terms of L (Theorem 3.1). Given a hypersurface S in each tangent space R^n_x a priori, by the well-known method (cf. Matsumoto [2, p 105]) we have a Finsler space whose indicatrix I_x is the given S. Thus the Gaussian curvature of S is expressed in terms of Finsler geometry. This fact seems interesting from the standpoint of application. In connection with two examples given in Theorem 3.2 and Theorem 3.3, in Section 4 we discuss the effects for the Gaussian curvature of the indicatrix by some changes of a Finsler metric (Theorem 4.1, Theorem 4.2.).

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As to the details of some discussions in the present paper and the treatment for a general Lagrange space, refer to [3].

2. The Gaussian curvature of a hypersurface

We return here to the case of $n=3$, and let a surface S in R^3 be parameterized as $x=x(u^1, u^2)$. At each point $x\in S$, two tangent vector fields $X_\alpha=\partial x/\partial u^\alpha$ $(\alpha=1, 2)$ constitute a basis of the tangent plane S_x, and the unit vector field $N=(X_1\wedge X_2)/|X_1\wedge X_2|$ is orthogonal to S_x. Suggested by the Weingarten equation

$$(2.1) \qquad N_\beta=-h_\beta^\alpha X_\alpha \qquad (N_\beta=\partial N/\partial u^\beta),$$

we define a linear transformation T of S_x by

$$(2.2) \qquad T: S_x\to S_x\,|\,v=v^\beta X_\beta\to T(v)=-v^\beta N_\beta.$$

Since $T(v)=(h_\beta^\alpha v^\beta)X_\alpha$, the Gaussian curvature $K=\det(h_\beta^\alpha)$ of S at x is the determinant of T. It is noted that the vector $v^\beta N_\beta$ in (2.2) is the derivative $\nabla_v N$ of N with respect to v.

Now, let (S, N) be an oriented hypersurface in R^n, where N is a unit vector field orthogonal to S. Let S_x be the tangent space of a point $x\in S$. The derivative $\nabla_v N$ of N is defined with respect to $v\in S_x$, and we have $\nabla_v N\in S_x$, so we can define a linear transformation T of S_x by

$$(2.3) \qquad T: S_x\to S_x\,|\,v\to T(v)=-\nabla_v N.$$

This is called the *Weingarten map* of (S, N) at x. The *Gaussian curvature* K of (S, N) at x is defined by the determinant of T.

In the case where a hypersurface S in R^n is implicitly defined by (1.4), for an orientation N of S we shall choose

$$(2.4) \qquad N=-\nabla f/|\nabla f|.$$

Then we have

Theorem 2.1. *Let (S, N) be an oriented hypersurface in R^n, where S and N are given by* (1.4) *and* (2.4) *respectively. Then the Gaussian curvature K of (S, N) is given by*

$$(2.5) \qquad K=-\begin{vmatrix} f_{ij} & f_i \\ f_j & 0 \end{vmatrix} / |\nabla f|^{n+1}.$$

Since for any $u=(u_i)$, $v=(v_i)\in S_x$ the Weingarten map T of (S, N) at $x\in S$ satisfies

(2.6) $$\boldsymbol{u}\cdot T(\boldsymbol{v})=(\sum_{i,j} f_{ij}u_i v_j)/|\nabla f|,$$

the proof of Theorem 2.1 is obtained from the following lemma by putting $a_{ij}=f_{ij}/|\nabla f|$, $n_i=-f_i/|\nabla f|$.

Lemma 2.1. *Let W be an $(n-1)$-dimensional subspace of an n-dimensional euclidean vector space R^n, $N=(n_i)$ a unit vector orthogonal to W, and T a linear transformation of W. If for any $\boldsymbol{u}=(u_i)$, $\boldsymbol{v}=(v_i)\in W$ the inner product $\boldsymbol{u}\cdot T(\boldsymbol{v})$ is expressed by a matrix $A=(a_{ij})$ as*

(2.7) $$\boldsymbol{u}\cdot T(\boldsymbol{v})={}^t\boldsymbol{u}\,A\,\boldsymbol{v}\,(=\sum_{i,j} a_{ij}u_i v_j),$$

then the determinant K of T is given by

(2.8) $$K=-\begin{vmatrix} A & N \\ {}^tN & 0 \end{vmatrix}\left(=-\begin{vmatrix} a_{ij} & n_i \\ n_j & 0 \end{vmatrix}\right).$$

Proof. In the proof the Greek indices take the values $1,\cdots,n-1$. We choose a basis $X_1,\cdots,X_{n-1}$ of W such that $X_1,\cdots,X_{n-1},N$ constitute an orthonormal basis of R^n, and represent T by an $(n-1)\times(n-1)$ matrix $(b_{\alpha\beta})$, where $T(X_\beta)=\sum_\alpha b_{\alpha\beta}X_\alpha$. Then the determinant K of T is obtained by definition as $K=\det(b_{\alpha\beta})$. It is noted that $b_{\alpha\beta}=X_\alpha\cdot T(X_\beta)$.

We define an $n\times n$ matrix X by $(X_1,\cdots,X_{n-1},N)$ and $(n+1)\times(n+1)$ matrices $\tilde{A}$, $\tilde{X}$ by

$$\tilde{A}=\begin{pmatrix} A & N \\ {}^tN & 0 \end{pmatrix},\quad \tilde{X}=\begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}.$$

X and $\tilde{X}$ are orthogonal. Then we have from $X_\alpha\cdot N=0$, $N\cdot N=1$

$$ {}^t\tilde{X}\,\tilde{A}\,\tilde{X}=\begin{pmatrix} {}^tX_\alpha\,A\,X_\beta & {}^tX_\alpha\,A\,N & 0 \\ {}^tN\,A\,X_\beta & {}^tN\,A\,N & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

from which we have $\det\tilde{A}=-\det({}^tX_\alpha\,A\,X_\beta)$. Paying attention to ${}^tX_\alpha\,A\,X_\beta=X_\alpha\cdot T(X_\beta)=b_{\alpha\beta}$, we have $\det\tilde{A}=-\det(b_{\alpha\beta})$. Q. E. D.

As a special case of Theorem 2.1 we have

Theorem 2.2. *Let (S, N) be an oriented hypersurface in R^n, where S is a regular quadratic hypersurface defined by*

(2.9) $$2f(x)=a_{ij}x^i x^j+2b_i x^i+c=0 \qquad (a_{ij}=a_{ji})$$

and N is a unit vector field orthogonal to S given by (2.4). *Then the Gaussian curvature K of (S, N) is given by*

(2.10)
$$K=-\begin{vmatrix} a_{ij} & b_i \\ b_j & c \end{vmatrix} /(\sum_i f_i^2)^{(n+1)/2},$$

where $f_i(x)=a_{ij}x^j+b_i$.

3. The indicatrix of a Finsler space

Let (R^n, L) be a Finsler space. We put $l_i=\dot{\partial}_i L$, $\dot{\nabla} L=(l_i)$, $g_{ij}=(\dot{\partial}_i\dot{\partial}_j L^2)/2$, $(g^{ij})=(g_{ij})^{-1}$, and $g=\det(g_{ij})$. The Finslerian length of the normalized supporting element $\dot{\nabla} L$ is $1: g^{ij}l_il_j=1$, but $|\dot{\nabla} L|=(\sum_i l_i^2)^{1/2}$ denotes the euclidean length.

If we define a function f by

(3.1)
$$2f(x, y)=L^2(x, y)-1,$$

and put $\dot{\nabla} f=(\dot{\partial}_i f)$, then the indicatrix I_x is expressed as

(3.2)
$$I_x=\{y\in R_x^n | f(x, y)=0\},$$

whereon we have $\dot{\nabla} f=\dot{\nabla} L\neq 0$.

At each $y\in I_x$ the vector field $\dot{\nabla} L$ is orthogonal to I_x. We shall assume that an orientation N of I_x is always

(3.3)
$$N=-\dot{\nabla} L/|\dot{\nabla} L|.$$

Since on the indicatrix we have

$$\begin{vmatrix} f_{ij} & f_i \\ f_j & 0 \end{vmatrix} = \begin{vmatrix} g_{ij} & l_i \\ l_j & 0 \end{vmatrix} = -g,$$

we have from Theorem 2.1

Theorem 3.1. *Let (R^n, L) be a Finsler space. At each point $x\in R^n$, the Gaussian curvature K of the indicatrix I_x oriented in the direction opposite to $\dot{\nabla} L=(l_i)$ is given by*

(3.4)
$$K=g/|\dot{\nabla} L|^{n+1}.$$

We can apply Theorem 2.2 for a Randers space and a Kropina space. Let $\alpha(x, y)=(a_{ij}(x)y^iy^j)^{1/2}$ be a Riemannian metric and $\beta(x, y)=b_i(x)y^i$ a non-vanishing 1-form in R^n. Then we have

Theorem 3.2. *Let (R^n, L) be a Randers space, where $L=\alpha+\beta$. At each point $x\in R^n$, the Gaussian curvature K of the indicatrix I_x oriented in the direction opposite to $\dot{\nabla} L=(l_i)$ is given by*

(3.5)
$$K=\det(a_{ij})/(\sum_i f_i^2)^{(n+1)/2},$$

where $f_i(x, y)=a_{ij}(x)y^j+\alpha(x, y)b_i(x)$ $(f_i=\alpha l_i)$.

Theorem 3.3. *Let (R^n, L) be a Kropina space, where $L=\alpha^2/\beta$. At each point $x \in R^n$, the Gaussian curvature K of the indicatrix I_x oriented in the direction opposite to $\dot{\nabla} L=(l_i)$ is given by*

$$K=2^{n-1}b^2\det(a_{ij})/(\sum_i f_i^2)^{(n+1)/2}, \tag{3.6}$$

where $b^2=g^{ij}b_ib_j$ and $f_i(x, y)=2a_{ij}(x)y^j-b_i(x)$ $(f_i=\alpha^2 l_i)$.

4. Changes of Finsler metrics

We shall here investigate how the Gaussian curvature of the indicatrix is effected under some changes of a Finsler metric L in R^n. Let $\beta(x, y) = b_i(x)y^i$ be a non-vanishing 1-form in R^n. We shall first consider the change

$$L\to\bar{L}=L+\beta \tag{4.1}$$

called a *Randers change* (cf. Matsumoto [1]).

The indicatrix $\bar{I}_x$ at $x\in R^n$ of a Finsler space $(R^n, \bar{L})$ satisfies

$$2\bar{f}(x, y)=L^2(x, y)-(1-\beta(x, y))^2=0. \tag{4.2}$$

Then we have $\bar{f}_i=Ll_i+(1-\beta)b_i$, $\bar{f}_{ij}=g_{ij}-b_ib_j$. Since on the indicarix $\bar{I}_x$ we have $\bar{f}_i=L\bar{l}_i$, where $\bar{l}_i=\dot{\partial}_i\bar{L}$, the vector $\dot{\nabla}\bar{f}=(\dot{\partial}_i\bar{f})$ has the same direction as $\dot{\nabla}\bar{L}=(\bar{l}_i)$. Thus the vector field $\bar{N}=-\dot{\nabla}\bar{f}/|\dot{\nabla}\bar{f}|$ gives the orientation assumed for a Finsler space. Since on the indicarix $\bar{I}_x$ we have

$$\begin{vmatrix} \bar{f}_{ij} & \bar{f}_i \\ \bar{f}_j & 0 \end{vmatrix} = \begin{vmatrix} g_{ij}-b_ib_j & L(l_i+b_i) \\ L(l_j+b_j) & 0 \end{vmatrix} = -g,$$

applying Theorem 2.1 to (4.2) we have the Gaussian curvature $\bar{K}$ of the indicatrix $\bar{I}_x$ of the Finsler space $(R^n, \bar{L})$ as

$$\bar{K}=g/(L|\dot{\nabla}\bar{L}|)^{n+1}. \tag{4.3}$$

Since the Gaussian curvature K of the indicatrix I_x of the Finsler space (R^n, L) is expressed as $K=g/|\dot{\nabla}L|^{n+1}$, we have

Theorem 4.1. *Let $(R^n, \bar{L})$ be the Finsler space obtained from a Finsler space (R^n, L) by a Randers change $L\to\bar{L}=L+\beta$. Then the Gaussian curvature of the indicatrix is changed as*

$$\bar{K}=(|\dot{\nabla}L|/L|\dot{\nabla}\bar{L}|)^{n+1}K. \tag{4.4}$$

In the same way, we can treat a change

$$L\to\bar{L}=L^2/\beta \tag{4.5}$$

called a *Kropina change* (cf. Shibata [4]). The indicatrix $\bar{I}_x$ at $x\in R^n$ of a Finsler space $(R^n, \bar{L})$ may be expressed as

(4.6) $$\bar{f}(x, y)=L^2(x, y)-\beta(x, y)=0.$$

Then we have $\bar{f}_i=2Ll_i-b_i$, $\bar{f}_{ij}=2g_{ij}$. Since on the indicatrix $\bar{I}_x$ we have $\bar{f}_i=L^2\bar{l}_i$, where $\bar{l}_i=\dot{\partial}_i\bar{L}$, the vector $\dot{\nabla}\bar{f}=(\dot{\partial}_i\bar{f})$ has the same direction as $\dot{\nabla}\bar{L}=(\bar{l}_i)$. Thus the vector field $\bar{N}=-\dot{\nabla}\bar{f}/|\dot{\nabla}\bar{f}|$ gives the orientation assumed for a Finsler space. Since on the indicatrix $\bar{I}_x$ we have

$$\begin{vmatrix} \bar{f}_{ij} & \bar{f}_i \\ \bar{f}_j & 0 \end{vmatrix} = \begin{vmatrix} 2g_{ij} & 2Ll_i-b_i \\ 2Ll_j-b_j & 0 \end{vmatrix} = -2^{n-1}b^2g,$$

applying Theorem 2.1 to (4.6) we have the Gaussian curvature $\bar{K}$ of the indicatrix $\bar{I}_x$ of the Finsler space $(R^n, \bar{L})$ as

(4.7) $$\bar{K}=2^{n-1}b^2g/(L^2|\dot{\nabla}\bar{L}|)^{n+1}.$$

Since the Gaussian curvature K of the indicatrix I_x of the Finsler space (R^n, L) is expressed as $K=g/|\dot{\nabla}L|^{n+1}$, we have

Theorem 4.2. *Let $(R^n, \bar{L})$ be the Finsler space obtained from a Finsler space (R^n, L) by a Kropina change $L\to\bar{L}=L^2/\beta$. Then the Gaussian curvature of the indicatrix is changed as*

(4.8) $$\bar{K}=2^{n-1}b^2(|\dot{\nabla}L|/L^2|\dot{\nabla}\bar{L}|)^{n+1}K.$$

Remark 4.1. Applying (4.3) and (4.7) to $L=\alpha$, we also have Theorem 3.2 and Theorem 3.3 respectively.

Remark 4.2. Let $(R^n, \bar{L})$ be the Finsler space obtained from a Finsler space (R^n, L) by a Randers change $L\to\bar{L}=L+\beta$. By Theorem 3.1 the Gaussian curvature of the indicatrix $\bar{I}_x$ of $(R^n, \bar{L})$ is given by $\bar{K}=\bar{g}/|\dot{\nabla}\bar{L}|^{n+1}$. If we compare this formula with (4.3), we have $\bar{g}=g/L^{n+1}$ on the indicatrix $\bar{I}_x$. Since $y/\bar{L}\in\bar{I}_x$ for any $y\in R^n_x$, we generally have $\bar{g}=(\bar{L}/L)^{n+1}g$. It is interesting that we can get $\bar{g}$ without knowing the concrete form of $\bar{g}_{ij}$. Especially, we have $g=(L/\alpha)^{n+1}\det(a_{ij})$ for a Randers space (R^n, L), where $L=\alpha+\beta$.

Let $(R^n, \bar{L})$ be the Finsler space obtained from a Finsler space (R^n, L) by a Kropina change $L\to\bar{L}=L^2/\beta$. In the same way, we have $\bar{g}=2^{n-1}b^2(\bar{L}/L)^{2(n+1)}g$. Especially, we have $g=2^{n-1}b^2(L/\alpha)^{2(n+1)}\det(a_{ij})$ for a Kropina space (R^n, L), where $L=\alpha^2/\beta$.

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