On the structure of iteration scheme of modular type of order n

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On the structure of iteration scheme of modular type of order n

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Abstract

In this paper we shall clarify the structure of a discrete dynamical system, called by the name "iteration scheme of modular type" in [1], see an isomorphism between an iteration scheme of modular type of order n and a product of some iteration scheme of modular type of different orders, which constitutes the prime number factorization of the number n, and show fundamental properties of the iteration schemes by example.

Key word: discrete dynamical system, finite graph, iteration process, limit cycle.

1. Notations and Definitions.

Let X be a finite set, and f a mapping from X into itself. We call the system $<X, f>$ an iteration scheme over X: starting with an x^0 from X, we are interested in the sequence of successive iterations to f defined by

$$x^{k+1}=f(x^k) \quad (k=0,1,2\cdots).$$

The fundamental problem is to investigate the behavior of this sequence, given particular assumptions for f.

In [1] we assume that X is a finite set $\mathbf{Z}_m=\{0,1,2\cdots,m-1\}$, and f a function of the following form:

$$f(x)=ax+b \mod m \ (a,b\in\mathbf{Z}_m)$$

or X a finite set $\mathbf{Z}_m^n$ and f a mapping from $\mathbf{Z}_m^n$ into itself defined by

$$f(x)=Ax+b \mod m,$$

where A is an $n\times n$ matrix with elements from $\mathbf{Z}_m$ and b a vector with elements from $\mathbf{Z}_m$.

In this paper we assume that X is a set $\mathbf{Z}_m=\{0,1,2\cdots,m-1\}$, and f a polynomial with coefficients in $\mathbf{Z}_m$, computed by the operation of mod m, that is,

$$f(x)=\sum_{i=0}^{n} a_i x^{n-i} \mod m \ (x\in\mathbf{Z}_m),$$

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where $a_0, a_1, \cdots, a_n$ are elements in $\mathbf{Z}_m$.

Definition 1.1. We call a system $\mathbf{P}_{n,m,\mathbf{a}} = <X, f>$ an iteration scheme of modular type of order n (scheme, for short), where n and m are natural numbers, $\mathbf{a} = (a_0, a_1, \cdots, a_n)$ is an $(n+1)$ dimensional vector with elements in $\mathbf{Z}_m$, X is a set $\mathbf{Z}_m = \{0,1,2\cdots, m-1\}$, and f is a polynomial of the following form:

$$f(x) = \sum_{i=0}^{n} a_i x^{n-i} \mod m \ (x \in \mathbf{Z}_m),$$

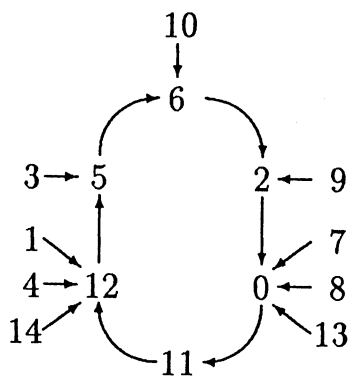
that is, the value $f(x)$ is computed by mod m operations.

Definition 1.2. The iteration graph of the scheme $\mathbf{P}_{n,m,\mathbf{a}} = <X, f>$ is the graph consisting of vertices which are elements of X and the following arcs: for all x in X, an arc conects x to $f(x)$.

Example 1.1. Let us take a scheme $\mathbf{P}_{2,15,(1,0,11)} = <X, f>$, where

$$X = \mathbf{Z}_{15} = \{0,1,2\cdots,14\}, \text{ and } f(x) = x^2 + 11 \mod 15 \ (x \in X)$$

The iteration graph of the scheme $\mathbf{P}_{2,15,(1,0,11)}$ is as follows:



The longest stable period of the scheme is 6, and scheme has no fixed points as seen in the iteration graph.

Definition 1.3. Let $<X_1, f_1>$ and $<X_2, f_2>$ be two iteration schemes. If there exists a bijection φ from X_1 onto X_2 such that the following diagram commutes, then the scheme $<X_1, f_1>$ is called to be isomorphic to the scheme $<X_2, f_2>$:

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & X_1 \arrow[d, "\varphi"] \\ X_2 \arrow[r, "f_2"] & X_2 \end{tikzcd},$$

that is,

$$\varphi \cdot f_1 = f_2 \cdot \varphi.$$

We write the above isomorphism as follows:

$$<X_1, f_1> \cong <X_2, f_2>.$$

2. Isomorphism of schemes.

Next we shall consider a product of two iteration schemes of modular type of order n_1 and n_2.

Definition 2.1. Let $\mathbf{P}_{n_1, m_1, \mathbf{a}_1} = <X_1, f_1>$ and $\mathbf{P}_{n_2, m_2, \mathbf{a}_2} = <X_2, f_2>$ be two iteration schemes of modular type of order n_1 and n_2, respectively, where n_1, m_1, n_2 and m_2 are natural numbers,

$$\mathbf{a}_1 = (a_0^{(1)}, a_1^{(1)}, \cdots, a_{n_1}^{(1)}),$$

and

$$\mathbf{a}_2 = (a_0^{(2)}, a_1^{(2)}, \cdots, a_{n_2}^{(2)}),$$

two (n_1+1) and (n_2+1) dimensional vectors with elements in $\mathbf{Z}_{m_1}$ and $\mathbf{Z}_{m_2}$, respectively, $X_1 = \mathbf{Z}_{m_1}$, $X_2 = \mathbf{Z}_{m_2}$ and

$$f_1(x) = \sum_{i=0}^{n_1} a_i^{(1)} x^{n_1 - i} \bmod m_1 \ (x \in \mathbf{Z}_{m_1}),$$

$$f_2(x) = \sum_{i=0}^{n_2} a_i^{(2)} x^{n_2 - i} \bmod m_2 \ (x \in \mathbf{Z}_{m_2}).$$

Then we call a scheme $<X, f>$ a product of two schemes $\mathbf{P}_{n_1, m_1, \mathbf{a}_1}$ and $\mathbf{P}_{n_2, m_2, \mathbf{a}_2}$, where

$$X = X_1 \times X_2 \ (= \mathbf{Z}_{m_1} \times \mathbf{Z}_{m_2})$$

and

$$f(x) := (f_1(x_1), f_2(x_2)) \ (x = (x_1, x_2) \in X).$$

We shall write the product by

$$\mathbf{P}_{n_1, m_1, \mathbf{a}_1} \times \mathbf{P}_{n_2, m_2, \mathbf{a}_2}.$$

Notice that product of finite number of schemes is defined similarly. We get the following result:

Theorem 2.1. *Let m_1 and m_2 be two relatively prime positive integers, and let m be the product: $m = m_1 \times m_2$.*

Given an iteration scheme of modular type of order n

$$\mathbf{P}_{n, m, \mathbf{a}} = <X, f>,$$

where $X = \mathbf{Z}_m$, $\mathbf{a} = (a_0, a_1, \cdots, a_n) \in X^{n+1}$ and

$$f(x)=\sum_{i=0}^{n} a_i x^{n-i} \mod m \ (x\in X),$$

we construct two iteration schemes of modular type of same order n:

$$\mathbf{P}_{n,m_1,\mathbf{a}_1}=<X_1, f_1> \text{ and } \mathbf{P}_{n,m_2,\mathbf{a}_2}=<X_2, f_2>,$$

where

$$\begin{gathered} X_1=\mathbf{Z}_{m_1},\ X_2=\mathbf{Z}_{m_2} \\ \boldsymbol{a}_1=(a_0^{(1)}, a_1^{(1)},\cdots,a_n^{(1)})\in X_1^{n+1}, \\ \boldsymbol{a}_2=(a_0^{(2)}, a_1^{(2)},\cdots,a_n^{(2)})\in X_2^{n+1}, \\ a_0\equiv a_0^{(1)} \pmod{m_1},\ a_0\equiv a_0^{(2)} \pmod{m_2}, \\ a_1\equiv a_1^{(1)} \pmod{m_1},\ a_1\equiv a_1^{(2)} \pmod{m_2}, \\ \vdots \\ a_n\equiv a_n^{(1)} \pmod{m_1},\ a_n\equiv a_n^{(2)} \pmod{m_2}, \end{gathered}$$

and

$$f_1(x)=\sum_{i=0}^{n} a_i^{(1)} x^{n-i} \mod m_1 \ (x\in X_1),$$

$$f_2(x)=\sum_{i=0}^{n} a_i^{(2)} x^{n-i} \mod m_2 \ (x\in X_2),$$

Then the scheme $\mathbf{P}_{n,m,\mathbf{a}}$ *is isomorphic to product* $\mathbf{P}_{n,m_1,\mathbf{a}_1}\times\mathbf{P}_{n,m_2,\mathbf{a}_2}$, *that is,*

$$\mathbf{P}_{n,m,\mathbf{a}}\cong\mathbf{P}_{n,m_1,\mathbf{a}_1}\times\mathbf{P}_{n,m_2,\mathbf{a}_2}.$$

Proof. The product of the schemes $\mathbf{P}_{n,m_1,\mathbf{a}_1}$ and $\mathbf{P}_{n,m_2,\mathbf{a}_2}$ is the scheme

$$\mathbf{P}_{n,m_1,\mathbf{a}_1}\times\mathbf{P}_{n,m_2,\mathbf{a}_2}=<X_1\times X_2, g>,$$

where

$$g(x_1, x_2)=(f_1(x_1), f_2(x_2)) \ (x_1\in X_1,\ x_2\in X_2).$$

It is sufficient to prove the existence of bijection φ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & X \arrow[d, "\varphi"] \\ X_1\times X_2 \arrow[r, "g"] & X_1\times X_2 \end{tikzcd}$$

For each $x\in X$ there exist uniquely two integers $x_1\in X_1$ and $x_2\in X_2$ such that

$$x\equiv x_1 \pmod{m_1}$$

and

$$x\equiv x_2 \pmod{m_2}.$$

Let φ be a mapping from X into $X_1 \times X_2$ such that

$$\varphi(x) = (x_1, x_2) \ (x \in X).$$

The mapping φ is a bijection from X onto $X_1 \times X_2$. It is sufficient to prove that the mapping φ is injective, since both sets X and $X_1 \times X_2$ are finite sets, and

$$\#(X) = \#(X_1 \times X_2).$$

Let us assume that $\varphi(x) = \varphi(y)$ $(x, y \in X)$. Letting $\varphi(x) = (x_1, x_2)$ and $\varphi(y) = (y_1, y_2)$, we get the following relations:

$$x_1 = y_1, \ x_2 = y_2 \text{ and}$$
$$x \equiv x_1 \pmod{m_1}, \ x \equiv x_2 \pmod{m_2},$$
$$y \equiv y_1 \pmod{m_1}, \ y \equiv y_2 \pmod{m_2},$$

Two integers m_1 and m_2 being relatively prime, and x and y belonging to X, we have $x = y$ by Chinese remainder theorem ([2]). We shall show that the mapping φ is commutative, that is, $\varphi \cdot f = g \cdot \varphi$. For each x in X, we have

$$g(\varphi(x)) = g(x_1, x_2) = (f_1(x_1), f_2(x_2)).$$

Furthermore we have for each $x \in X$

$$x \equiv x_1 \pmod{m_1} \text{ and } x \equiv x_2 \pmod{m_2}$$

So we get ([3])

$$x^k \equiv x_1^k \pmod{m_1} \text{ and } x^k \equiv x_2^k \pmod{m_2}, \ k = 2, 3, 4, \cdots.$$

And, from the relation $a_i \equiv a_i^{(1)} \pmod{m_1}$, and, $a_i \equiv a_i^{(2)} \pmod{m_2}$ $(i = 0, 1, 2, \cdots, n)$, we obtain

$$\sum_{i=0}^{n} a_i x^{n-i} \equiv \sum_{i=0}^{n} a_i^{(1)} x_1^{n-i} \pmod{m_1}$$

and

$$\sum_{i=0}^{n} a_i x^{n-i} \equiv \sum_{i=0}^{n} a_i^{(2)} x_2^{n-i} \pmod{m_2}$$

So we have

$$f(x) \equiv f_1(x_1) \pmod{m_1}$$

and

$$f(x) \equiv f_2(x_2) \pmod{m_2}.$$

The numbers $f_1(x_1)$ and $f_2(x_2)$ are uniquely determined in X_1 and X_2, respectively. Hence we have

$$\varphi(f(x)) = (f_1(x_1), f_2(x_2)) \ (x \in X).$$

The conclusion follows:

$$g(\varphi(x))=\varphi(f(x)) \quad (x\in X).$$

The schemes $\mathbf{P}_{n,\,m,\,\mathbf{a}}$ is, therefore, isomorphic to the product $\mathbf{P}_{n,\,m_1,\,\mathbf{a}_1}\times\mathbf{P}_{n,\,m_2,\,\mathbf{a}_2}$.

$$\mathbf{P}_{n,\,m,\,\mathbf{a}}\cong\mathbf{P}_{n,\,m_1,\,\mathbf{a}_1}\times\mathbf{P}_{n,\,m_2,\,\mathbf{a}_2}\,.$$

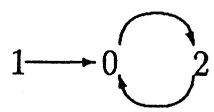
(Q. E. D.)

Example **2.1.**

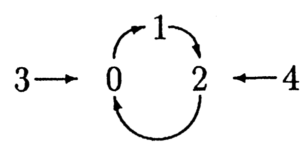
$$\mathbf{P}_{2,15,(1,0,11)}\cong\mathbf{P}_{2,3,(1,0,2)}\times\mathbf{P}_{2,5,(1,0,1)}.$$

Now $m=15$, so $m_1=3$ and $m_2=5$,

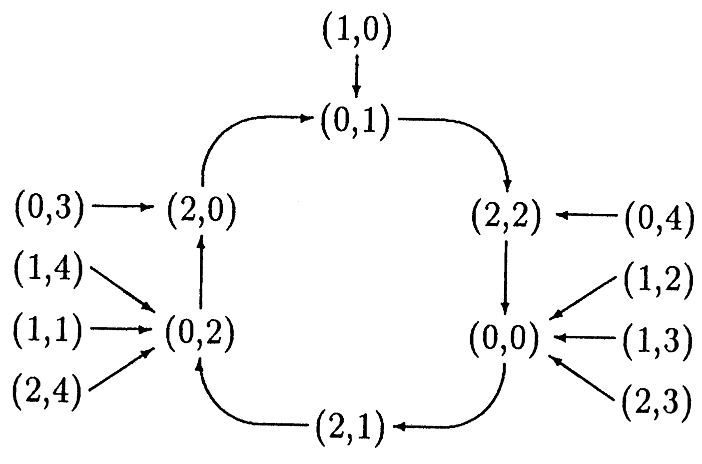
$$\mathbf{P}_{2,3,(1,0,2)}: f_1(x)=x^2+2 \quad \bmod 3$$



$$\mathbf{P}_{2,5,(1,0,1)}: f_2(x)=x^2+1 \quad \bmod 5$$



We have the following iteration graph of the product of schemes $\mathbf{P}_{2,3,(1,0,2)}$ and $\mathbf{P}_{2,5,(1,0,1)}$



This is isomorphic to the iteration graph of the scheme $\mathbf{P}_{2,15,(1,0,11)}$ (See Example 1.1).

3. Main theorem.

We have the following theorem which seems to be important and fundamental to analyse behaviors of schemes.

Theorem 3.1. *If m is a positive integer of the form*

$$m=p_1^{l_1}p_2^{l_2}\cdots p_k^{l_k},$$

where $\{p_i\}$ are prime numbers such that $p_1<p_2\cdots<p_k$ and l_i are positive integers $(i=1,2,\cdots,k;\ k>1)$, then an iteration scheme $\mathbf{P}_{n,m,\mathbf{a}}=<X, f>$ of modular type of order n defined by

$$f(x)=\sum_{i=0}^{n} a_i x^{n-i} \mod m\ (x\in X),$$

where $X=\mathbf{Z}_m$, and $\mathbf{a}=(a_0,a_1\cdots,a_n)$ in X^{n+1}, is isomorphic to the product of iteration schemes $\mathbf{P}_{n,m_j,\mathbf{a}_j}=<X_j, f_j>$ defined by

$$f_j(x)=\sum_{i=0}^{n} a_i^{(j)} x^{n-i} \mod m_j\ (x\in X_j),$$

where $m_j=p_j^{l_j}$, $X_j=\mathbf{Z}_{m_j}$ and $\mathbf{a}_j=(a_0^{(j)}a_1^{(j)},\cdots,a_n^{(j)})$ is in X_j^{n+1} satisfying the relations

$$a_i\equiv a_i^{(j)}\ (\mathrm{mod}\ m_j)\ (i=0,1,2,\cdots,n;\ j=1,2\cdots k),$$

that is,

$$\mathbf{P}_{n,m,\mathbf{a}}\cong\prod_{j=1}^{k}\mathbf{P}_{n,m_j,\mathbf{a}_j}.$$

Proof The product $\prod_{j=1}^{k}\mathbf{P}_{n,m_j,\mathbf{a}_j}$ is a system $<Y, g>$, where $Y=\prod_{j=1}^{k}X_j$ and g is a mapping from Y into itself such that for each $y=(x_1, x_2,\cdots,x_k)\in Y$,

$$g(y)=(f_1(x_1), f_2(x_2),\cdots, f_k(x_k))\in Y.$$

It is sufficient to prove that there exists a bijection φ from X onto Y such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \varphi\downarrow & \circlearrowright & \downarrow\varphi \\ Y & \xrightarrow{g} & Y \end{array},$$

that is, $g\cdot\varphi=\varphi\cdot f$. For each x in X there exists uniquely a vector $(x_1, x_2,\cdots,x_k)$ in Y such that

$$x\equiv x_j\quad (\mathrm{mod}\ m_j)\ (j=1,2,\cdots,k).$$

Let us define a mapping φ such that for each $x\in X$

$$\varphi(x)=(x_1, x_2,\cdots,x_k)\ (\in Y).$$

The mapping φ is an injection as easily shown by the Chinese remainder theorem, so is also surjective, thus, the mapping φ is bijective. And the mapping φ is commutative, since for each $x\in X$

$$g(\varphi(x)) = g(x_1, x_2, \cdots, x_k)$$
$$= (f_1(x_1), f_2(x_2), \cdots, f_k(x_k)) \in Y,$$

and

$$\varphi(f(x)) = (f_1(x_1), f_2(x_2), \cdots, f_k(x_k)),$$

the proof of which follows similarly from the proof of Theorem 2.1. Hence the scheme $\mathbf{P}_{n, m, \mathbf{a}}$ is isomorphic to the product of schemes $\prod_{j=1}^{k} \mathbf{P}_{n, m_j, \mathbf{a}_j}$.

(Q. E. D.)

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