Rep. Fac. Sci. Kagoshima Univ. (Math., Phys. & Chem.),
No. 26, 23-32, 1993.

A Note on Definable Subsets of N^k

Kiwamu AOYAMA*

(Received August 18, 1993)

Abstract

We give some remarks of the class of definable subsets of N^k in some formal language. In [1] we studied a characterization, multiple eventually periodic, of the definable subset in fragments of the first order arithmetic which contains the equivalence relation, the order relation, the modular relation and the successor function. In [4] Péladeau gives a nice characterization, semi-base-simple, of the class of definable subsets in the first order logic extended the modulo quantifier with the order relation. We see some relations between Péladeau's and our characterizations in this paper.

Key words: Semi-base-simple, Multiple eventually periodic.

1. Preliminaries

1.1. Basic notion and notation

The set of non negative integers is denoted by N. We denote the number zero, the successor function, the addition function, the order relation, and the binary relation of congruence modulo $q(1 \leq q)$ by 0, s, $+$, $<$, and $\equiv_q$, respectively. For a positive integer k, the Cartesian product N^k is defined inductively as follows; $N^1=N$, $N^{k+1}=N^k \times N$.

A *monoid* M is a set equipped with an associative binary operation (or product) and an identity element. For any subset S of a monoid M with product $*$, the submonoid generated by S is denoted by S^*. Let k be a positive integer. N^k is a monoid with componentwise addition, also write $+$, as binary operation and 0 vector as identity element. Since the product of N^k is $+$, S^* is also denoted by $S^{\oplus}$ for $S \subset N^k$. For $S \subset N^k$ and $V \subset N^k$,

$$S+V=\{x \mid \exists s \exists v (s \in S \wedge v \in V \wedge x=s+v)\}.$$

When S or V is a certain element of N^k, we abuse of above notation. For example, for $u \in N^k$

* Department of Mathematics, Faculty of Science, Kagoshima University, 1-21-35 Korimoto, Kagoshima 890, Japan.

and $V \subset N^k$,

$$u + V^{\oplus} \quad (= \{u\} + V^{\oplus}),$$

and for u, $v \in N^k$,

$$u + v^{\oplus} \quad (= \{u\} + \{v\}^{\oplus}),$$

and so on.

1.2. Formal language with quantifier

In [4], formal language with quantifier is called theory. To be familiar with [4], we will use 'theory' in this sence.

The *first order modular theory of* $<$, which we denote by $Th_{1+mod}[<]$, is the set of formulas obtained from

- variables x_1, x_2, x_3, $\cdots$;
- the less-than predicate $<$;
- Boolean connectives $\wedge$, $\vee$, $\neg$;
- quantifiers $\exists$, and $\exists_q^p$ for $1 \leq q$, $0 \leq p < q$.

The variables are interpreted as natural numbers. The binary predicate $<$ has its usual meaning. The formula $\exists_q^p x\phi(x)$ is true iff the number n of natural numbers i, such that ϕ is ture when we replace x by i, is congruent to p modulo q. $Th_1[<]$ is that $\exists_q^p$ take off the $Th_{1+mod}[<]$, and Th_{mod} is that restriction of first order take off the $Th_{1+mod}[<]$. The first order theory of s and $=$, denoted $Th_1[s,=]$, is the set of formulas obtained from the above definition of $Th_1[<]$ in which, instead of using the predicate $<$, we use the function s and the predicate $=$.

The definitions above are in [4]. The definition of Th_{mod} is felt inclear. We will state later, do not know whether it is a reason for or not, there exists a state in [4] be not understood. Remark that $Th_{1+mod}[<]$ must be sub-theory of $Th_{mod}[<]$ since $Th_{mod}[<]$ is given by taken off the restriction from $Th_{1+mod}[<]$.

We introduce other 'theory' more natural by usual way. The *first order language* $L[R_1, R_2, \cdots; f_1, f_2, \cdots; c_1, c_2, \cdots]$ is the set of formulas obtained from

- variables x_1, x_2, $\cdots$;
- predicates R_1, R_2, $\cdots$;
- functions f_1, f_2, $\cdots$;
- constant's c_1, c_2, $\cdots$;
- Boolean conectives $\wedge$, $\neg$;
- quantifier $\forall$.

The variables are interpreted as natural numbers. Predicates, functions, and constants are interpreted as usual meaning.

We only deal with sub-language of $L[=, <, \equiv_1, \equiv_2, \cdots; s; 0]$. For a natural number n, the *numeral* $\bar{n}$ is defined by $\bar{0}=0$, $\overline{n+1}=s(\bar{n})$. For a natural number n and a variable v, $\overline{v+n}$ is defined by $\overline{v+0}=v$, $\overline{v+(n+1)}=s(\overline{v+n})$. The $\bar{n}$ of $\overline{+n}$ in this case is also called numeral.

1.3. Formal language without quantifier

Let $\gamma_{t,q}$ be the congruence on N defined by $i\gamma_{t,q}\, k$ iff $i<t$ implies $i=j$, and $t\leq i$ implies $t\leq j$ and $i\equiv_q j$. The *language of congruence arithmetic*, denote as LCA_{1+mod}, is the set of formulas obtained from

- variables $x_1, x_2, \cdots$;
- unary predicate $C_{n,t,q}$ for $0\leq t$, $1\leq q$ and $0\leq n<t+q$;
- binary predicate $D_{n,t,q}$ for $0\leq t$, $1\leq q$ and $0\leq n<t+q$;
- logical connerctives $\wedge$, $\vee$, $\neg$.

The predicate $C_{n,t,q}(x)$ is true iff $x\gamma_{t,q}\, n$ and the predicate $D_{n,t,q}(x, y)$ is true iff $y<x$ and $C_{n,t,q}(x-y-1)$. We use LCA_1 and LCA_{mod} to denote the restrictions of LCA_{1+mod} when q is fixed to 1 and t is fixed to 0, respectively.

The definitions above are in [4]. These are very technical. Remark that LCA_{mod} is sub-language of LCA_{1+mod}.

We will give some quantifier free language more natural by usual way. The *quantifier free first order language* $QFL[R_1, R_2, \cdots; f_1, f_2, \cdots; c_1, c2, \cdots]$ is the set of formulas obtained from

- variables $x_1, x_2, \cdots$;
- predicates $R_1, R_2, \cdots$;
- functions $f_1, f_2, \cdots$;
- constants $c_1, c_2, \cdots$;
- logical conectives $\wedge$, $\neg$.

The variables are interpreted as natural numbers. Predicates, functions, and constants are interpreted as usual meaning.

We only deal with sub-language of $QFL[=, <, \equiv_1, \equiv_2, \cdots; s; 0]$. A logical operator which is not in language is ususal abbreviation. For example, in $QFL[=; s; 0]$, $\phi \rightarrow \varphi$ means $\neg(\phi\wedge\neg\varphi)$, and so on.

2. Definable sets and quantifier elimination

Let L be a formal language, or 'theory', and k a positive integer. A vector $v\in N^k$ is said to *satisfy* a formula $\phi\ (x_1, \cdots, x_k)$, where the x_i are free variables, if $\phi\ (v_1, \cdots, v_k)$ is true, where v_i is the i-th component of vector v. So, a subset $S\in N^k$ is said to *definable* in L if there exists a formula ϕ in L with k free variables such that

$$S=\{v\in N^k \mid v \textit{ satisfies } \phi\}.$$

We will confuse a formal language L with the class of definable subsets in L. The following is well known (see [1], [3]).

Theorem 2.1 (Quantifier elimination)

1. $L[=;\ s;\ 0]\ =QFL[=;\ s;\ 0]$.

2. $L[=,\ <;\ s;\ 0]\ =QFL[=,\ <;\ s;\ 0]$.

3. $L[=,\ \equiv_1,\ \equiv_2,\ \cdots;\ s;\ 0]\ =QFL[=,\ \equiv_1,\ \equiv_2,\ \cdots;\ s;\ 0]$.

4. $L[=,\ <,\ \equiv_1,\ \equiv_2,\ \cdots;\ s;\ 0]\ =QFL[=,\ <,\ \equiv_1,\ \equiv_2,\ \cdots;\ s;\ 0]$.

Péladeau state the following theorem.

Theorem 2.2 (Theorem 2.2 in [4])

1. $Th_{1+mod}[<]=LCA_{1+mod}$.

2. $Th_1[<]=LCA_1$.

3. $Th_{mod}[<]=LCA_{mod}$.

From this theorem, we get $Th_{1+mod}[<]=Th_{mod}[<]$ and $LCA_{1+mod}=LCA_{mod}$ since $Th_{1+mod}[<]$ is sub-theory of $Th_{mod}[<]$ and $LCA_{mod}[<]$ is sub-language of LCA_{1+mod}. Unfortunately, this contradicts to Theorem 4.2 in [4]. We will ***not*** reffer to $Th_{mod}[<]$ from now on. We will see other properties.

Theorem 2.3 *1.* $Th_1[s,\ =]\ =QFL[=;\ s;\ 0]$.

2. $LCA_{1+mod}=QFL[=,\ <,\ \equiv_1,\ \equiv_2,\ \cdots;\ s;\ 0]$.

3. $LCA_1=QFL[=,\ <;\ s;\ 0]$.

Proof 1. It is suffices to show that $x=\bar{n}$ is definable in $Th_1[s,\ =]$ for any natural number n. This can be carry out by the following way,

- $x=\bar{0}\ \leftrightarrow\ \neg\exists y\ (x=s(y))$,
- $x=\bar{1}\ \leftrightarrow\ \neg\exists y\ (x=s(s(y)))\ \wedge x\neq\bar{0}$,
- $x=\bar{2}\ \leftrightarrow\ \neg\exists y\ (x=s(s(s(y))))\ \wedge x\neq\bar{0}\wedge x\neq\bar{1}$,

and so on. 3. $LCA_1\subset QFL[=,\ <;\ s;\ 0]$ is easy. We show that $QFL[=,\ <;\ s;\ 0]\subset LCA_1$. It is suffices to show that a definable subset by an atomic formula in $QFL[=,\ <;\ s;\ 0]$ is definable in LCA_1. This can be seen by the following;

- $x=y\ \leftrightarrow\ \neg D_{0,0,1}(x,y)\vee\neg D_{0,0,1}(y,x)$,
- $x=\bar{n}\ \leftrightarrow\ C_{n,n+1,1}(x)$,
- $\bar{m}=\bar{n}\ \leftrightarrow\ \begin{cases} x=\bar{0}\wedge\neg x=\bar{0} & \text{if } m\neq n, \\ x=\bar{0}\vee\neg x=\bar{0} & \text{if } m=n, \end{cases}$
- $x=\overline{y+n}\ \ (n\neq 0)\ \leftrightarrow\ D_{n-1,n,1}(x,y)$,
- $y<x\ \leftrightarrow\ D_{0,0,1}(x,y)$,
- $x<\bar{n}\ \leftrightarrow\ \begin{cases} x=\bar{0}\vee\ \cdots\ \vee x=\overline{n-1} & \text{if } n\neq 0, \\ x=\bar{0}\wedge\neg x=\bar{0} & \text{if } n=0, \end{cases}$
- $\bar{n}<x\ \leftrightarrow\ \neg\ (x<\bar{n}\vee x=\bar{n})$,

- $\overline{y+n}<x \leftrightarrow D_{n,n,1}(x, y)$,
- $\bar{m}<\bar{n} \leftrightarrow \begin{cases} x=\bar{0}\wedge\neg x=\bar{0} & \text{if } m\neq n, \\ x=\bar{0}\vee\neg x=\bar{0} & \text{if } m=n, \end{cases}$
- $y<\overline{x+n}\ (n\neq 0) \leftrightarrow y=x\vee y=\overline{x+1}\vee \cdots \vee y=\overline{x+(n-1)}\vee y<x.$

2. is similar. □

LCA_{mod} can not be reduced to a usual first order language of fragment of arithmetic. In this sence, LCA_{mod} is not simple. We introduce the restricted order relation $<^*$ which is usual order relation with the following restriction;

both left and right arguments are only variables,

and is interpreted as usual order. For example, $x_1<^*x_2$ is allowd formula but neither $x_1<^*s(0)$ nor $s(x_2)<^*x_1$.

Theorem 2.4 $LCA_{mod}=QFL[<^*, \equiv_1, \equiv_2, \cdots; s; 0]$.

Proof It is suffices to show that a definable subset by an atomic formula in $QFL[<^*, \equiv_1, \equiv_2, \cdots; s; 0]$ is definable in LCA_{mod}. This can be seen by the following;

- $y<^*x \leftrightarrow D_{0,0,1}(x, y)$,
- $x\equiv_1 \bar{n} \leftrightarrow C_{n,0,q}(x)$,
- $x\equiv_1 y \leftrightarrow D_{0,0,1}(x, y)\vee\neg D_{0,0,1}(x, y)$,
- $x\equiv_q y\ (1<q) \leftrightarrow (\neg D_{0,0,q}(x, y)\wedge \cdots \wedge\neg D_{q-2,0,q}(x, y))\vee(\neg D_{0,0,q}(y, x)\wedge \cdots \wedge\neg D_{q-2,0,q}(y, x))$,
- $x\equiv_q \overline{y+n}\ (n\neq 0) \leftrightarrow D_{n-1,0,q}(x, y)\vee D_{n-1,0,q}(y, x)$.

The converse is easy. □

3. Characterizations

3.1. Semi-base-simple

In [4], Péladeau gives nice characterizations of the definable subsets in LCA_{1+mod}, LCA_1 and LCA_{mod}. We study his characterizations in this section.

Let k be a positive integer, and $[k]$ means the set $\{1, \cdots, k\}$. *A strict-ordering formula* ρ on the variables $x_1, \cdots, x_k$ is a formula of the form

$$x_{\sigma(1)}\ c_1\ \cdots\ c_{k-1}\ x_{\sigma(k)},$$

where $\sigma: [k] \to [k]$ is a permutation, and each c_i is either an $=$ or a $<$. The *rank* of a strict-order formula ρ, denoted as $rk(\rho)$, is the number of $<$ plus one. The formula ρ partitions the set $[k]$ into disjoint subsets $I_1, \cdots, I_{rk(\rho)}$ such that $v\in N^k$ satisfies ρ iff $i, i'\in I_j$ implies $v_i=v_{i'}$, and $i\in I_j$, $i'\in I_{j'}$ and $j<j'$ implies $v_i<v_{i'}$. Given a partitioning of $[k]$ into $I_1, \cdots, I_l$, we denote $I_j^{\uparrow}=\cup_{j'=j}^{l} I_j$ for $j\in[l]$. Let $E=\{e_1, \cdots, e_k\}$ be the natural base of N^k. If $I\subset[k]$, then e_I denotes $\sum_{i\in I} e_i$. A subset of N^k

$$X=u+\sum_{j=1}^{rk(\rho)}(q_j e_{I_j^{\uparrow}})^{\oplus},$$

where $u\in N^k$, $0\leq q_j$ is said to be *bese-simple* if u satisfies a strict-ordering formula ρ whose associated partitioning of $[k]$ is $I_1, \cdots, I_{rk(\rho)}$.

A finit disjoint union of base-simple sets is said to be *semi-base-simple.* The set of base-simple subsets of N^k is denoted by $BS(N^k)$ and the semi-base-simple subsets of N^k by $SBS(N^k)$. $BS_1(N^k)$ is the set of base-simple subsets of N^k where in the definition each $q_i\in\{0, 1\}$. $BS_{mod}(N^k)$ is the set of base-simple subset of N^k where in the definition each $q_i\geq 1$, $0\leq u_i<q_1$ for each $i\in I_1$, and $0\leq u_i-u_{i'}-1<q_j$ for each $1<j<rk(\rho)$, $i\in I_j$ and $i'\in I_{j-1}$. $SBS_1(N^k)$ (or $SBS_{mod}(N^k)$) denotes the subsets of N^k which are finit disjoint unions of sets in $BS_1(N^k)$ (or $BS_{mod}(N^k)$), respectively.

We define $SBS_{s,=}(N^k)$ to be subsets of N^k of the form $X=\bigcup_{s=1}^{t} X_s$, with the union being disjoint and such that the $X_s\in BS_1(N^k)$ satisfy the following condition. Let

$$X_s=v+\sum_{j=1}^{rk(\rho)}(q_j e_{I_j^{\uparrow}})^{\oplus},$$

then for each permutation $\sigma : [rk(\rho)] \to [rk(\rho)]$ such that $q_i=0$ implies $\sigma(j)=j$, there is an $s_\sigma\in[t]$ such that

$$X_{s_\sigma}=v+\sum_{j=1}^{rk(\rho)}(q_j e_{I_{\sigma(j)}^{\uparrow}})^{\oplus},$$

where $I^{\uparrow}_{\sigma(j)}=\bigcup_{j'=\sigma(j)}^{rk(\rho)} I_{j'}$, $q_1=0$ implies $u_i=v_i$ for each $i\in I_1$, and for $j>1$, $q_j=0$ implies $u_i-u_{i'}=v_i-v_{i'}$ for each $i\in I_j$ and $i'\in I_{j-1}$.

Lemma 3.1 (c.f. Lemma 3.2 and Lemma 5.1 in [4]) *Let* $X\in SBS(N^k)$.

1. $X\times N\in SBS(N^{k+1})$.
2. $N\times X\in SBS(N^{k+1})$.
3. $\{(x_1, \cdots, y, \cdots, x_k)\mid y\in N\wedge(x_1, \cdots, x_k)\in X\}\in SBS(N^{k+1})$.

The above lemma also holds for $SBS_1(N^k)$, $SBS_{mod}(N^k)$ and $SBS_{s,=}(N^k)$.

Lemma 3.2 (c.f. Lemma 3.4 and Lemma 5.3 in [4]) $SBS(N^k)$ *is a Boolean algebra with respect to union, intersection and complementation. Also* $SBS_1(N^k)$, $SBS_{mod}(N^k)$ *and* $SBS_{s,=}(N^k)$.

The class of definable subsets of N^k in LCA_{1+mod} is denoted by $LCA_{1+mod}(N^k)$. $LCA_1(N^k)$ and $LCA_{mod}(N^k)$ are similar.

Theorem 3.3 (Theorem 3.3 and Theorem 5.2 in [4])

1. $LCA_{1+mod}(N^k)=SBS(N^k)$.
2. $LCA_1(N^k)=SBS_1(N^k)$.
3. $LCA_{mod}(N^k)=SBS_{mod}(N^k)$.
4. $Th_1[s, =] = SBS_{s,=}(N^k)$.

3.2. Multiple eventually periodic

In this section, we study *multiple eventually periodic* introduced in [1].

Let S be a subset of N^{k+1}. For a positive integer j $(\leq k+1)$ and a natural number n, the subset $S_{j\text{-}th=n}$ of N^k is

$$\{(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_{k+1}) \mid (x_1, \cdots, x_{j-1}, n, x_{j+1}, \cdots, x_{k+1}) \in S\}.$$

We denote $n<x_i$ for all $i=1, \cdots, k$ by $n<(x_1, \cdots, x_k)$, and $x_i \equiv_q y_i$ for all $i=1, \cdots, k$ by $(x_1, \cdots, x_k) \equiv_q (y_1, \cdots, y_k)$.

For a subset S of N^k and positive integers b and q, 'S is $MEP[b, q](N^k)$' which is read that S *is multiple eventually periodic with bound* b *and period* q is defined inductively on k as follows;

1. $k=1: x\in S \leftrightarrow x+q\in S$ if $b<x$.
2. $k>1:$
   - (a) $(x_1, \cdots, x_k)\in S \leftrightarrow (x_1+q, \cdots, x_k+q)\in S$ if $b<(x_1, \cdots, x_k)$,
   - (b) $S_{j\text{-}th=n}$ is $MEP[b+n, q](N^{k-1})$ for any natural number n and any positive integer j $(\leq k)$.

Under the same situation, $MEP^-[b, q](N^k)$ is also defined as follows;

1. $k=1: x\in S \leftrightarrow x+q\in S$ if $b<x$.
2. $k>1:$
   - (a) i. $(x_1, \cdots, x_k)\in S \leftrightarrow (x_1+q, \cdots, x_k+q)\in S$ if $b<(x_1, \cdots, x_k)$.
     ii. $(x_1, \cdots, x_k)\in S \leftrightarrow (y_1, \cdots, y_k)\in S$ if $b<(x_1, \cdots, x_k)$, $b<(y_1, \cdots, y_k)$, $(x_1, \cdots, x_k)\equiv_q(y_1, \cdots, y_k)$, $b<|x_i-x_j|$ and $b<|y_i-y_j|$ for $1\leq i\neq j\leq k$.
   - (b) $S_{j\text{-}th=n}$ is $MEP^-[b+n, q](N^{k-1})$ for any natural number n and any positive integer j $(\leq k)$.

In MEP^- the superscript '$-$' means 'without the order relation'. For $S\subset N^k$, if there exist b and q such that S is $MEP[b, q](N^k)$, then we say that S is in $MEP_{<+mod}(N^k)$. $MEP_{mod}(N^k)$, $MEP_<(N^k)$ and $MEP(N^k)$ are similar. More precisely,

- $MEP_{<+mod}(N^k)=\{S \mid S\subset N^k \wedge \exists b \exists q\ (S \text{ is } MEP[b, q](N^k))\}$.
- $MEP_{mod}(N^k)=\{S \mid S\subset N^k \wedge \exists b \exists q\ (S \text{ is } MEP^-[b, q](N^k))\}$.
- $MEP_<(N^k)=\{S \mid S\subset N^k \wedge \exists b\ (S \text{ is } MEP[b, 1](N^k))\}$.
- $MEP(N^k)=\{S \mid S\subset N^k \wedge \exists b\ (S \text{ is } MEP^-[b, 1](N^k))\}$.

Lemma 3.4 *Let* $X \in MEP_{<+mod}(N^k)$.

1. $X \times N \in MEP_{<+mod}(N^{k+1})$.
2. $N \times X \in MEP_{<+mod}(N^{k+1})$.
3. $\{(x_1, \cdots, y, \cdots, x_k) \mid y \in N \wedge (x_1, \cdots x_k) \in X\} \in MEP_{<+mod}(N^{k+1})$.

The above lemma also holds for $MEP_{mod}(N^k)$, $MEP_<(N^k)$ and $MEP(N^k)$.

Lemma 3.5 $MEP_{<+mod}(N^k)$ *is a Boolean algebra with respect to union, intersection and complementation. Also* $MEP_{mod}(N^k)$, $MEP_<(N^k)$ *and* $MEP(N^k)$.

Proof The proof is strightforword but tedious work. □

The class of definable subsets of N^k in $L[=, <, \equiv_1, \equiv_2, \cdots; s; 0]$ is denoted by $L[=, <, \equiv_1, \equiv_2, \cdots; s; 0](N^k)$. $L[=, \equiv_1, \equiv_2, \cdots; s; 0](N^k)$, $L[=, <, \equiv_1, \equiv_2, \cdots; s; 0](N^k)$ and $L[=; s; 0](N^k)$ are similar.

Theorem 3.6 (c.f. [1], [2], [5])

1. $QFL[=, <, \equiv_1, \equiv_2, \cdots; s; 0](N^k) = MEP_{<+mod}(N^k)$.
2. $QFL[=, \equiv_1, \equiv_2, \cdots; s; 0](N^k) = MEP_{mod}(N^k)$.
3. $QFL[=, <; s; 0](N^k) = MEP_<(N^k)$.
4. $QFL[=; s; 0](N^k) = MEP(N^k)$.

Proof For any formula, a bound b is the maximum of all numerals occuring in the formula, and a period q is the least common multiple of all t's occuring of the form $\equiv_t$ in the formula. The converse is by induction on k. □

4. Conclusion

SBS is the union of $SBS(N^k)$ by k. SBS_1, SBS_{mod}, $SBS_{s,=}$, $MEP_{<+mod}$, MEP_{mod}, $MEP_<$ and MEP are similar. The following equations are immediate consequences from previous theorems.

1. $SBS = LCA_{1+mod} = Th_{1+mod}[<] = L[=, <, \equiv_1, \equiv_2, \cdots; s; 0] = QFL[=, <, \equiv_1, \equiv_2, \cdots; s; 0] = MEP_{<+mod}$,
2. $SBS_1 = LCA_1 = Th_1[<] = L[=, <; s; 0] = QFL[=, <; s; 0] = MEP_<$,
3. $L[=, \equiv_1, \equiv_2, \cdots; s; 0] = QFL[=, \equiv_1, \equiv_2, \cdots; s; 0] = MEP_{mod}$,
4. $SBS_{mod} = LCA_{mod} = QFL[<^*, \equiv_1, \equiv_2, \cdots; s; 0]$,
5. $SBS_{s,=} = Th_1[s, =] = L[=; s; 0] = QFL[=; s; 0] = MEP$.

We see properness of inclusion to each class.

Lemma 4.1 (Theorem 4.2 in [4]) *The following holds, and each inclusion is proper.*

1. $SBS_1 \subset SBS$.
2. $SBS_{mod} \subset SBS$.

Lemma 4.2 (c.f. Corollary 1 and 2 in [1]) *The following holds, and each inclusion is proper.*

1. $MEP \subset MEP_{<} \subset MEP_{<+mod}$.
2. $MEP \subset MEP_{mod} \subset MEP_{<+mod}$.

Proof All inclusion are clear by the definition. Assume that $Odd=\{x \mid x \equiv_2 1\}$ of N^1 is in $MEP_{<}$, or in MEP. By the definition of $MEP_{<}$, or of MEP, there exists a bound b such that $x \in Odd \leftrightarrow x+1 \in Odd$ for $b<x$. This is a contradiction. That is to say, Odd is in neither $MEP_{<}$ nor MEP. Hence MEP is a proper subset of MEP_{mod}, and $MEP_{<}$ is a proper subset of $MEP_{<+mod}$. Next, we assume that the subset $Ord=\{(x, y) \mid x<y\}$ of N^2 is in MEP_{mod}, or in MEP. By the definition of MEP_{mod}, or of MEP, there exist a bound b and a period q such that $(x, y) \in Ord \leftrightarrow (u, v) \in Ord$ for $b<(x, y)$, (u, v) and $(x, y) \equiv_q (u, v)$ and $b<|x-y|, |u-v|$. Especially, $(2 \cdot (b+1) \cdot q, (b+1) \cdot q) \in Ord$. This is a contradiction. That is to say, Ord is in neither MEP_{mod} nor MEP. Hence MEP is a proper subset of $MEP_{<}$, and MEP_{mod} is a proper subset of $MEP_{<+mod}$. □

Lemma 4.3 *$MEP_{<}$, SBS_{mod}* and *MEP_{mod} are incomparable under inclusion. Also SBS_{mod} and MEP.*

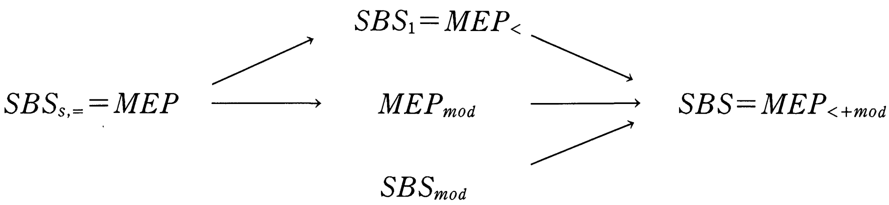
Proof We consider $QFL[<^*, \equiv_1, \equiv_2, \cdots; s; 0]$. Assume that the equivalence relation $=$ is definable in this language. Then this language becomes to $QFL[=, <, \equiv_1, \equiv_2, \cdots; s; 0]$ since the no restricted order $<$ is definable in this by the following way;

$$\overline{y+n}<x \leftrightarrow y<^*x \wedge y \neq x \wedge \overline{y+1} \neq x \wedge \cdots \wedge \overline{y+n} \neq x,$$

$$y<\overline{x+n} \quad (n \neq 0) \leftrightarrow y<^*x \vee y=x \vee y=\overline{x+1} \vee \cdots \vee y=\overline{x+(n-1)},$$

and so on. But this contradicts the theorem 4.1. Thus the equivalence relation $=$ is not definable in $QFL[<^*, \equiv_1, \equiv_2, \cdots; s; 0]$. Hence SBS_{mod} includes neither $MEP_{<}$ nor MEP_{mod}. And futher, this also does not include MEP. By the proof of the previous lemma, $MEP_{<}$ includes neither SBS_{mod} nor MEP_{mod} since Odd is not in $MEP_{<}$, and MEP_{mod} includes neither SBS_{mod} nor $MEP_{<}$ since Ord is not in MEP_{mod}, and since Odd is not in MEP then MEP does not include SBS_{mod}. □

We get the following figure. $S \rightarrow S'$ means that S is a proper subset of S'. Any arrow can not be added in the figure by previous lemmata.



We know neither SBS-type characterization for $QFL[=, \equiv_1, \equiv_2, \cdots; s; 0]$ nor MEP-type characterization for $QFL[<^*, \equiv_1, \equiv_2, \cdots; s; 0]$. We do not know the first order theory, or the first order language with quantifier, corresponding to $QFL[<^*, \equiv_1, \equiv_2, \cdots; s; 0]$. *SBS*-type characterization is useful to getting a positive result, that is, to show that a subset is definable in. *MEP*-type characterization is useful to getting a negative result, that is, to show that a subset is not definable in. An importance is that we get both *SBS*- and *MEP*-type characterizations for $Rec(N^k)$ and for $Rat(N^k)$ (c.f. [4]).

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