ON LEBESQUE'S BOUNDED CONVERGENCE THEOREM

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§ **1. Introduction.** One knows Lebesque's bounded convergence theorem ([1] 26.D). John W. Pratt has shown that a convergent sequence of integrable functions permits exchange of lim and $\int$ if it is bracketed by two sequences which permit this exchange [2]. In this paper we treat these theorems. It is assumed throughout this note that the underlying space X is a measure space (X, S, μ).

§ **2. Theorems.**

THEOREM 1. *If $\{f_n\}$ is a sequence of intergrable functions which converges in measure to f [or else converges to f a.e.], and if, g and h are integrable functions such that $g(x) \leqq f_n(x) \leqq h(x)$ a.e., $n = 1,2,\cdots\cdots$, then f is integrable and the sequence $\{f_n\}$ converges to f in the mean.*

COROLLARY. If $\{f_n\}$ is a sequence of integrable functions which converges in measure to f [or else converges to f a.e.], and if g is an integrable function such that $|f_n(x)| \leqq |g(x)|$ a.e., $n=1,2,\cdots\cdots$, then f is integrable and the sequence $\{f_n\}$ converges to f in the mean. ([1] 26.D Lebesque's bounded convergence theorem).

To prove this theorem we need some lemmas.

LEMMA 1. The indefinite integral of an integrable function is absolutely continuous. ([1] 23.H)

LEMMA 2. A sequence $\{f_n\}$ of integrable functions converges in the mean to the integrable function f if and only if $\{f_n\}$ converges in measure to f and the indefinite integrals of $|f_n|$, $n=1,2,\cdots\cdots$, are uniformly absolutely continuous and equicontinuous from about at 0. ([1] 26.C)

PROOF OF THEOREM 1. (1) In the case of convergence in measure,

(a) it is assumed that $\{f_n\}$ converges in measure to f,

(b) uniformly absolutely continuity,

by assumption, g and h are inteegrable, then $|g|$ and $|h|$ are integrable also. Therefore, from lemma 1, for every positive number $\in$ there exist a positive number δ such that $\int_E |g| d\mu < \in$ and $\int_E |h| d\mu < \in$, for every measurable set E for which $\mu(E) < \delta$, then we have

$$\int_E |f_n| d\mu \leqq \max \left(\int_E |g| d\mu, \int_E |h| d\mu \right) < \in$$

for every positive integer n,

(c) equicontinuity,

if $\{E_m\}$ is a decreasing sequence of measurable sets with an empty intersection, then there exists a positive integer m_0 such that, $m > m_0$,

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$|\int_{E_m} f_n \, d\mu| \leqq \max(\int_{E_m} |g| d\mu, \int_{E_m} |h| d\mu) < \in \ (n=1,2,\cdots\cdots)$,

the desired result follows from lemma 2.

(2) The case of convergence a.e.,

since $|f_n - f| < \in$ a.e.,

$|f_n| \leqq \max(|g|, |h|)$ and $|f| \leqq \max(|g|, |h|)$ a.e.,

hence, if we assume, as we may without any loss of generality, that

$|f_n(x)| \leqq \max\{|g(x)|, |h(x)|\}$ and $|f(x)| \leqq \max\{|g(x)|, |h(x)|\}$ for every x in X, then we have, for every fixed positive number $\in$,

$$E_n = \bigcup_{i=n}^{\infty} \{x: |f_i(x) - f(x)| \geqq \in\} \subset \{x: \max(|g|, |h|) \geqq \tfrac{1}{2} \in\},$$

(because $|f_i(x) - f(x)| \leqq |f_i(x)| + |f(x)| \leqq 2 \max(|g|, |h|)$),

and therefore $\mu(E_n) < \infty$, $n = 1,2,\cdots\cdots$. Since the assumption of convergence a.e. implies that $\mu(\bigcap_{n=1}^{\infty} E_n) = 0$, it follows that

$$\limsup_n \mu(\{x: |f_n(x) - f(x)| \geqq \in\}) \leqq \lim_n \mu(E_n) = \mu(\lim_n E_n) = 0,$$

hence $\mu(\{x: |f_n(x) - f(x)| \geqq \in\}) \to 0 \ (n \to \infty)$,

the desired result follows from (1). ▮

PROOF OF COROLLARY.

Applying Theorem 1 to $|g(x)|$ in place of $\max(|g(x)|, |h(x)|)$, the desired result follows. ▮

THEOREM 2. *If $\{f_n\}$ is a sequence of integrable functions which converges in measure to f [or also converges to f a.e.] and if, $\{g_n\}$ and $\{h_n\}$ are mean fundamental sequence of integrable functions such that*

$g_n(x) \leqq f_n(x) \leqq h_n(x)$ a.e. $n=1,2,\cdots\cdots$, then f is integrable and the sequence $\{f_n\}$ converges to f in the mean.

To prove this theorem we need some lemmas.

LEMMA 1. If $\{f_n\}$ is a mean fundamental sequence of integrable functions, then there exists an integrable function f such that $\rho(f_n, f) \to 0$ (and consequently $\int f_n d\mu \to \int f d\mu$) as $n \to \infty$. ([1] 26.B)

LEMMA 2. A mean fundamental sequence $\{f_n\}$ of integrable functions is fundamental in measure. ([1] 24.A)

PROOF OF THEOREM 2. By lemma 1 and lemma 2, for $\in > 0$, there exists an integer N such that $n > N$ implies $|f_n - f| < \in$, that is $f - \in < f_n < f + \in \ (n = N+1, N+2, \cdots\cdots)$, if we write $g = \min(f_1, f_2, \cdots\cdots, f_N, f - \in, f + \in)$

and $h = \max(f_1, f_2, \cdots\cdots, f_N, f - \in, f + \in)$,

it is clear that g and h are integrable.

Hence the desired result follows from Theorem 1. ▮

Moreover, if we use the following lemma, we see that this Theorem 2 is equal to Pratt's Theorem. Namely, Theorem 2 is another form of Pratt's Theorem and the proof of Theorem 2 is another proof of Pratt's Theorem.

LEMMA. A sequence $\{f_n\}$ of integrable functions converges to f in the mean if and only if $\int_A f_n d\mu \rightarrow \int_A f d\mu$ uniformly for $A \varepsilon$ **S**. ([3] Appendix).

REFERENCES

[1] Paul R. Halmos, *Measure Theory*, D. Van Nostrand, New York, 1950.

[2] John W. Pratt, "On interchanging limit and integrals," Ann. Math. Stat., Vol. 31, No.1(1960), 74--77.

[3] Lehmann, E.L. (1959) *Testing Statistical Hypotheses*. Wiley, New York.