The Norm and the Semi-norm over the Sobolev Spaces

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§1. Introduction

Let V be a Hilbert space, $a(\cdot,\cdot)$: $V\times V\to\boldsymbol{R}$ be a continuous *V-elliptic* bilinear form, and f: $V\to\boldsymbol{R}$ be a continuous linear form. Then the abstract variational problem is: *Find an element u such that*

$$u\in V \quad and \quad \forall v\in V, \qquad a(u, v)=f(v). \tag{1.1}$$

By the well known *Lax-Milgram lemma*, the problem has one and only one solution.

Given a bounded domain Ω with boundary Γ in $\boldsymbol{R}^n$, the space $\mathscr{D}(\Omega)$ consists of infinitely differentiable functions v: $\Omega\to\boldsymbol{R}$ with compact supports. For each integer $m\geqslant 0$, the Sobolev space $H^m(\Omega)$ consists of functions $v\in L^2(\Omega)$, for which all partial derivatives $\partial^\alpha v$ (in the distribution sense) $|\alpha|\leqslant m$, belong to the space $L^2(\Omega)$, i.e. for each multi-index α with $|\alpha|\leqslant m$, there exists a function $\partial^\alpha v\in L^2(\Omega)$ which satisfies

$$\forall\phi\in\mathscr{D}(\Omega), \qquad \int_\Omega \partial^\alpha v\phi dx=(-1)^{|\alpha|}\int_\Omega v\partial^\alpha\phi dx.$$

We note the space $H^m(\Omega)$ is provided with the norm

$$\|v\|_{m,\Omega}=(\sum_{|\alpha|<m}\int_\Omega|\partial^\alpha v|^2dx)^{\frac{1}{2}},$$

and

$$|v|_{m,\Omega}=(\sum_{|\alpha|=m}\int_\Omega|\partial^\alpha v|^2dx)^{\frac{1}{2}}$$

is a semi-norm over it. Now corresponding to the Sobolev space $H^m(\Omega)$ another Sobolev space is effectively considered, which is $H_0^m(\Omega)=\overline{\mathscr{D}(\Omega)}$, closure being taken with respect to the norm $\|\cdot\|_{m,\Omega}$.

Now as an explanatory example of the aim of this report we mention the problem (1.1) with the following specifications:

$$\begin{cases} V=H_0^1(\Omega), \\ a(u, v)=\int_\Omega(\sum_{i=1}^n\partial_i u\partial_i v+auv)\ dx, \\ f(v)=\int_\Omega fv\ dx, \end{cases} \tag{1.2}$$

where $a\in L^2(\Omega)$, $a\geqslant 0$ a.e. on Ω, $f\in L^2(\Omega)$. In order that the problem (1.2) satisfies the condition of *Lax-Milgram lemma*, it is useful to use the fact that the semi-norm $|\cdot|_{1,\Omega}$ is a norm over the space $H_0^1(\Omega)$, equivalent to the norm $\|\cdot\|_{1,\Omega}$, (cf. theorem 2–1). Indeed, by dint of the relation

$$\forall v \in H_0^1(\Omega), \qquad a(v, v) \geqslant \int_\Omega \sum_{i=1}^n (\partial_i v)^2 dx = |v|_{1,\Omega}^2,$$

it holds that $a(\cdot,\cdot)$ is *V-elliptic*.

Furthermore, to the solution of (1.2) we can associate the solution of the following boundary value problem:

$$\begin{cases} -\Delta u + au = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \Gamma. \end{cases} \tag{1.3}$$

In this paper, we shall study various relations between the norm and the semi-norm over several Sobolev spaces which are found useful in the finite element method. (cf. Ciarlet [2] and [3]).

§2. The norm and the semi-norm over the Sobolev space $H^m(\Omega)$

Theorem 2–1 is well known but since it forms the basis of our argument, we dare to give its proof. Theorem 2–2 is about a mixed boundary value problem, and Theorem 2–3 deals with the space of the type $H^m(\Omega) \cap H_0^{m-1}(\Omega)$.

PROPOSITION 1. *Let a function f defined on an interval I be locally integrable. Assume that the distributional derivative g of the function f is also a locally integrable function. Then the function f is absolutely continuous on I and it holds $f'(x) = g(x)$ a.e. in I.* □

The proof may be found in Shibagaki [8] or Liusternik and Sobolev [6].

Throughout this paper, let C (or $C(\Omega)$) denote constant, not necessarily the same in its various occurrences.

THEOREM 2–1. *Let m be an integer $\geqslant 1$. Let Ω be a bounded domain in $\boldsymbol{R}^n$. Then the semi-norm $|\cdot|_{m,\Omega}$ is a norm over the space $H_0^m(\Omega)$, equivalent to the norm $\|\cdot\|_{m,\Omega}$.*

Proof. Ω being bounded, there exists a constant $C(\Omega)$ such that

$$\forall v \in H_0^1(\Omega), \qquad |v|_{0,\Omega} \leqslant C(\Omega)\,|v|_{1,\Omega}, \tag{2.1}$$

the inequality being known as Poincaré-Friedrichs inequality.

Now since the relation $\forall v \in H_0^2(\Omega)$, $\partial^\alpha v \in H_0^1(\Omega)$ with $|\alpha| = 1$ holds, we can write $|\partial^\alpha v|_{0,\Omega} \leqslant C(\Omega)\,|\partial^\alpha v|_{1,\Omega}$ for $|\alpha| = 1$ by means of the above inequality (2.1). Thus we have immediately

$$|v|_{1,\Omega} \leqslant C(\Omega)\,|v|_{2,\Omega}$$

and consequently

$$|v|_{0,\Omega} \leqslant C(\Omega)\,|v|_{2,\Omega}.$$

By repeating the same procedure, we obtain

$$|v|_{m-1,\Omega} \leqslant C(\Omega)\,|v|_{m,\Omega} \tag{2.2}$$

and

$$|v|_{0,\Omega} \leqslant C(\Omega)\,|v|_{m,\Omega}. \tag{2.3}$$

Assume that v is a function $H_0^m(\Omega)$ which satisfies $|v|_{m,\Omega}=0$. Owing to (2.2) and (2.3), it follows $|v|_{0,\Omega}=0$, which implies that $v=0$ a.e. in Ω, so that $v=0$ in the space $H_0^m(\Omega)$. (More precisely, by Proposition 1 we can conclude that the function v is absolutely continuous, so that $v=0$ in Ω.) Thus $|\cdot|_{m,\Omega}$ is a norm over the space $H_0^m(\Omega)$.

Next we show that the norm $|\cdot|_{m,\Omega}$ is equivalent to $\|\cdot\|_{m,\Omega}$, i.e. that there exists a constant $C(\Omega)$ such that

$$\forall v \in H_0^m(\Omega), \qquad \|v\|_{m,\Omega} \leqslant C(\Omega)\,|v|_{m,\Omega}. \tag{2.4}$$

Using (2.1), we have for $\forall v \in H_0^1(\Omega)$,

$$\|v\|_{1,\Omega}^2 = |v|_{0,\Omega}^2 + |v|_{1,\Omega}^2 \leqslant C^2(\Omega)\,|v|_{1,\Omega}^2 + |v|_{1,\Omega}^2. \tag{2.5}$$

Thus $\qquad \|v\|_{1,\Omega} \leqslant C(\Omega)\,|v|_{1,\Omega}.$

Assume that for $\forall v \in H_0^{m-1}(\Omega)$,

$$\|v\|_{m-1,\Omega} \leqslant C(\Omega)\,|v|_{m-1,\Omega}.$$

Then using (2.5) and (2.2), we have (2.4) for $\forall v \in H_0^m(\Omega)$. So the proof is complete. □

The following theorem may be considreed as an extension of Theorem 2–1.

THEOREM 2–2. *Let Ω be a bounded domain in $\boldsymbol{R}^n$. Let Γ be the boundary of Ω such that $\Gamma=\Gamma_0 \cup \Gamma_1$, and $\Gamma_0 \cap \Gamma_1=\emptyset$. Let $V=\{v \in H^1(\Omega);\ v=0$ on $\Gamma_0\}$, which is a subspace of $H^1(\Omega)$.*

Then if Γ_0 has a strictry positive measure, i.e. meas$(\Gamma_0)>0$, *the semi-norm $|\cdot|_{1,\Omega}$ is a norm over the space V, equivalent to the norm $\|\cdot\|_{1,\Omega}$.*

Proof. First we show that the space V is a closed subspace of $H^1(\Omega)$. Let $\{v_k\}$ be a sequence of functions $v_k \in V$ which ocnverges to an element $v \in H^1(\Omega)$ with respect to the norm $\|\cdot\|_{1,\Omega}$, i.e. $v_k \in V \to v \in H^1(\Omega)$. From the trace theory, we have that

$$\forall v \in H^1(\Omega), \qquad \|t_r v\|_{0,\Gamma} \leqslant C(\Omega)\,\|v\|_{1,\Omega},$$

where t_r denotes the trace operator. Thus the inequality

$$\|t_r v_k - t_r v\|_{0,\Gamma} \leqslant C(\Omega)\,\|v_k - v\|_{1,\Omega}$$

holds, so that we have

$$\|t_r v_k - t_r v\|_{0,\Gamma} \to 0 \qquad (k \to \infty).$$

Therefore by a theorem in Lebesgue integration theory, there exists a subsequence $\{v_l\}$ such that $t_r v_l \to t_r v$ a.e. on Γ. Since $v_l \in V$, we have $v_l=0$ on Γ_0, so that we obtain $v=0$ on Γ_0.

Thus $v \in V$, which proves that V is a closed subspace of $H^1(\Omega)$.

Next let us show that the semi-norm $|\cdot|_{1,\Omega}$ is a norm over the space V. Let v be a function in the space V which satisfies $|v|_{1,\Omega}=0$. $|v|_{1,\Omega}=0$ implies $\partial^\alpha v=0$ with $|\alpha|=1$ a.e. in Ω, but since by Proposition 1, the function v is absolutely continuous, the function v is a constant in Ω and hence $t_r v$ is the same constant. That this constant is zero follows from the fact that $v=0$ on Γ_0 and meas$(\Gamma_0)>0$.

Finally we show that the two norms $|\cdot|_{1,\Omega}$ and $\|\cdot\|_{1,\Omega}$ are equivalent over the space V. Assume that the two norms are not equivalent over V. Then for an arbitrary large constant C, there must exist some function $v \in V$ such that $\|v\|_{1,\Omega} > C|v|_{1,\Omega}$. Then we may find a sequence $\{v_k\}$ of functions $v_k \in V$ such that on the one hand it holds for any k $\|v_k\|_{1,\Omega}=1$, and on the other hand $\lim_{k\to\infty}|v_k|_{1,\Omega}=0$. Rellich's theorem tells us that any bounded sequence in the space $H^1(\Omega)$ contains a sequence which converges in $L^2(\Omega)$. Thus we can conclude that there exists a subsequence $\{v_l\}$ of functions $v_l \in V$ which converges in the space $L^2(\Omega)$ and which is such that $\lim_{l\to\infty}|v_l|_{1,\Omega}=0$. Now since $|v_l-v_{l'}|_{0,\Omega}\to 0$ $(l, l'\to\infty)$ and $|v_l-v_{l'}|_{1,\Omega}\leqslant |v_l|_{1,\Omega}+|v_{l'}|_{1,\Omega}\to 0$ $(l, l'\to\infty)$, we have $\|v_l-v_{l'}\|_{1,\Omega}\to 0$ $(l, l'\to\infty)$, so that the sequence $\{v_l\}$ is a Cauchy sequence in the complete space V. Therefore it converges in the norm $\|\cdot\|_{1,\Omega}$ to an element $v \in V$. Since $|v|_{1,\Omega}=\lim_{l\to\infty}|v_l|_{1,\Omega}=0$, it follows that $v=0$, which is in contradiction with $\|v_k\|_{1,\Omega}=1$ for $\forall k$. □

Next we consider the space of the type $H^m(\Omega)\cap H_0^{m-1}(\Omega)$ for any integer $m\geqslant 2$. These spaces are important in the analysis of boundary value problem of biharmonic equation.

THEOREM 2–3. *Let Ω be a bounded domain in $\boldsymbol{R}^n$. Let $V=H^m(\Omega)\cap H_0^{m-1}(\Omega)$ for any integer $m\geqslant 2$. Then the semi-norm $|\cdot|_{m,\Omega}$ is a norm over the space V, equivalent to the norm $\|\cdot\|_{m,\Omega}$.*

Proof. First we show that the semi-norm $|\cdot|_{m,\Omega}$ is a norm over the space V. Let v be a function $\in V$, which satisfies $|v|_{m,\Omega}=0$. $|v|_{m,\Omega}=0$ implies $\partial^\alpha v=0$ with $|\alpha|=m$ a.e. in Ω, so that by Proposition 1, the function v is a polynomial of degree $\leqslant m$. And since $\partial^\alpha v=0$ with $|\alpha|\leqslant m-2$ on Γ, we have $v=0$ in Ω. Thus $|\cdot|_{m,\Omega}$ is a norm over the space V.

Further we can show by the same procedure as the final part of the proof of Theorem 2–2 that the semi-norm $|\cdot|_{m,\Omega}$ is equivalent to the norm $\|\cdot\|_{m,\Omega}$. But in this case, we must use the more general Kondrasov-Rellich theorem. □

§ 3. The norm and the semi-norm about a three-dimensional elasticity problem

We consider the product space $(H^1(\Omega))^3$ for a bounded domain Ω in $\boldsymbol{R}^3$ with Lipshitz continuous boundary. Let

$$V=\{\boldsymbol{v}=(v_1, v_2, v_3)\in(H^1(\Omega))^3;\ v_i=0 \quad \text{on} \quad \Gamma_0,\ 1\leqslant i\leqslant 3\}$$

where Γ_0 is a subboundary of Γ. The space V is provided with the product norm

$$\|\boldsymbol{v}\|_{1,\Omega}=\Big(\sum_{i=1}^{3}\|v_i\|_{1,\Omega}^2\Big)^{\frac{1}{2}} \qquad \text{for} \quad \forall\boldsymbol{v}=(v_1, v_2, v_3)\in(H^1(\Omega))^3.$$

Let $\varepsilon_{ij}(\boldsymbol{v})=\varepsilon_{ji}(\boldsymbol{v})=\frac{1}{2}(\partial_j v_i+\partial_i v_j), \qquad (i, j=1, 2, 3),$

and we define the semi-norm

$$|\boldsymbol{v}|=(\sum_{i,j=1}^{3}|\varepsilon_{ij}(\boldsymbol{v})|_{0,\Omega}^2)^{\frac{1}{2}}.$$

Now the following inequality is known as Korn's inequality, whose proof may be found in Duvaut and Lions [4], which tells us that there exists a constant $C(\Omega)$ such that

$$\forall \boldsymbol{v}=(v_1, v_2, v_3)\in(H^1(\Omega))^3,$$

$$\|\boldsymbol{v}\|_{1,\Omega}\leqslant C(\Omega)(\sum_{i,j=1}^{3}|\varepsilon_{ij}(\boldsymbol{v})|_{0,\Omega}^2+\sum_{i=1}^{3}|v_i|_{0,\Omega}^2)^{\frac{1}{2}} \tag{3.1}$$

THEOREM 3-1. *Let $\Gamma_0\subset\Gamma$ and Γ_0 have strictly positive measure. Then the semi-norm $|\boldsymbol{v}|$ is a norm over the space V, equivalent to the norm $\|\boldsymbol{v}\|_{1,\Omega}$.*

Proof. The fact that V is a closed subspace of $(H^1(\Omega))^3$ may be proved by the same procedure as the first part of the proof of Theorem 2–2.

It is shown that the vector $\boldsymbol{v}\in(H^1(\Omega))^3$ which satisfies $|\boldsymbol{v}|=0$ is of the form $\boldsymbol{v}(\boldsymbol{x})=\boldsymbol{a}\times\boldsymbol{x}+\boldsymbol{b}$ for some constant vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. (cf. Hlaváček and Nečas [5]). Since $\boldsymbol{v}=0$ on Γ_0 and meas$(\Gamma_0)>0$, $\boldsymbol{a}=\boldsymbol{b}=0$, so that $\boldsymbol{v}(\boldsymbol{x})=0$. Thus $|\boldsymbol{v}|$ is a norm over the space V.

The fact that $\varepsilon_{ij}(\boldsymbol{v})$ only involves certain combinations of first derivatives, namely $\varepsilon_{ij}(\boldsymbol{v})=\frac{1}{2}(\partial_j v_i+\partial_i v_j)$, while $\|\boldsymbol{v}\|_{1,\Omega}$ involves all first derivatives tells us immediately the relation: There exists a constant C such that for $\forall\boldsymbol{v}\in(H^1(\Omega))^3$,

$$\sum_{i,j=1}^{3}|\varepsilon_{ij}(\boldsymbol{v})|_{0,\Omega}^2+\sum_{i=1}^{3}|v_i|_{0,\Omega}^2\leqslant C\|\boldsymbol{v}\|_{1,\Omega}^2.$$

Thus the opposite inequality to (3.1) is deduced, so that (3.1) is equilvaent to saying that $(|\boldsymbol{v}|^2+\sum_{i=1}^{3}|v_i|_{0,\Omega}^2)^{\frac{1}{2}}$ is a norm over the space V, equivalent to $\|\boldsymbol{v}\|_{1,\Omega}$.

Next we show the existence of a $C_0>0$ such that for $\forall\boldsymbol{v}=(v_1, v_2, v_3)\in V$,

$$|\varepsilon_{ij}(\boldsymbol{v})|_{0,\Omega}\geqslant C_0|v_i|_{0,\Omega} \qquad (i, j=1, 2, 3) \tag{3.2}$$

By replacing v_i by $v_i|v_i|_{0,\Omega}^{-1}$, we may assume that $|v_i|_{0,\Omega}=1$. Then we have to prove the existence of a $C_0>0$ such that $\forall\boldsymbol{v}\in V$, $|\varepsilon_{ij}(\boldsymbol{v})|_{0,\Omega}\geqslant C_0$. We argue by contradiction. If the result were false, there should exist a sequence$\boldsymbol{v}_n=(v_{n1}, v_{n2}, v_{n3})$ with $|v_{ni}|_{0,\Omega}=1$ and $|\varepsilon_{ij}(\boldsymbol{v}_n)|_{0,\Omega}\to0$ $(n\to\infty)$ $(i, j=1, 2, 3)$. According to (3.1), we then have $\|\boldsymbol{v}_n\|_{1,\Omega}\leqslant$ constant. Then by Rellich's theorem, we can select a subsequence $\{\boldsymbol{v}_l\}$ of functions $\boldsymbol{v}_l\in(H^1(\Omega))^3$ which converges in the space $(L^2(\Omega))^3$ and which is such that $\lim_{l\to\infty}|\boldsymbol{v}_l|^2=\lim_{l\to\infty}\sum_{i,j=1}^{3}|\varepsilon_{ij}(\boldsymbol{v}_l)|_{0,\Omega}=0$. Thus the sequence $\{\boldsymbol{v}_l\}$ is a Cauchy sequence in the complete space $(H^1(\Omega))^3$ and therefore it converges in the norm $\|\cdot\|_{1,\Omega}$ to an element $\boldsymbol{v}\in(H^1(\Omega))^3$. Since $|\boldsymbol{v}|=\lim_{l\to\infty}|\boldsymbol{v}_l|=0$, we have $\boldsymbol{v}=0$. This contradicts with the assumption $|v_{ni}|_{0,\Omega}=1$. Thus we have (3.2), which shows by (3.1) that there exists a constant $C(\Omega)$ such that for $\forall\boldsymbol{v}\in V$,

$$\|\boldsymbol{v}\|_{1,\Omega}\leqslant C(\Omega)\,|\boldsymbol{v}|.$$

The proof is complete. □

§4. The norm and the semi-norm over the space $W^{m,p}(\Omega)$

For any integer $m \geqslant 0$ and any number p satisfying $1 \leqslant p < \infty$, the Sobolev space $W^{m,p}(\Omega)$ consists of those functions $v \in L^p(\Omega)$, for which all distributional derivatives $\partial^\alpha v$ with $|\alpha| \leqslant m$ belong to the space $L^p(\Omega)$.

We provide $W^{m,p}(\Omega)$ with the norm

$$\|v\|_{m,p,\Omega} = \Big(\sum_{|\alpha| \leqslant m} \int_\Omega |\partial^\alpha v|^p dx \Big)^{\frac{1}{p}}$$

and

$$|v|_{m,p,\Omega} = \Big(\sum_{|\alpha| = m} \int_\Omega |\partial^\alpha v|^p dx \Big)^{\frac{1}{p}}.$$

We define the Sobolev space $W_0^{m,p}(\Omega) = \overline{\mathscr{D}(\Omega)}$, closure being taken with respect to the norm $\| \cdot \|_{m,p,\Omega}$.

In this section, we study the relation of the norm and the semi-norm over the space $W^{m,p}(\Omega)$ as an extension of the case of the space $H^m(\Omega)$.

THEOREM 4–1. *Let Ω be a bounded domain in $\boldsymbol{R}^n$. The semi-norm $| \cdot |_{m,p,\Omega}$ is a norm over the space $W_0^{m,p}(\Omega)$, equivalent to the norm $\| \cdot \|_{m,p,\Omega}$.*

Proof. First we prove the next inequality — generalized Poincaré inequality: There exists a constant $C(\Omega)$ such that for $\forall v \in W_0^{m,p}(\Omega)$,

$$|v|_{0,p,\Omega} \leqslant C(\Omega) |v|_{1,p,\Omega}. \tag{4.1}$$

Let $x = (x', x_n)$, where $x' = (x_1, \ldots, x_{n-1})$ and let $[a, b]$ be the bound of Ω of abscissa along the axis x_n. Then for $\forall v \in \mathscr{D}(\Omega)$, we can write

$$v(x) = \int_a^{x_n} \partial_n v(x', t) dt.$$

Here, using Hölder's inequality,

$$|v(x)| \leqslant \Big(\int_a^b |v(x', t)|^p dt \Big)^{\frac{1}{p}} \Big(\int_a^b dx \Big)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. And there exists a constant C such that

$$|v(x)|^p \leqslant C \int_a^b |\partial_n v(x', t)|^p dt.$$

Therefore
$$|v|_{0,p,\Omega}^p = \int_\Omega |v(x)|^p dx$$
$$\leqslant C \int_\Omega \Big(\int_a^b |\partial_n v(x', t)|^p dt \Big) dx$$

$$\leqslant C\left(\int_a^b dx_n\right)\left(\int_{\Omega|_{R_{n-1}}}\left(\int_a^b |\partial_n v(x', t)|^p dt\right) dx'\right)$$

$$\leqslant C|\partial_n v|_{0,p,\Omega}^p$$

$$\leqslant C|v|_{1,p,\Omega}^p.$$

Thus since $\mathscr{D}(\Omega)$ is dense in $W_0^{m,p}(\Omega)$, the inequality (4.1) is shown.

Now, using (4.1) instead of (2.1), the Theorem 4–1 can be proved by the same method as Theorem 2–1. □

Next from the trace theory (cf. Adams [1]), we have the following inequality on the $W^{m,p}(\Omega)$: *There exists a constant* $C(\Omega)$ *such that for* $\forall v \in W^{m,p}(\Omega)$,

$$\|t_r v\|_{0,p,\Gamma} \leqslant C(\Omega)\, \|v\|_{m,p,\Omega}.$$

Therefore we have the following theorems, i.e. Theorem 4–2 and Theorem 4–3, as the extension of Theorem 2–2 and Theorem 2–3. Their proofs will be given by the same procedure as Theorem 2–2 and 2–3.

THEOREM 4–2. *Let* Ω *be a bounded domain in* $\boldsymbol{R}^n$. *Let* Γ *be the boundary of* Ω *such that* $\Gamma = \Gamma_0 \cup \Gamma_1$ *and* $\Gamma_0 \cap \Gamma_1 = \emptyset$. *Let*

$$V = \{v \in W^{1,p}(\Omega);\ v = 0 \text{ on } \Gamma_0\}.$$

Then if Γ_0 *has strictly positive measure, the semi-norm* $|\cdot|_{1,p,\Omega}$ *is a norm over the space V, equivalent to the norm* $\|\cdot\|_{1,p,\Omega}$. □

THEOREM 4–3. *Let* Ω *be a bounded domain in* $\boldsymbol{R}^n$. *Let* $V = W^{m,p}(\Omega) \cap W_0^{m-1,p}(\Omega)$ *for any integer* $m \geqslant 2$.

Then the semi-norm $|\cdot|_{m,p,\Omega}$ *is a norm over the space V, equivalent to the norm* $\|\cdot\|_{m,\Omega}$. □

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