Optimal Filtering Algorithm Using Covariance Information in Linear Continuous Distributed Parameter Systems

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Abstract

This paper proposes an optimal filtering algorithm using covariance information in linear continuous distributed parameter system. It is assumed that observation noise is a white Gaussian process. Autocovariance function of a signal, variance of white Gaussian noise and observed value are used in the filtering algorithm. It is an advantage that current filtering algorithm can be applied to the case where a partial differential equation, which generates the signal process, is unknown in linear continuous stochastic distributed parameter systmes.

1. Introduction

Of usual estimation problems in linear stochastic distributed parameter systems, a partial differential equation, which generates a state-vector function, is known with associate boundary conditions (Sawaragi, Soeda and Omatu, 1978). An estimation problem using covariance information also has been researched in linear lumped parameter systems (Nakamori and Sugisaka, 1977; Nakamori and Hataji, 1982). However, there seems to be few studies on estimation procedure using covariance information in linear distributed parameter systems.

By the way, stochastic partial differential equations have been analyzed by Heine (1955) for obtaining covariance functions realized by partial differential equations. It is reported that constant coefficient second-order hyperbolic partial differential equation of certain type has a separable autocovariance function for a two-dimensional signal

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(Jain and Jain, 1978).

In this paper, an optimal filtering algorithm using covariance information is designed in linear continuous distributed parameter systems. It is assumed that observation noise is a white Gaussian process. The autocovariance function of a signal, the variance of white Gaussian noise and the observed value are used in the filtering algorithm. The autocovariance function of the signal is expressed by a semi-degenerate kernel. The semi-degenerate kernel has a separable form and is given as a finite sum of products of two nonrandom functions. It is advantageous that current filtering algorithm can be applied to the case where a partial differential equation, which generates the signal, is unknown in linear continuous stochastic distributed parameter systems.

2. Two-dimensional filtering problems

Let D be a connected bounded open domain of an r-dimensional Euclidean space R^r. The spacial coordinate vector is denoted by $x=(x_1, x_2, \dots, x_p)\in D$ and let S be the sufficiently smooth boundary of D. Let $u(t, x)$ be an n-dimensional zero-mean signal vector:

$$u(t, x) = Col[u_1(t, x), \dots, u_n(t, x)]. \tag{1}$$

Let us assume that the measurement date are taken at fixed m points $x^1, x^2, \dots, x^m$ of $\bar{D} = D \cup S$. Furthermore, let us define an mn-dimensional column vector

$$u_m(t) = Col[u(t, x^1), \dots, u(t, x^m)] = \begin{bmatrix} u(t, x^1) \\ \vdots \\ u(t, x^m) \end{bmatrix}. \tag{2}$$

Assume the observation equation is described by

$$z(t) = H(t)u_m(t) + v(t), \tag{3}$$

where z(t) is r-dimensional measurement vector at the points $x^1, \dots, x^m$, $H(t)$ is a known $r\times mn$ matrix function, and $v(t)$ is a vector-valued white Gaussian process. $v(t)$ is uncorrelated with $u_m(t)$. The mean and covariance of $v(\cdot)$ are given by

$$E[v(t)] = O,\ E[v(t)v^T(s)] = R(t)\,\delta(t-s). \tag{4}$$

As in the Kalman filter approach, an estimate $\hat{u}(t, x)$ of $u(t, x)$ is denoted by

$$\hat{u}(t, x) = \int_0^t h(t, x, s)z(s)\,ds,\ x\in\bar{D} \tag{5}$$

through a linear integral operation on the past of the measurement data. The filtering estimate which minimizes the mean-square value of the estimation error $u(t, x)-\hat{u}(t, x)$

$$E\left[\|u(t, x)-\hat{u}(t, x)\|^2\right] \qquad (6)$$

is said to be optimal, where $\|\|$ denotes the Euclidean norm. Minimizing (6) leads to the Wiener-Hopf integral equation

$$E\left[u(t, x) z^T(s)\right]=\int_0^t h(t, x, s') E\left[z(s') z^T(s)\right] ds', \ 0 \leqq s < t, \ x \in \bar{D}. \qquad (7)$$

Substituting (3) into (7) and using (4), we obtain

$$h(t, x, s) R(s)=B_m(t, x, s) H^T(s)-\int_0^t h(t, x, s') H(s') Q_m(s', s) \ \mathrm{H}^T(s) ds', \qquad (8)$$

where

$$B_m(t, x, s)=E\left[u(t, x) u_m^T(s)\right], \ Q_m(t, s)=\left[u_m(t) u_m^T(s)\right]. \qquad (9)$$

Let us assume that the autocovariance function of the signal $u(t, x)$ is expressed by

$$K(t, x, s, y)=E\left[u(t, x) u^T(s, y)\right]$$
$$=\begin{cases} \alpha(t, x, y) \beta^T(s, x, y), & 0 \leqq s \leqq t, \\ \quad \vdots & \\ \beta(t, x, y) \alpha^T(s, x, y), & 0 \leqq t \leqq s, \end{cases} \qquad (10)$$

where $\alpha(t, x, y)$ and $\beta(s, x, y)$ are $n \times m'$ bounded matrices. Then $B_m(t, x, s)$ is written as

$$B_m(t, x, s)=E\left[u(t, x) u_m^T(s)\right]$$

$$=E\left[u(t, x) u^T(s, x^1) \cdot \cdot \ u(t, x) u^T(s, x^m)\right]$$

$$=\begin{cases} \left[\alpha(t, x, x^1) \beta^T(s, x, x^1) \cdot \cdot \ \alpha(t, x, x^m) \beta^T(s, x, x^m)\right], \\ \qquad 0 \leqq s \leqq t, \\ \left[\beta(t, x, x^1) \alpha^T(s, x, x^1) \cdot \cdot \ \beta(t, x, x^m) \alpha^T(s, x, x^m)\right], \\ \qquad 0 \leqq t \leqq s. \end{cases} \qquad (11)$$

Also, $Q_m(t, s)$ is denoted as

$$Q_m(t, s)=E\left[u_m(t) u_m^T(s)\right]$$

$$
=\begin{cases}
\begin{bmatrix}
\alpha(t, x^1, x^1)\,\beta^T(s, x^1, x^1) & \cdots & \alpha(t, x^1, x^m)\,\beta^T(s, x^1, x^m) \\
\vdots & & \vdots \\
\alpha(t, x^m, x^1)\,\beta^T(s, x^m, x^1) & \cdots & \alpha(t, x^m, x^m)\,\beta^T(s, x^m, x^m),
\end{bmatrix}, & 0 \leqq s \leqq t, \\
\begin{bmatrix}
\beta(t, x^1, x^1)\,\alpha^T(s, x^1, x^1) & \cdots & \beta(t, x^1, x^m)\,\alpha^T(s, x^1, x^m) \\
\vdots & & \vdots \\
\beta(t, x^m, x^1)\,\alpha^T(s, x^m, x^1) & \cdots & \beta(t, x^m, x^m)\,\alpha^T(s, x^m, x^m),
\end{bmatrix}, & 0 \leqq t \leqq s.
\end{cases} \tag{12}
$$

It is desirable that $h(t, x, s)$ in (8) is calculated recursively. In the succeeding section, sequential algorithm for calculating the linear least-squares filtering estimate of $u(t, x)$ is derived.

3. Derivation of optimal filtering algorithm

In this section, a Cauchy system for the optimal filtering estimate is obtained by using an invariant imbedding method (Kagiwada and Kalaba, 1970).

From (8) and (11) we have

$$
h(t, x, s)R(s) = [\alpha(t, x, x^1)\,\beta^T(s, x, x^1) \cdots \alpha(t, x, x^m)\,\beta^T(s, x, x^m)]
$$
$$
H^T(s) - \int_0^t h(t, x, s')H(s')Q_m(s', s)H^T(s)\,ds'. \tag{13}
$$

Let us introduce an auxiliary $m' \times r$ matrix function $J_l(t, x, s)$, which satisfies

$$
J_l(t, x, s)R(s) = \beta^T(s, x, x^l)H_l^T(s) - \int_0^t J_l(t, x, s')H(s')Q_m(s', s)H^T(s)\,ds', \tag{14}
$$

where $H_l(s)$, $l=1, \ldots, m$, are $r \times n$ matrix elements of the observation matrix $H(s)$ as

$$
H(s) = [H_1(s) \cdots H_m(s)]. \tag{15}
$$

Then

$$
h(t, x, s) = \sum_{l=1}^{m} \alpha(t, x, x^l)J_l(t, x, s). \tag{16}
$$

Differentiating (14) with respect to t yields

$$
\partial J_l(t, x, s)/\partial t R(s) = -J_l(t, x, t)H(t)Q_m(t, s)H^T(s) -
$$
$$
\int_0^t \partial J_l(t, x, s')/\partial t H(s')Q_m(s', s)H^T(s)\,ds'. \tag{17}
$$

Taking into consideration of the semi-degenerate kernel of (12), we rewrite (17) as

$$\partial J_l(t, x, s)/\partial t R(S) = -J_l(t, x, t) H(t) \begin{bmatrix} \alpha(t, x^1, x^1)\beta^T(s, x^1, x^1) & \cdots & \alpha(t, x^1, x^m)\beta^T(s, x^1, x^m) \\ \vdots & & \vdots \\ \alpha(t, x^m, x^1)\beta^T(s, x^m, x^1) & \cdots & \alpha(t, x^m, x^m)\beta^T(s, x^m, x^m) \end{bmatrix} \begin{bmatrix} H_1^T(s) \\ \vdots \\ H_m^T(s) \end{bmatrix} - \int_0^t \partial J_l(t, x, s')/\partial t H(s') Q_m(s', s) H^T(s) ds'. \quad (18)$$

If we introduce auxiliary functions $L_{ln}(t, s)$ which satisfy

$$L_{ln}(t, s) R(s) = \beta^T(s, x^l, x^n) H_n^T(s) - \int_0^t L_{ln}(t, s') H(s') Q_m(s', s) H^T(s) ds', \quad l, n = 1, \ldots, m, \quad (19)$$

we have

$$\partial J_l(t, x, s)/\partial t = -J_l(t, x, t) \sum_{p=1}^{m} \sum_{n=1}^{m} H_p(t)\, \alpha(t, x^p, x^n) L_{pn}(t, s). \quad (20)$$

If we differentiate (19) with respect to t, we have

$$\partial L_{ln}(t, s)/\partial t R(s) = -L_{ln}(t, t) H(t) Q_m(t, s) H^T(s) - \int_0^t \partial L_{ln}(t, s')/\partial t H(s') Q_m(s', s) H^T(s) ds'. \quad (21)$$

Substituting (12) into (21), we have

$$\partial L_{ln}(t, s)/\partial t R(s) = -L_{ln}(t, t) H(t) \begin{bmatrix} \alpha(t, x^1, x^1)\beta^T(s, x^1, x^1) & \cdots & \alpha(t, x^1, x^m)\beta^T(s, x^1, x^m) \\ \vdots & & \vdots \\ \alpha(t, x^m, x^1)\beta^T(s, x^m, x^1) & \cdots & \alpha(t, x^m, x^m)\beta^T(s, x^m, x^m) \end{bmatrix} \begin{bmatrix} H_1^T(s) \\ \vdots \\ H_m^T(s) \end{bmatrix} - \int_0^t \partial L_{ln}(t, s')/\partial t H(s') Q_m(s', s) H^T(s) ds'. \quad (22)$$

It follows from (19) and (22) that

$$\partial L_{ln}(t, s)/\partial t = -L_{ln}(t, t) \sum_{p=1}^{m} \sum_{n'=1}^{m} H_p(t)\, \alpha(t, x^p, x^{n'}) L_{pn'}(t, s). \quad (23)$$

From (14) $J_l(t, x, t)$ in (20) is written as follows.

$$J_l(t, x, t) R(t) = \beta^T(t, x, x^l) H_l^T(t) - \int_0^t J_l(t, x, s') H(s') Q_m(s', t) H^T(t) ds' \quad (24)$$

From (12) and (24) we obtain

$$J_l(t, x, t)R(t) = \beta^T(t, x, x^l)H_l^T(t) -$$

$$\int_0^t J_l(t, x, s')H(s') \begin{bmatrix} \beta(s', x^1, x^1)\,\alpha^T(t, x^1, x^1) & \cdots & \beta(s', x^1, x^m)\,\alpha^T(t, x^1, x^m) \\ \vdots & & \vdots \\ \beta(s', x^m, x^1)\,\alpha^T(t, x^m, x^1) & \cdots & \beta(s', x^m, x^m)\,\alpha^T(t, x^m, x^m) \end{bmatrix} \begin{bmatrix} H_1^T(t) \\ \vdots \\ H_m^T(t) \end{bmatrix} ds'. \tag{25}$$

If we introduce new functions

$$r_{lkn}(t, x) = \int_0^t J_l(t, x, s')H_k(s')\,\beta(s', x^k, x^n)\,ds',\ l, k, n = 1, \ldots, m, \tag{26}$$

and substitute (26) into (25), we have

$$J_l(t, x, t)R(t) = \beta^T(t, x, x^l)H_l^T(t) - \sum_{p=1}^{m}\sum_{n=1}^{m} r_{lpn}(t, x)\,\alpha^T(t, x^p, x^n)H_n^T(t). \tag{27}$$

Now putting $s \to t$ in (19) and using (12), we obtain

$$L_{ln}(t, t)R(t) = \beta^T(t, x^l, x^n)H_n^T(t) - \int_0^t L_{ln}(t, s')H(s')\ Q_m(s', t)\ H^T(t)\ ds'$$
$$= \beta^T(t, x^l, x^n)H_n^T(t) -$$

$$\int_0^t L_{ln}(t, s')H(s') \begin{bmatrix} \beta(s', x^1, x^1)\,\alpha^T(t, x^1, x^1) & \cdots & \beta(s', x^1, x^m)\,\alpha^T(t, x^1, x^m) \\ \vdots & & \vdots \\ \beta(s', x^m, x^1)\,\alpha^T(t, x^m, x^1) & \cdots & \beta(s', x^m, x^m)\,\alpha^T(t, x^m, x^m) \end{bmatrix} \begin{bmatrix} H_1^T(t) \\ \vdots \\ H_m^T(t) \end{bmatrix} ds'. \tag{28}$$

Let us introduce new functions given by

$$b_{lnkp}(t) = \int_0^t L_{ln}(t, s')H_k(s')\,\beta(s', x^k, x^p)\,ds',\ l, n, k, p = 1, \ldots, m. \tag{29}$$

Substituting (29) into (28) yields

$$L_{ln}(t, t)R(t) = \beta^T(t, x^l, x^n)H_n^T(t) - \sum_{p=1}^{m}\sum_{n'=1}^{m} b_{lnpn'}(t)\,\alpha^T(t, x^p, x^{n'})H_{n'}^T(t). \tag{30}$$

Let us differentiate (26) with respect to t.

$$\partial r_{lkn}(t, x)/\partial t = J_l(t, x, t) H_k(t)\,\beta\,(t, x^k, x^n) + \int_0^t \partial J_l(t, x, s')/\partial t H_k(s')\,\beta\,(s', x^k, x^n)\,ds' \tag{31}$$

If we substitute (20) into (31), we have

$$\partial r_{lkn}(t, x)/\partial t = J_l(t, x, t) H_k(t)\,\beta\,(t, x^k, x^n) - J_l(t, x, t) \sum_{p=1}^{m} \sum_{n'=1}^{m} H_p(t)\,\alpha\,(t, x^p, x^{n'}) \int_0^t L_{pn'}(t, s') H_k(s')\,\beta\,(s', x^k, x^n)\,ds'. \tag{32}$$

It follows from (29) and (32) that

$$\partial r_{lkn}(t, x)/\partial t = J_l(t, x, t)\,(H_k(t)\,\beta\,(t, x^k, x^n) - \sum_{p=1}^{m} \sum_{n'=1}^{m} H_p(t)\,\alpha\,(t, x^p, x^{n'})\,b_{pn'kn}(t)). \tag{33}$$

The initial condition on the partial differential equation (33) at $t=0$ *is*

$$r_{lkn}(0, x) = 0 \tag{34}$$

from (26).

Let us differentiate (29) with respect to t.

$$db_{lnkp}(t)/dt = L_{ln}(t, t) H_k(t)\,\beta\,(t, x^k, x^p) + \int_0^t \partial L_{ln}(t, s')/\partial t H_k(s')\,\beta\,(s', x^k, x^p)\,ds' \tag{35}$$

If we substitute (23) into (35) and use (29), we obtain

$$\begin{aligned} db_{lnkp}(t)/dt &= L_{ln}(t, t)\,(H_k(t)\,\beta\,(t, x^k, x^p) - \sum_{p'=1}^{m} \sum_{n'=1}^{m} H_{p'}(t)\,\alpha\,(t, x^{p'}, x^{n'}) \int_0^t L_{p'n'}(t, s') H_k(s')\,\beta\,(s', x^k, x^p)\,ds') \\ &= L_{ln}(t, t)\,(H_k(t)\,\beta\,(t, x^k, x^p) - \sum_{p'=1}^{m} \sum_{n'=1}^{m} H_{p'}(t)\,\alpha\,(t, x^{p'}, x^{n'})\,b_{p'n'kp}(t)). \end{aligned} \tag{36}$$

The initial condition on the differential equation (36) at $t=0$ is

$$b_{lnkp}(0) = 0 \tag{37}$$

from (29).

If we substitute (16) into (5), we have

$$\hat{u}(t, x) = \sum_{l=1}^{m} \alpha(t, x, x^l) \int_0^t J_l(t, x, s) z(s) ds. \tag{38}$$

Introducing the function

$$e_i(t, x) = \int_0^t J_i(t, x, s) z(s) ds, \ i = 1, \ldots, m, \tag{39}$$

we have

$$\hat{u}(t, x) = \sum_{l=1}^{m} \alpha(t, x, x^l) e_l(t, x) \tag{40}$$

from (38) and (39). If we differentiate (39) with respect to t, we have

$$\partial e_i(t, x) / \partial t = J_i(t, x, t) z(t) + \int_0^t \partial J_i(t, x, s) / \partial t z(s) ds. \tag{41}$$

Substituting (20) into (41) yields

$$\partial e_i(t, x) / \partial t = J_i(t, x, t) \left(z(t) - \sum_{p=1}^{m} \sum_{n=1}^{m} H_p(t) \alpha(t, x^p, x^n) \int_0^t L_{pn}(t, s) z(s) ds\right). \tag{42}$$

Let us introduce new functions given by

$$g_{ij}(t) = \int_0^t L_{ij}(t, s) z(s) ds, \ i, j = 1, \ldots, m. \tag{43}$$

It follows from (42) and (43) that

$$\partial e_i(t, x) / \partial t = J_i(t, x, t) \left(z(t) - \sum_{p=1}^{m} \sum_{n=1}^{m} H_p(t) \alpha(t, x^p, x^n) g_{pn}(t)\right). \tag{44}$$

If we differentiate (43) with respect to t, we have

$$dg_{ij}(t) / dt = L_{ij}(t, t) z(t) + \int_0^t \partial L_{ij}(t, s) / \partial t z(s) ds. \tag{45}$$

Substituting (23) into (45) yields

$$dg_{ij}(t) / dt = L_{ij}(t, t) z(t) - L_{ij}(t, t) \sum_{p=1}^{m} \sum_{n'=1}^{m} H_p(t) \alpha(t, x^p, x^{n'}) \int_0^t L_{pn'}(t, s) z(s) ds. \tag{46}$$

If follows from (43) and (46) that

$$dg_{ij}(t) / dt = L_{ij}(t, t) \left(z(t) - \sum_{p=1}^{m} \sum_{n'=1}^{m} H_p(t) \alpha(t, x^p, x^{n'}) g_{pn'}(t)\right). \tag{47}$$

Let us summarize the above filtering algorithm in [Theorem 1].

[Theorem 1]

Let the autocovariance function of the signal $u(t, x)$ be given by (10) in the semi-degenerate kernel form, and let the variance of the white Gaussian observation noise be $R(t)$, then the sequential algorithm for the linear least-squares filtering estimate consists of (48) ~ (54).

$$\hat{u}(t, x) = \sum_{l=1}^{m} \alpha(t, x, x^l) e_l(t, x) \tag{48}$$

$$\partial e_i(t, x)/\partial t = J_i(t, x, t)\left(z(t) - \sum_{p=1}^{m}\sum_{n=1}^{m} H_p(t)\, \alpha(t, x^p, x^n) g_{pn}(t)\right),\ i = 1, \ldots, m \tag{49}$$

$$dg_{ij}(t)/dt = L_{ij}(t, t)\left(z(t) - \sum_{p=1}^{m}\sum_{n'=1}^{m} H_p(t)\, \alpha(t, x^p, x^{n'}) g_{pn'}(t)\right),\ i, j = 1, \ldots, m \tag{50}$$

$$J_l(t, x, t) = \left(\beta^T(t, x, x^l) H_l^T(t) - \sum_{p=1}^{m}\sum_{n=1}^{m} r_{lpn}(t, x)\, \alpha^T(t, x^p, x^n) H_n^T(t)\right) R^{-1}(t),\ l = 1, \ldots, m \tag{51}$$

$$L_{ln}(t, t) = \left(\beta^T(t, x^l, x^n) H_n^T(t) - \sum_{p=1}^{m}\sum_{n'=1}^{m} b_{lnpn'}(t)\, \alpha^T(t, x^p, x^{n'}) H_{n'}^T(t)\right) R^{-1}(t),\ l, n = 1, \ldots, m \tag{52}$$

$$\partial r_{lkn}(t, x)/\partial t = J_l(t, x, t)\left(H_k(t)\, \beta(t, x^k, x^n) - \sum_{p=1}^{m}\sum_{n'=1}^{m} H_p(t)\, \alpha(t, x^p, x^{n'}) b_{pn'kn}(t)\right),\ l, k, n = 1, \ldots, m \tag{53}$$

$$db_{lnkp}(t)/dt = L_{ln}(t, t)\left(H_k(t)\, \beta(t, x^k, x^p) - \sum_{p'=1}^{m}\sum_{n'=1}^{m} H_{p'}(t)\, \alpha(t, x^{p'}, x^{n'}) b_{p'n'kp}(t)\right),\ l, n, k, p = 1, \ldots, m \tag{54}$$

The initial conditions on the differential equations (49), (50), (53) and (54) at $t = 0$ are $e_i(0, x) = 0$, $g_{ij}(0) = 0$, $r_{lkn}(0, x) = 0$ and $b_{lnkp}(0) = 0$.

Also, the sequential algorithm for the optimal impulse response function $h(t, x, s)$ consists of (51) ~ (57).

$$h(t, x, s) = \sum_{l=1}^{m} \alpha(t, x, x^l) J_l(t, x, s) \tag{55}$$

$$\partial J_l(t, x, s)/\partial t = -J_l(t, x, t) \sum_{p=1}^{m}\sum_{n=1}^{m} H_p(t)\, \alpha(t, x^p, x^n) L_{pn}(t, s),\ l = 1, \ldots, m \tag{56}$$

$$\partial L_{ln}(t, s)/\partial t = -L_{ln}(t, t) \sum_{p=1}^{m}\sum_{n'=1}^{m} H_p(t)\, \alpha(t, x^p, x^{n'}) L_{pn'}(t, s),\ l, n=1, \ldots, m \tag{57}$$

The initial conditions on the partial differential equations (56) and (57) at $t=0$ are $J_l(0, x, s) = \beta^T(s, x, x^l) H_l^T(s) R^{-1}(s)$ and $L_{ln}(0, s) = \beta^T(s, x^l, x^n) H_n^T(s) R^{-1}(s)$.

4. Filtering error covariance function

Let us derive an equation for a filtering error covariance function. The filtering error covariance function is defined by

$$P(t, x, s, y) = E\,[(u(t, x) - \hat{u}(t, x))(u(s, y) - \hat{u}(s, y))^T]. \tag{58}$$

From an orthogonal projection lemma that smoothing error $u(t, x) - \hat{u}(t, x)$ is orthogonal to $\hat{u}(s, y)$, we obtain

$$P(t, x, s, y) = K(t, x, s, y) - E\,[\hat{u}(t, x) u^T(s, y)],\ 0 \leqq s < t, \textit{for all } x, y \in \bar{D}. \tag{59}$$

Substituting (5) into (59), and using (3) with the uncorrelation property of $u(\cdot, \cdot)$ with $v(\cdot)$, we obtain

$$P(t, x, s, y) = K(t, x, s, y) - \int_0^t h(t, x, s') H(s') B_m^T(s, y, s')\, ds. \tag{60}$$

If we substitute (55) into (60), introduce new functions given by

$$S_l(t, x, s, y) = \int_0^t J_l(t, x, s') H(s') B_m^T(s, y, s')\, ds',\ l = 1, \dots, m, \tag{61}$$

and take into consideration of the expression for the semi-degenerate kernel of (10), we obtain

$$\begin{aligned} P(t, x, s, y) &= \alpha(t, x, y)\, \beta^T(s, x, y) - \\ &\qquad \sum_{l=1}^{m} \alpha(t, x, x^l) \int_0^t J_l(t, x, s') H(s') B_m^T(s, y, s')\, ds' \\ &= \alpha(t, x, y)\, \beta^T(s, x, y) - \sum_{l=1}^{m} \alpha(t, x, x^l) S_l(t, x, s, y). \end{aligned} \tag{62}$$

If we differentiate (61) with respect to t, use (56) and introduce new functions given by

$$T_{pn}(t, s, x) = \int_0^t L_{pn}(t, s') H(s') B_m^T(s, x, s')\, ds',\ p, n = 1, \dots, m, \tag{63}$$

we obtain

$$\partial S_l(t, x, s, y)/\partial t = J_l(t, x, t)H(t)B_m^T(s, y, t) +$$

$$\int_0^t \partial J_l(t, x, s')/\partial t H(s')B_m^T(s, y, s')ds'$$

$$= J_l(t, x, t)\,(H(t)B_m^T(s, y, t) -$$

$$\sum_{p=1}^{m}\sum_{n=1}^{m} H_p(t)\,\alpha\,(t, x^p, x^n)\int_0^t L_{pn}(t, s')H(s')B_m^T(s, y, s')ds')$$

$$= J_l(t, x, t)\,(H(t)B_m^T(s, y, t) -$$

$$\sum_{p=1}^{m}\sum_{n=1}^{m} H_p(t)\,\alpha\,(t, x^p, x^n)\,T_{pn}(t, s, y)). \tag{64}$$

If we differentiate (63) with respect to t and use (11), (57) and (63), we obtain

$$\partial T_{pn}(t, s, x)/\partial t = L_{pn}(t, t)H(t)B_m^T(s, x, t) +$$

$$\int_0^t \partial L_{pn}(t, s')/\partial t H(s')B_m^T(s, x, s')ds'$$

$$= L_{pn}(t, t)\,(\sum_{k=1}^{m} H_k(t)\,\alpha\,(t, x, x^k)\,\beta^T(s, x, x^k) -$$

$$\sum_{p'=1}^{m}\sum_{n'=1}^{m} H_{p'}(t)\,\alpha\,(t, x^{p'}, x^{n'})\,T_{p'n'}(t, s, x)). \tag{65}$$

Therefore, the sequential algorithm for the filtering error covariance function $P(t, x, s, y)$ consists of (62), (64) and (65).

The initial conditions on the differential equations (64) and (65) at $t=0$ are $S_l(0, x, s, y) = 0$ and $T_{pn}(0, s, x) = 0$ from (61) and (63).

Now, the filtering error covariance function $P(t, x, s, y)$ is written as

$$P(t, x, s, y) = K(t, x, s, y) - E[\hat{u}(t, x)\hat{u}^T(s, y)] = K(t, x, s, y) - P_u(t, x, s, y), \tag{66}$$

where $P_u(t, x, s, y)$ denotes an autocovariance function of the filtering estimate $\hat{u}(t, x)$.$P_u(t, x, s, y)$ is a positive semi-definite matrix, and the filtering error covariance function is also positive semi-definite. Therefore, we notice that the relationship

$$0 \leqq P_u(t, x, s, y) \leqq K(t, x, s, y) \tag{67}$$

is valid. According to a discussion on stability problems (Kailath, 1976), (67) ensures that the present filtering algorithm has a unique solution, since $P_u(t, x, s, y)$ is both lower and upper bounded.

5. A numerical simulation example

Let us consider two digital simulation examples.

5-1. Deterministic signal case

A deterministic signal to be estimated is given by

$$u(t, x) = A\cos(wt)\cos(wx),\ A = 4.5,\ w = 20\pi. \tag{68}$$

The autocovariance function of $u(t, x)$ is given by

$$K(t, x, s, y) = A^2\cos(w(t-s))\cos(w(x-y))/4. \tag{69}$$

Then it follows from (10) that

$$\left.\begin{aligned} \alpha(t, x, y) &= [A^2\cos(wt)/4 \quad A^2\sin(wt)/4], \\ \beta^T(s, x, y) &= \begin{bmatrix} \cos(ws)\cos(w(x-y)) \\ \sin(ws)\cos(w(x-y)) \end{bmatrix}. \end{aligned}\right\} \tag{70}$$

The observation equation is given by

$$z(t) = H(t)u(t, x^1) + v(t),\ H(t) = 1.5, \tag{71}$$

where $u(t, x)$ is observed at the point x^1.

The linear least-squares filtering estimate of $u(t, x)$ is calculated sequentially by substituting the covariance information of the signal, given by (70), the variance of white Gaussian observation noise, the observed value and $H(t)$ $(=1.5)$ into [Theorem 1]. Fig. 1 depicts the filtering esimate $\hat{u}(t, 0.1)$ *vs.* t. Graph (a) illustrates the signal process $u(t, 0.1)$. Graphs (b), (c), (d) and (e) illustrate the filtering estimate $\hat{u}(t, 0.1)$ for white Gaussian observation noises $N(0, 0.1^2)$, $N(0, 0.3^2)$, $N(0, 0.5^2)$ and $N(0, 0.7^2)$ respectively. Table 1 shows the mean-square value (M. S. V.) of filtering error $u(t, x) - \hat{u}(t, x)$, $\sum_{i=1}^{500}(u(i\Delta, x) - \hat{u}(i\Delta, x))^2/500$, $\Delta = 0.001$, for $x = 0.0$, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40, 0.45 and 0.50 when the observation point is $x^1 = 0.1$ and white Gaussian observation noises are $N(0, 0.1^2)$, $N(0, 0.3^2)$, $N(0, 0.5^2)$, $N(0, 0.7^2)$ and $N(0, 1)$. Table 2, Table 3 and Table 4 show the M. S. V. of the filtering error $u(t, x) - \hat{u}(t, x)$

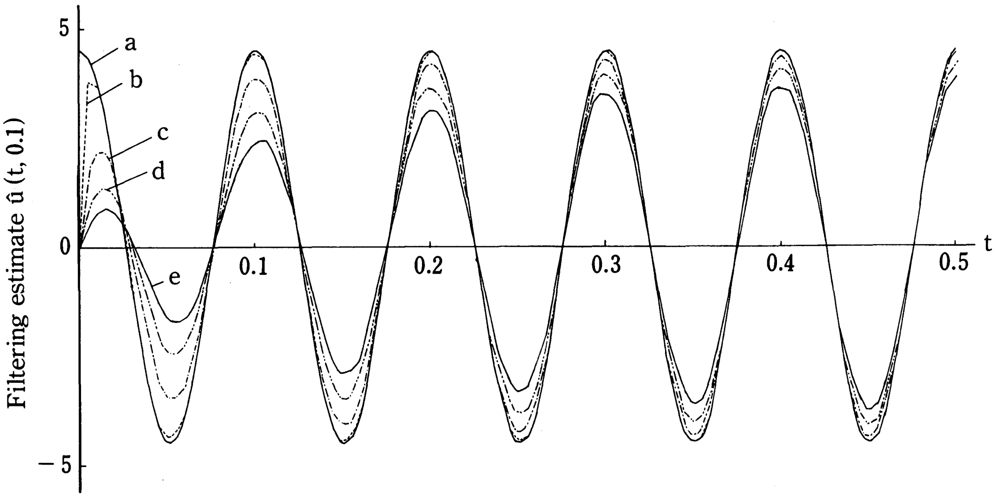

Fig. 1 Filtering estimate $\hat{u}(t, 0.1)$ vs. t.
Graph a …Signal process $u(t, 0.1)$ vs. t.
Graph b …Filtering estimate $\hat{u}(t, 0.1)$ vs. t for white Gaussian observation noise $N(0, 0.1^2)$.
Graph c …Filtering estimate $\hat{u}(t, 0.1)$ vs. t for white Gaussian observation noise $N(0, 0.3^2)$.
Graph d …Filtering estimate $\hat{u}(t, 0.1)$ vs. t for white Gaussian observation noise $N(0, 0.5^2)$.
Graph e …Filtering estimate $\hat{u}(t, 0.1)$ vs. t for white Gaussian observation noise $N(0, 0.7^2)$.

Table 1 Mean-square values of filtering error $u(t, x)-\hat{u}(t, x)$, $\sum_{i=1}^{500}(u(i\Delta, x)-\hat{u}(i\Delta, x))^2/500$, $\Delta=0.001$, for $x=0.0$, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40, 0.45 and 0.50 when the observation point is $x^1=0.1$.

Value of x	White Gaussian observation noise				
	$N(0, 0.1^2)$	$N(0, 0.3^2)$	$N(0, 0.5^2)$	$N(0, 0.7^2)$	$N(0, 1)$
0.0	0.27241×10^{-1}	0.30056	0.83014	1.5338	2.7336
0.05	0.25235×10^{-1}	0.28873	0.78923	1.4542	2.5995
0.10	0.02422×10^{-1}	0.28051	0.77573	1.4356	2.5735
0.15	0.24210×10^{-1}	0.27588	0.75750	1.4041	2.5285
0.20	0.26011×10^{-1}	0.28528	0.78053	1.4410	2.5798
0.25	0.25067×10^{-1}	0.27271	0.73748	1.3550	2.4267
0.30	0.27000×10^{-1}	0.29247	0.79366	1.4557	2.5911
0.35	0.27086×10^{-1}	0.29188	0.79332	1.4574	2.5980
0.40	0.27525×10^{-1}	0.28159	0.76657	1.4180	2.5481
0.45	0.26031×10^{-1}	0.29273	0.80430	1.4796	2.6334
0.50	0.26900×10^{-1}	0.29577	0.80961	1.4923	2.6639

Table 2 Mean-square values of filtering error $u(t, x) - \hat{u}(t, x)$, $\sum_{i=1}^{500} (u(i\Delta, x) - \hat{u}(i\Delta, x))^2/500$, $\Delta = 0.001$, for $x = 0.0$, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40, 0.45 and 0.50 when the observation point is $x^1 = 0.05$.

	White Gaussian observation noise				
Value of x	$N(0, 0.1^2)$	$N(0, 0.3^2)$	$N(0, 0.5^2)$	$N(0, 0.7^2)$	$N(0, 1)$
0.0	0.24860×10^{-1}	0.26954	0.72714	1.3365	2.4008
0.05	0.26715×10^{-1}	0.28015	0.76518	1.4118	2.5293
0.10	0.27590×10^{-1}	0.28761	0.77728	1.4285	2.5532
0.15	0.27749×10^{-1}	0.29303	0.79673	1.4614	2.5995
0.20	0.25794×10^{-1}	0.28256	0.77173	1.4219	2.5452
0.25	0.27023×10^{-1}	0.29714	0.81910	1.5144	2.7068
0.30	0.24800×10^{-1}	0.27551	0.75889	1.4076	2.5343
0.35	0.24698×10^{-1}	0.27610	0.75929	1.4060	2.5276
0.40	0.24534×10^{-1}	0.28817	0.78962	1.4505	2.5836
0.45	0.25885×10^{-1}	0.27570	0.74905	1.3847	2.4930
0.50	0.24975×10^{-1}	0.27286	0.74448	1.3735	2.4650

Table 3 Mean-square values of filtering error $u(t, x) - \hat{u}(t, x)$, $\sum_{i=1}^{500} (u(i\Delta, x) - \hat{u}(i\Delta, x))^2/500$, $\Delta = 0.001$, for $x = 0.0$, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40, 0.45 and 0.50 when the observation point is $x^1 = 0.01$.

	White Gaussian observation noise				
Value of x	$N(0, 0.1^2)$	$N(0, 0.3^2)$	$N(0, 0.5^2)$	$N(0, 0.7^2)$	$N(0, 1)$
0.0	1.3327	1.8930	2.6569	3.4807	4.6733
0.05	1.3152	1.8420	2.5692	3.3591	4.5137
0.10	1.3107	1.8286	2.5501	3.3356	4.4851
0.15	1.3043	1.8097	2.5182	3.2939	4.4353
0.20	1.3123	1.8328	2.5546	3.3404	4.4900
0.25	1.2916	1.7704	2.4491	3.1939	4.2937
0.30	1.3161	1.8414	2.5651	3.3496	4.4946
0.35	1.3166	1.8435	2.5696	3.3571	4.5068
0.40	1.3083	1.8185	2.5311	3.3106	4.4555
0.45	1.3201	1.8556	2.5924	3.3888	4.5471
0.50	1.3247	1.8690	2.6141	3.4206	4.5948

Table 4 Mean-square values of filtering error $u(t, x)-\hat{u}(t, x)$, $\sum_{i=1}^{500}(u(i\Delta, x)-\hat{u}(i\Delta, x))^2/500$, $\Delta=0.001$, for $x=0.0$, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40, 0.45 and 0.50 when the observation point is $x^1=0.02$.

Value of x	White Gaussian observation noise				
	$N(0, 0.1^2)$	$N(0, 0.3^2)$	$N(0, 0.5^2)$	$N(0, 0.7^2)$	$N(0, 1)$
0.0	8.4492	8.6430	8.8689	9.0845	9.3608
0.05	8.4327	8.5980	8.8017	9.0028	9.2683
0.10	8.4287	8.5888	8.7894	8.9886	9.2528
0.15	8.4224	8.5722	8.7666	8.9626	9.2254
0.20	8.4295	8.5905	8.7915	8.9908	9.2547
0.25	8.4092	8.5344	8.7075	8.8871	9.1346
0.30	8.4328	8.5956	8.7954	8.9926	9.2535
0.35	8.4332	8.5980	8.8002	8.9993	9.2623
0.40	8.4247	8.5781	8.7749	8.9724	9.2358
0.45	8.4372	8.6096	8.8174	9.0198	9.2849
0.50	8.4414	8.6215	8.8364	9.0453	9.3173

similarly to Table 1 when the observation points are $x^1 = 0.05$, 0.01 and 0.02. The estimation accuracies for $x^1 = 0.01$ and 0.02 decrease compared with those for $x^1 = 0.1$ and 0.05. This decrease might come from the fact that the observation at the point where the greatest value of the amplitude of the wave form in the spacial domain yields the minimum estimation error.covariance (Sawaragi, Soeda and Omatu, 1978).

In the computation of the differential equations (49), (50), (53) and (54), the fourth-order Runge-Kutta method is adopted, where the sampling interval for the numerical integration is 0.001.

5-2. Stationary stochastic signal case

We shall consider the second-order linear stochastic hyperbolic partial differential equation

$$\partial u(t, x)/\partial t = \partial^2 u(t, x)/\partial x^2 + w(t, x) \tag{72}$$

driven by a white noise $w(t, x)$ with an autocovariance function

$$E[w(t, x)w(s, y)] = 0.7^2 \delta(t-s)\delta(x-y). \tag{73}$$

The initial condition at $x=0$ is $u(t, 0)=5\sin(\pi t/20)$ and boundary conditions are $u(0, x)=0$ and $u(t, 1)=0$. The observation equation is same with (71). The auto-covariance function of $u(t, x)$ is $K(t, x, s, y)=1/2$ from Heine (1955), so that we find that $\alpha(t, x, y)=1/2$, $\beta(s, x, y)=1$. The filtering estimate of the stochastic signal generated by (72) is calculated by substituting the covariance information into [Theorem 1]. Fig. 2 depicts the filtering estimate $\hat{u}(t, 0.15)$ *vs.* t for white Gaussian observation noises $N(0, 0.1^2)$ (graph (b)), $N(0, 0.2^2)$ (graph (c)) and $N(0, 0.3^2)$ (graph (d)). Graph (a) illustrates the signal process $u(t, 0.15)$. Fig. 3 and Fig. 4 depict the filtering estimates $\hat{u}(t, 0.45)$ and $\hat{u}(\mathrm{t}, 0.7)$. The present filter is compared with widely known estimation procedure based on spacial discretization technique (Sage and White, 1977) applied to the Kalman filter which is often adopted in lumped parameter systems. Table 5 shows the M. S. V. of filtering error $u(t, x)-\hat{u}(t, x)$ for $x=0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$ and 0.9 with one hundred data in the interval of $0<t\leqq 1$, provided that the sampling interval of numerical integration by the Runge-Kutta method is 0.001. Here, the filtering estimate $\hat{u}(t, x)$ is calculated at the observation points $x^1=0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$ and 0.9 respectively for observation noises $N(0, 0.1^2)$, $N(0, 0.2^2)$, $N(0, 0.3^2)$ and $N(0, 0.5^2)$. Initial error variances are $0.3^2 \cdot \mathrm{I}$ for case 1 and I for case 2, where I is an identity matrix of order 18, since x, $0\leqq x\leqq 1$, is spacially partitioned. It should be noted that the M. S. V. of filtering error for white observation noise $N(0, 0.1^2)$ diverges.

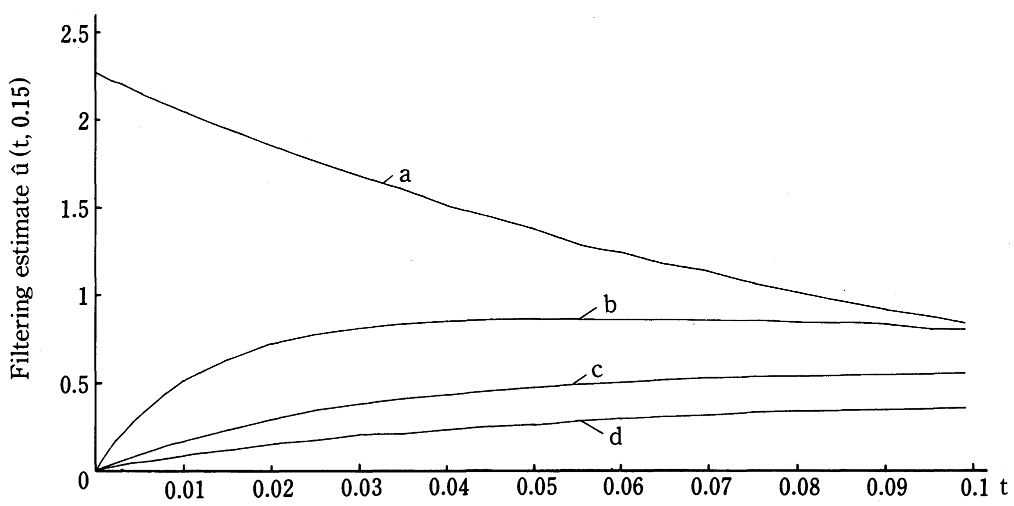

Fig. 2　Filtering estimate $\hat{u}(t, 0.15)$ vs. t.
Graph a …Signal process $u(t, 0.15)$ vs. t.
Graph b …Filtering estimate $\hat{u}(t, 0.15)$ vs. t for white Gaussian observation noise $N(0, 0.1^2)$.
Graph c …Filtering estimate $\hat{u}(t, 0.15)$ vs. t for white Gaussian observation noise $N(0, 0.2^2)$.
Graph d …Filtering estimate $\hat{u}(t, 0.15)$ vs. t for white Gaussian observation noise $N(0, 0.3^2)$.

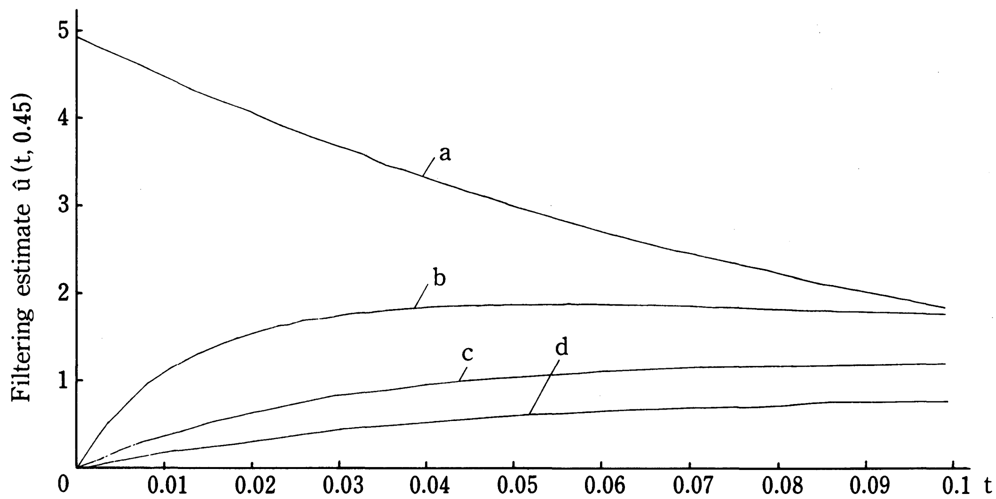

Fig. 3 Filtering estimate $\hat{u}(t, 0.45)$ vs. t.
Graph a ···Signal process $u(t, 0.45)$ vs. t.
Graph b ···Filtering estimate $\hat{u}(t, 0.45)$ vs. t for white Gaussian observation noise $N(0, 0.1^2)$.
Graph c ···Filtering estimate $\hat{u}(t, 0.45)$ vs. t for white Gaussian observation noise $N(0, 0.2^2)$.
Graph d ···Filtering estimate $\hat{u}(t, 0.45)$ vs. t for white Gaussian observation noise $N(0, 0.3^2)$.

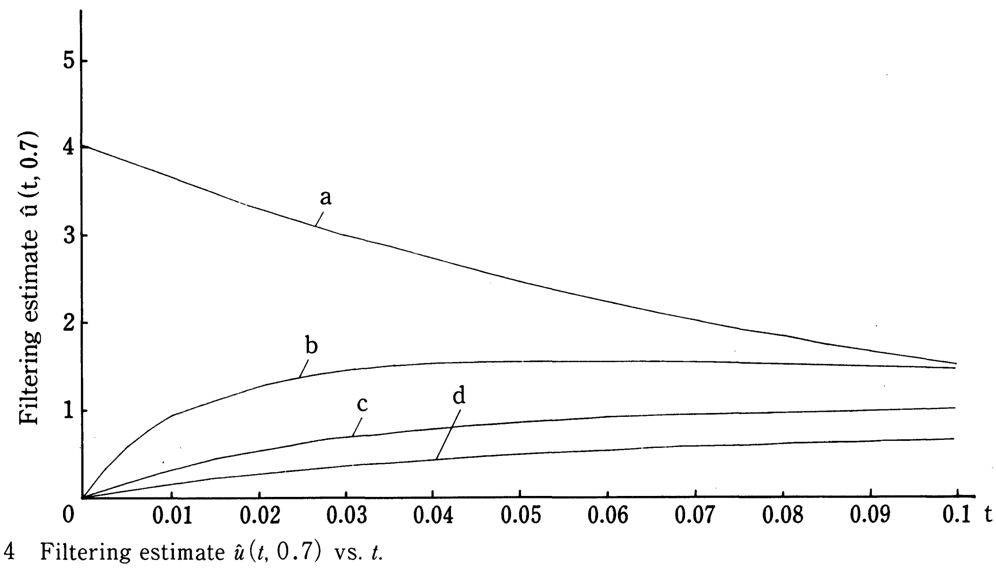

Fig. 4 Filtering estimate $\hat{u}(t, 0.7)$ vs. t.
Graph a ···Signal process $u(t, 0.7)$ vs. t.
Graph b ···Filtering estimate $\hat{u}(t, 0.7)$ vs. t for white Gaussian observation noise $N(0, 0.1^2)$.
Graph c ···Filtering estimate $\hat{u}(t, 0.7)$ vs. t for white Gaussian observation noise $N(0, 0.2^2)$.
Graph d ···Filtering estimate $\hat{u}(t, 0.7)$ vs. t for white Gaussian observation noise $N(0, 0.3^2)$.

Table 5　Mean-square values of filtering error $u(t, x)-\hat{u}(t, x)$, $\sum_{i=1}^{500}(u(i\Delta, x)-\hat{u}(i\Delta, x))^2/500$, $\Delta=0.001$, for $x=0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$ and 0.9 when the observation points are $x^1=0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$ and 0.9 respectively.

Value of x^1	observation noise	Present method	Case 1	Case 2
0.1	$N(0, 0.1^2)$	0.35182	Divergence	Divergence
0.1	$N(0, 0.2^2)$	0.62592	0.83238	0.62743
0.1	$N(0, 0.3^2)$	0.78515	1.1378	0.88969
0.1	$N(0, 0.5^2)$	0.92507	1.4999	1.2793
0.2	$N(0, 0.1^2)$	1.2477	Divergence	Divergence
0.2	$N(0, 0.2^2)$	2.2269	3.422	2.5321
0.2	$N(0, 0.3^2)$	2.8018	4.7119	3.6168
0.2	$N(0, 0.5^2)$	3.315	6.2658	5.2653
0.3	$N(0, 0.1^2)$	2.3775	Divergence	Divergence
0.3	$N(0, 0.2^2)$	4.2387	2.2739	3.4316
0.3	$N(0, 0.3^2)$	5.3273	10.844	2.034
0.3	$N(0, 0.5^2)$	6.2954	12.8483	9.7184
0.4	$N(0, 0.1^2)$	3.2879	Divergence	Divergence
0.4	$N(0, 0.2^2)$	5.8534	9.2044	6.8531
0.4	$N(0, 0.3^2)$	7.3539	12.647	9.7714
0.4	$N(0, 0.5^2)$	8.6912	16.769	14.17
0.5	$N(0, 0.1^2)$	3.6461	Divergence	Divergence
0.5	$N(0, 0.2^2)$	6.4979	10.21	7.6119
0.5	$N(0, 0.3^2)$	8.1627	14.024	10.849
0.5	$N(0, 0.5^2)$	9.6388	18.585	15.724
0.6	$N(0, 0.1^2)$	3.2841	Divergence	Divergence
0.6	$N(0, 0.2^2)$	5.8472	9.1938	6.8473
0.6	$N(0, 0.3^2)$	7.3482	12.632	9.7582
0.6	$N(0, 0.5^2)$	8.6868	16.751	14.151
0.7	$N(0, 0.1^2)$	2.3522	Divergence	Divergence
0.7	$N(0, 0.2^2)$	4.2057	6.5361	4.8162
0.7	$N(0, 0.3^2)$	5.2952	9.0148	6.8952
0.7	$N(0, 0.5^2)$	6.2694	12.007	10.067
0.8	$N(0, 0.1^2)$	1.2489	Divergence	Divergence
0.8	$N(0, 0.2^2)$	2.2275	3.4855	2.5848
0.8	$N(0, 0.3^2)$	2.8013	4.7958	3.6893
0.8	$N(0, 0.5^2)$	3.3132	6.3712	5.3641
0.9	$N(0, 0.1^2)$	0.34696	Divergence	Divergence
0.9	$N(0, 0.2^2)$	0.6165	0.97044	0.72302
0.9	$N(0, 0.3^2)$	0.77401	1.3323	1.0312
0.9	$N(0, 0.5^2)$	0.91478	1.7643	1.4926

In this example, just four differential equations are included in the present algorithm, whereas the Kalman filter via the spacial discretization procedure has to solve 189 number of differential equations. Thus, the current filter needs less computer storage memory than the conventional method.

From these simulation results, we find that the filtering estimate approaches the signal process gradually as time t increases. It can also be seen that the filtering estimate for additive observation noise with the smaller noise variance is better in estimation accuracy than that with the larger values.

6. Conclusions

In this paper, a new type of filtering algorithm was devised in linear continuous distributed parameter systems. The proposed estimator used the covariance information of the signal and white Gaussian observation noise, and needs not the information of a signal generating model.

A numerical simulation result has shown that the current filter is quite feasible.

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