Image Restoration Technique by RLS Wiener Fixed-Point Smoother and Filter

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Abstract

This paper proposes an image restoration technique by the recursive least-squares (RLS) Wiener fixed-point smoother and filter in linear discrete-time stochastic systems. The RLS Wiener estimators use the auto-covariance function of the signal, the system matrix Φ for the n by 1 zero-mean signal vector $s(i,j)$, which is obtained by subtracting the mean of the image signal $x(i,j)$ from the signal values of the image, the variance of $s(i,j)$ and the variance of the white observation noise. Here, for the two-dimensional zero-mean values $s(i,j)$, $i=1, 2, \cdots, M, j=1, 2, \cdots, N$, the n by 1 vector consisting of the components $s(i,j)$, $i=1, 2, \cdots, n$, are estimated along the horizontal direction for $j=1, 2, \cdots, N$. Next, the n by 1 vector consisting of $s(i,j)$, $i=n+1, n+2, \cdots, 2n$, are estimated, recursively from the left to the right column, similarly. This procedure is continued, at last, to the last column j=N for the bottom n by 1 vector of $s(i,j)$. Finally, the estimates of the image are obtained by adding the mean to the estimates of $s(i,j)$.

Keyword : Wiener-Hopf equation, linear discrete-time systems, recursive Wiener filter, auto-covariance function, image restoration

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1. Introduction

In [1], an estimation technique for the signal is studied, by applying the Kalman filter to the state-space model, whose state vector is transformed by the wavelet transformation, in linear discrete-time stochastic systems. This filter is called as the wavelet Kalman filter [1]. The filter bank by the wavelet transformation consists of the decomposition and the composition in the octave division, where the number of the sub-band decomposition is chosen for a data block with length $n=2^2=4$. In the wavelet Kalman filter, the scaling coefficients and the wavelet coefficients, which are given as the output variables in the sub-band decomposition, are estimated by the Kalman filter. The filtering estimate of the signal is obtained by composing the scaling and wavelet coefficients, namely by the inverse wavelet transformation of the estimates of the scaling coefficients and the wavelet coefficients. In the simulation example of [1], a scalar first-order stochastic signal process known as the Brownian random walk is estimated. However, even for the first-order model, the derivation of the variance of the input vector in the state equation is not straightforward. For the autoregressive (AR) model with the order $n \geq 2$, it is not an easy task to get the state-space model. From this viewpoint, the wavelet Kalman filter might not be suitable for the estimation of the stochastic signal modeled by the AR model with the order $n \geq 2$. Since the wavelet and inverse wavelet transformations are linear, it is seen that the estimation accuracy using the identity matrix transformation instead of the wavelet transformation is same as that with the wavelet transformation.

In [2], [3], using the covariance information of the signal and observation noise, the recursive least-squares (RLS) Wiener fixed-point smoother and filter are designed in linear stochastic systems. In [3], the continuous-time RLS Wiener filter is shown. This paper proposes an image restoration technique by the RLS Wiener fixed-point smoother and filter in linear discrete-time stochastic systems. The RLS Wiener estimators use the system matrix Φ for the n by 1 zero-mean signal vector $s(i,j)$, which is obtained by subtracting the mean of the image signal $x(i,j)$ from the signal values of the image, the variance of $s(i,j)$, the variance of the white observation noise and the observation matrix. Here, for the two-dimensional zero-mean values $s(i,j)$, $i=1, 2, \cdots, M$, $j=1, 2, \cdots, N$, the n by 1 vector consisting of the components $s(i,j)$, $i=1, 2, \cdots, n$, are estimated along the horizontal direction for $j=1, 2, \cdots, N$. Next, the n by 1 vector consisting of $s(i,j)$, $i=n+1, n+2, \cdots, 2n$, are estimated, recursively from the left to the right column, similarly. This procedure is continued, to the last column $j=N$ for the bottom n by 1 vector of $s(i,j)$. Finally, the estimates of the image $x(i,j)$ are obtained by adding the mean to the estimates of $s(i,j)$.

A numerical simulation example is shown and the estimation accuracies of the RLS Wiener fixed-point smoother and filter, for $n=1$, $n=4$ and $n=8$, are compared.

2. RLS Wiener fixed-point smoother and filter

Let a vector observation equation be given by

$$y(k)=z(k)+v(k), \tag{1}$$

in linear discrete-time stochastic systems, where $z(k)$ is an n by 1 signal, and $v(k)$ is white observation noise. It is assumed that the signal and the observation noise are mutually independent. Let the variance of $v(k)$ be R.

$$E[v(k)v^{\mathrm{T}}(s)]=R\delta_{\mathrm{K}}(k-s) \tag{2}$$

Here, $\delta_{\mathrm{K}}(\cdot)$ denotes the Kronecker δ function.

Let a fixed-point smoothing estimate $\hat{z}(k,L)$ of $z(k)$ be given by

$$\hat{z}(k,L)=\sum_{i=1}^{L} h(k,i,L)y(i) \tag{3}$$

as a linear transformation of the set of the observed values $\{y(i),\ 1\leq i\leq L\}$. The impulse response function $h(k,s,L)$, which minimizes the mean-square value of the filtering error $z(k)-\hat{z}(k,L)$,

$$J=E[\| z(k)-\hat{z}(k,L)\|^2], \tag{4}$$

[4] satisfies

$$h(k,s,L)R=K(k,s)-\sum_{i=1}^{L} h(k,i,L)K(i,s). \tag{5}$$

Here, $K(k,s)$ represent the auto-covariance function of the signal $z(k)$. From $K(k,s)$, the system matrix in the state-space model for $z(k)$ is given by

$$\Phi=K(k+1,k)K^{-1}(k,k). \tag{6}$$

2. RLS fixed-point smoothing and filtering alg orithms

The RLS Wiener fixed-point and filtering algorithms [2], using the covariance information, are shown in **Theorem 1.**

Theorem 1. Let the observation equation be given by (1). Let the auto-covariance function $K(k,s)$ of the signal and the variance of white observation noise be given. Then the RLS Wiener fixed-point smoothing and filtering equations consist of (7)-(9) in linear discrete-time stochastic systems.

Fixed-point smoothing estimate of $z(k)$: $\hat{z}(k,L)$

$$\hat{z}(k,L)=\hat{z}(k,L-1)+h(k,L,L)(y(L)-\Phi\hat{z}(L-1,L-1)) \quad (7)$$

$$h(k,L,L)=(k(k,k)(\Phi^T)^{L-K}H^T-q(k,L-1)\Phi^T H^T)(R+K(L,L)-\Phi S(L-1)\Phi^T)^{-1} \quad (8)$$

$$q(k,L)=q(k,L-1)\Phi^T+h(k,L,L)(K(L,L)-\Phi S(L-1)\Phi^T),\ q(L,L)=S(L) \quad (9)$$

Filtering estimate of : $z(k)$: $\hat{z}(k,k)$

$$\hat{z}(k,k)=\Phi\hat{z}(k-1,k-1)+G(k)(y(k)-\Phi\hat{z}(k-1,k-1)),\ \hat{z}(0,0)=0 \quad (10)$$

Filter gain: $G(k)$

$$G(k)=(K(k,k)-\Phi S(k-1)\Phi^T)(R+K(k,k)-\Phi S(k-1)\Phi^T)^{-1} \quad (11)$$

$$S(k)=\Phi S(k-1)\Phi^T+G(k)(K(k,k)-H\Phi S(k-1)\Phi^T),\ S(0)=0 \quad (12)$$

Here, the system matrix Φ, concerned with the signal $z(k)$, is calculated by $\Phi=K(k+1,k)K^{-1}(k,k)$.

3. A numerical simulation example

Let us consider of estimating the two-dimensional image "Lena.tif". The image consists of 512 × 512 pixels. For the n by 1 zero-mean signal vector $z(k)$, $z(k)$ is estimated along the horizontal direction, from the left 1st column to the 512th right end column, then the next n lines' $z(k)$ is estimated similarly, from the left to the right column. After these iterations, at last, the bottom n by 1 signal vector $z(k)$ is estimated to the 512th column. Fig.1 shows the original gray image "Lena.tif". The mean, the variance (after subtracting the mean from the pixel level) and the standard deviation of the image are $\bar{x}$=134.3997, $K_{\bar{x}}$=1715.2 and $\sigma_x=\sqrt{K_x}$=41.47 respectively. The values

Fig.1 Original gray image "Lena.tif".

of the standard deviation (S/N [dB]) of the additional white observation noise are 14.0688 (S/N=20[dB]), 25.0182 (S/N=15[dB]), 44.4893 (S/N=10[dB]) and 79.1144 (S/N=5[dB]). Let $s(i,j)$, $i=1, 2, \cdots, 512$, $j=1, 2, \cdots, 512$, be two-dimensional zero-mean data, which were obtained by subtracting the mean $\bar{x}=134.3997$ from the image levels $x(i,j)$, $i=1, 2, \cdots, 512$, $j=1, 2, \cdots, 512$. Let the vector elements of the signal $z(k)$ be given by $z(k)=[z_1(k), z_2(k), \cdots, z_n(k)]^T$. Here, for $i=1, 2, \cdots, 64$, $j=1, 2, \cdots, 512$, in the case of 8 by 1 vector of $z(k)$, the vector components are given by

$z_1(512(i-1)+j)=s(8(i-1)+1,j)$,

$z_2(512(i-1)+j)=s(8(i-1)+2,j)$,

$z_2(512(i-1)+j)=s(8(i-1)+3,j)$,

$z_4(512(i-1)+j)=s(8(i-1)+4,j)$,

$z_5(512(i-1)+j)=s(8(i-1)+5,j)$,

$z_6(512(i-1)+j)=s(8(i-1)+6,j)$,

$z_7(512(i-1)+j)=s(8(i-1)+7,j)$,

$z_8(512(i-1)+j)=s(8(i-1)+8,j)$,

The system matrix Φ is calculated by $\Phi=K(k+1,k)K^{-1}(k,k)$ in terms of the auto-covariance function of the signal $z(k)$. For the wide-sense stationary stochastic systems, $K(k,k)=K(k-k)=K(0)$ and $K(k+1,k)=K(1)$ are valid. For $n=8$, Φ and $K(k,k)=K(0)$ are computed as

$$\Phi = \begin{bmatrix} 0.6669 & 0.1643 & 0.0462 & 0.0552 & 0.0126 & 0.0179 & 0.0006 & 0.0181 \\ 0.2400 & 0.3277 & 0.2567 & 0.1027 & 0.0080 & 0.0179 & 0.0081 & 0.0220 \\ 0.0807 & 0.1339 & 0.3503 & 0.3065 & 0.0449 & 0.0199 & 0.0008 & 0.0461 \\ 0.0358 & 0.0488 & 0.1052 & 0.4212 & 0.2349 & 0.0232 & -0.0117 & 0.0702 \\ 0.0277 & 0.0228 & -0.0078 & 0.1965 & 0.3350 & 0.0829 & 0.0135 & 0.0968 \\ 0.0238 & 0.0216 & -0.0388 & 0.0785 & 0.1247 & 0.3022 & 0.1979 & 0.1810 \\ 0.0292 & 0.0037 & -0.0105 & 0.0279 & 0.0248 & 0.3969 & 0.2602 & 0.4788 \\ 0.0292 & 0.0162 & -0.0267 & 0.0354 & 0.0130 & 0.1696 & -0.1264 & 0.9545 \end{bmatrix},$$

$$K(0)=\begin{bmatrix} 1713.9 & 1688.9 & 1647.3 & 1605.4 & 1564.8 & 1528.1 & 1497.7 & 1470.4 \\ 1688.9 & 1718.2 & 1694.8 & 1653.9 & 1607.7 & 1565.0 & 1530.7 & 1501.1 \\ 1647.3 & 1694.8 & 1722.7 & 1699.5 & 1653.5 & 1604.9 & 1565.5 & 1533.0 \\ 1605.4 & 1653.9 & 1699.5 & 1726.9 & 1700.1 & 1652.5 & 1607.0 & 1569.8 \\ 1.5648 & 1.6077 & 1653.5 & 1.7001 & 1725.5 & 1697.6 & 1652.0 & 1608.8 \\ 1528.1 & 1565.0 & 1604.9 & 1652.5 & 1697.6 & 1723.6 & 1697.9 & 1654.7 \\ 1497.7 & 1530.7 & 1565.5 & 1607.0 & 1652.0 & 1697.9 & 1725.2 & 1701.1 \\ 1470.4 & 1501.1 & 1533.0 & 1569.8 & 1608.8 & 1654.7 & 1701.1 & 1729.0 \end{bmatrix}.$$

By substituting Φ , $K(k,k)=K(0)$ and the zero-mean observed values, which are calculated by subtracting the mean of the image from the original observed image levels, into the RLS Wiener fixed-point smoother and filter of [Theorem 1], the fixed-point smoothing estimate $\hat{z}(k,L)$, $L=k+Lag$, and the filtering estimates $\hat{z}(k,k)$ are calculated. After obtaining the estimates of $s(i,j)$ from $\hat{z}(k,k+Lag)$ and $\hat{z}(k,k)$, the fixed-point smoothing estimate and the filtering estimate of the image are calculated by adding the mean to the estimates of the zero-mean two-dimensional signal $s(i,j)$. Fig.2 shows the observed image with additive white observation noise for the signal to noise ratio SNR=10[dB]. Fig.3 shows the restorated image by the RLS Wiener filter for SNR=10[dB]. Fig.4 shows the restorated image by the fixed-point smoother for Lag=5 and SNR=10[dB]. In comparison of Fig.4 with Fig.3, the smoothing effect is seen in Fig.4. The mean-square values (MSVs) of the fixed-point smoothing errors and the filtering errors are evaluated respectively by

Fig.2 Observed image for SNR=10 [dB].

Fig.3 Filtering estimate of the image for SNR=10 [dB].

Fig.4 Fixed-point smoothing estimate of the image for Lag=5 and SNR=10 [dB].

$MSE(FPS) = \frac{1}{512\times 512/n}\sum_{i=1}^{512\times 512/n}(z(i)-\hat{z}(i,i))^T(z(i)-\hat{z}(i,i))$ and $MSE(FIL) = \frac{1}{512\times 512/n-Lag}\sum_{i=1}^{512\times 512/n-Lag}(z(i)-\hat{z}(i,i+Lag))^T(z(i)-\hat{z}(i,i+Lag))$. In the [dB] expressions, the MSVs of the fixed-point smoothing errors and the filtering errors are evaluated by $10\log_{10}(\frac{MSE(FPS)}{K_x})$ [dB] and $10\log_{10}(\frac{MSE(FIL)}{K_x})$ [dB], K_x =1715.2, respectively. Fig.5 illustrates the MSVs in [dB] of the estimation errors to the signal variance vs. Lag for the $SNRs$ = 5, 10, 15, 20 [dB] . From the MSV of the filtering errors for Lag=0, the MSV decreases for Lag=1. For the $SNRs$ = 5, 10, 15, 20 [dB], as the value of Lag increases, the MSV decreases gradually and the estimation accuracy of the image restoration is improved. Fig.6, in the case of n=4, illustrates the MSVs [dB] of the estimation errors to the signal variance vs. Lag for the $SNRs$ = 5, 10, 15, 20 [dB]. The MSVs of the fixed- point smoothing errors and the filtering errors for n=4, in comparison with the case for n=8, are degraded. Fig.7, in the case of n=1, illustrates the MSVs [dB] of the estimation errors to the signal variance vs. *Lag* for *SNRs* = 5, 10, 15, 20 [dB]. From Fig.5, Fig.6 and Fig.7, it is seen, as n increases, that the MSVs of the estimation errors decrease and the estimation accuracy is improved. For n=1, 4, 8, the estimation accuracy of the fixed-point smoother is superior to that of the filter. Fig.8 shows the restorated image by the two-dimensional filter in [5] for *SNR*=10[dB]. The MSVs of the filtering errors to signal variance by the filter [5], for *SNRs* = 5, 10, 15, 20 [dB], are -3.8509, -6.1344, -8.8724 and -12.0719 [dB] respectively. The MSVs of the filtering errors to signal variance by the RLS Wiener filter for n=1 are -3.8130, -6.0736, -8.8040 and -12.0251[dB] respectively. Hence, the estimation accuracy of the recursive Wiener filter is almost same with that of the two-dimensional filter [5].

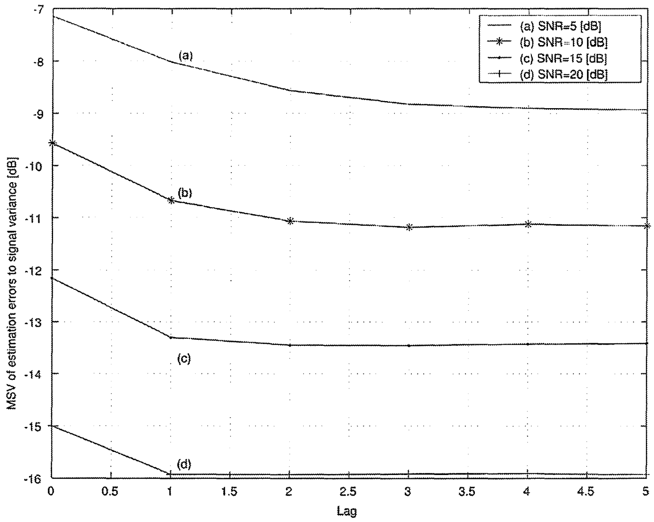

Fig.5 MSVs in [dB] of the estimation errors to the signal variance vs. *Lag* for the *SNRs* = 5, 10, 15, 20 [dB] in the case of n=8.

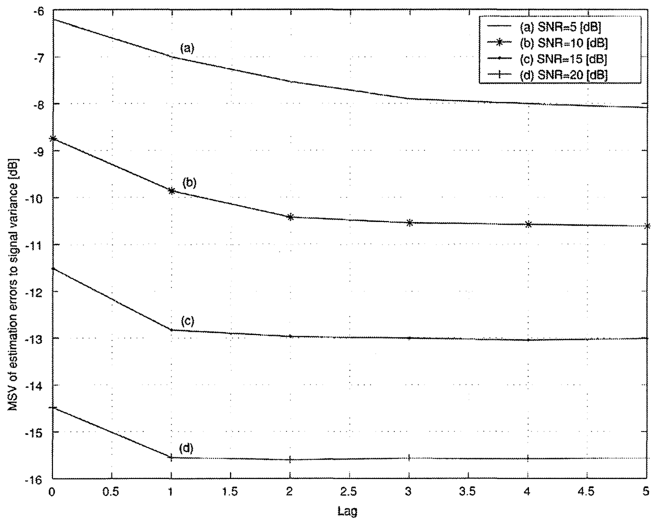

Fig.6 MSVs [*dB*] of the estimation errors to the signal variance vs. *Lag* for the *SNRs* = 5, 10, 15, 20 [*dB*] in the case of n=4.

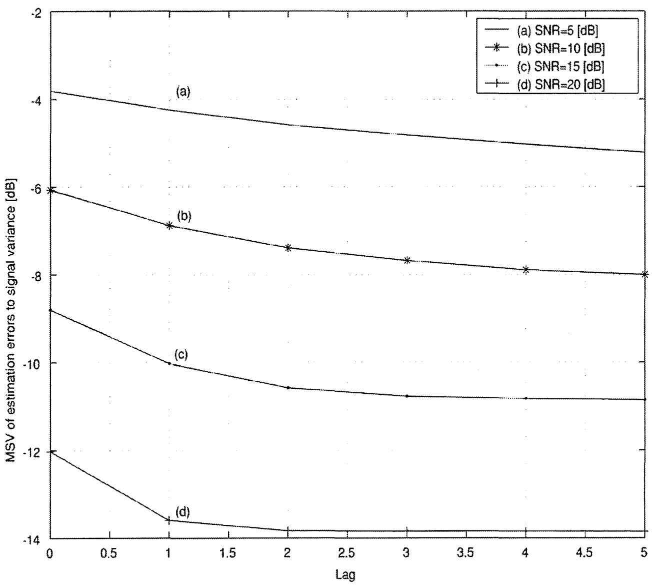

Fig.7 MSVs [*dB*] of the estimation errors to the signal variance vs. *Lag* for the *SNRs* = 5, 10, 15, 20 [*dB*] in the case of n=1.

Fig.8 Restorated image by the two-dimensional filter in [5] for *SNR*= 10[*dB*].

5. Conclusions

In this paper, the image restoration technique based on the RLS Wiener fixed-point smoother and filter is devised. The numerical simulation example showed the following characteristics.

(1) The estimation accuracy is improved as the value of n increases. The RLS Wiener filter for n=1 has almost the same estimation accuracy with the two-dimensional filter [5].

(2) The estimation accuracy of the fixed-point smoother is superior to the filter.

(3) The proposed estimation technique might also be suitable for the signal estimation of the n channel signals in linear discrete-time stochastic systems.

(4) The proposed estimation technique based on the RLS Wiener estimators require only the system matrix Φ , calculated by the auto-covariance function of $z(k)$, the variance of $z(k)$, the variance of white observation noise besides the observed value.

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