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Enumerating embeddings of n-manifolds in Euclidean $(2n-1)$-space

Dedicated to Professor Minoru Nakaoka on his 60th birthday

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Introduction.

Throughout this paper, "n-manifold" and "embedding" will mean closed connected differentiable manifold of dimension n and differentiable embedding, respectively. Let $[M\subset R^{m}]$ denote the set of isotopy classes of embeddings of a manifold M into Euclidean m-space.

It is known that for an n-manifold M,

(1) (Whitney [24]) the set $[M\subset R^{2n+2}]$ consists of only one element if $n\geqq 1$,

(2) (Wu [25]) the set $[M\subset R^{2n+1}]$ consists of only one element if $n\geqq 2$,

(3) (Haefliger [6], Bausum [1], Rigdon [13] etc.) if $n\geqq 4$, then, as a set,

$$[M\subset R^{2n}]=\begin{cases} H^{n-1}(M;Z) & \text{for } n\equiv 1\,(2),\ w_1(M)=0,\\ H^{n-1}(M;Z_2) & \text{for } n\equiv 1\,(2),\ w_1(M)\neq 0,\\ & \quad\text{or } n\equiv 0\,(2),\ w_1(M)=0,\\ Z\times\rho_2 H^{n-1}(M;Z) & \text{for } n\equiv 0\,(2),\ w_1(M)\neq 0. \end{cases}$$

The purpose of this paper is to inquire into the question of whether or not the set $[M\subset R^{2n-1}]$ for an n-manifold M, if it is not empty, can be described in terms of the cohomology of M, its characteristic classes and the cohomology operations. We shall study $[M\subset R^{2n-1}]$ along the lines of Haefliger [5], [6].

Let X^2 be the product $X\times X$ of X and let ΔX be the diagonal in X^2. The cyclic group of order 2, Z_2, acts on X^2 via the map $t: X^2\rightarrow X^2$ defined by $t(x, y)=(y, x)$, where ΔX is the fixed point set of this action. The quotient space

$$X^*=(X^2-\Delta X)/Z_2$$

is called the reduced symmetric product of X. Let P^m denote the real projective space of dimension m $(m\leqq\infty)$ and let

$$\xi: X^*\longrightarrow P^{\infty}$$

denote the classifying map of the double covering $X^2-\Delta X\rightarrow X^*$. Then the first Stiefel-Whitney class of this double covering is given by

$$v=\xi^*(u) \qquad (u \text{ being the generator of } H^1(P^\infty;\ Z_2)).$$

For a finitely generated abelian group G, let G' denote the new abelian group associated with G defined as follows: A given group G is of the form

$$G=\sum_{i=1}^{b} Z_{r(i)}\langle a_i\rangle+G_o \qquad \text{(direct sum)},$$

$$r(i)=\begin{cases} \infty & \text{for } 1\leqq i\leqq a\,, \\ 2^{s(i)}\ (s(i)\geqq 1) & \text{for } a<i\leqq b\,, \end{cases}$$

where $Z_r\langle c\rangle$ denotes a cyclic group of order r ($r\leqq\infty$) generated by c and G_o denotes the odd torsion subgroup of G. Then G' is defined by

$$G'=\sum_{i=1}^{b} Z_{2r(i)}\langle(1/2)a_i\rangle+G_o\,.$$

Under these notations, we shall prove the following theorem.

THEOREM 1. *Let $n\geqq 6$ and let M be an n-manifold. Assume that M is orientable or n is even and that there exists an embedding of M into R^{2n-1}. Then, as a set, it follows that*

$$[M\subset R^{2n-1}]=(1+(-1)^n t^*)(H^{n-1}(M;\ Z)\otimes H^{n-1}(M;\ Z))\times \operatorname{Coker}\Theta$$

$$\times\begin{cases} [H^{n-1}(M;\ Z)]'\times H^{n-2}(M;\ Z) & \text{if } n \text{ is even and } w_1(M)=0, \\ [H^{n-1}(M;\ Z)]'\times H^{n-2}(M;\ Z_2) & \text{if } n \text{ is even and } w_1(M)\neq 0, \\ H^{n-2}(M;\ Z_2) & \text{if } n \text{ is odd and } w_1(M)=0, \end{cases}$$

where

$$\Theta=\Big(Sq^2+\binom{2n-1}{2}v^2\Big)\tilde{\rho}_2:\ \ H^{2n-3}(M^*;\ Z[v])\longrightarrow H^{2n-1}(M^*;\ Z_2),$$

v is the first Stiefel-Whitney class of the double covering $M^2-\Delta M\rightarrow M^*$, $Z[v]$ is the sheaf of coefficients over M^*, locally isomorphic to Z, twisted by v, and $\tilde{\rho}_2$ is the reduction mod 2 *twisted by v. The following information is sufficient to determine* Coker Θ:

(i) *the integral cohomology groups $H^i(M;\ Z)$ for $n-3\leqq i\leqq n$,*

(ii) *the actions of Sq^2 on $H^i(M;\ Z_2)$ for $i=n-3,\ n-2$,*

(iii) *the action of $w_1(M)$ on $H^{n-2}(M;\ Z_2)$ for $n\equiv 2\ (4)$.*

REMARK. (1) For an n-manifold M ($n>4$), there is an embedding of M into R^{2n-1} if M is orientable or if n is not a power of 2 (cf. [**10**, (1) and (2) in §1]).

(2) It is known (see (2.5) below) that

$$Sq^2x=\begin{cases} (w_2(M)+w_1(M)^2)x & \text{if } x\in H^{n-2}(M;\ Z_2), \\ (w_2(M)+w_1(M)^2+w_1(M)Sq^1)x & \text{if } x\in H^{n-3}(M;\ Z_2). \end{cases}$$

The following corollary is an immediate consequence of the theorem, since $H^{2n-1}(M^*; Z_2)$ is isomorphic to $H^{n-1}(M; Z_2)$ by Thomas [17, § 2].

COROLLARY (Haefliger [6]). *If* $H_1(M; Z)=0$, *then*

$$[M\subset R^{2n-1}]=\begin{cases} H_2(M; Z) & \text{for even } n, \\ H_2(M; Z_2) & \text{for odd } n. \end{cases}$$

Further, Propositions 2-4 will be shown as corollaries to the theorem.

PROPOSITION 2. *If M is a spin n-manifold* ($n\geqq 6$), *then*

$$[M\subset R^{2n-1}]=(1+(-1)^n t^*)(H^{n-1}(M; Z)\otimes H^{n-1}(M; Z))$$

$$\times\begin{cases} H^{n-2}(M; Z)\times[H^{n-1}(M; Z)]'\times H^{n-1}(M; Z_2)/Sq^1H^{n-2}(M; Z_2) & n\equiv 0\,(4), \\ H^{n-2}(M; Z)\times[H^{n-1}(M; Z)]'\times H^{n-1}(M; Z_2) & n\equiv 2\,(4), \\ H^{n-2}(M; Z_2)\times H^{n-1}(M; Z_2) & n\equiv 1\,(2). \end{cases}$$

PROPOSITION 3. *If M is an orientable n-manifold* ($n\geqq 6$) *and if either* $Sq^2\rho_2 H^{n-2}(M; Z)\neq 0$ *or* $Sq^2\rho_2 H^{n-3}(M; Z)=H^{n-1}(M; Z_2)$, *then*

$$[M\subset R^{2n-1}]=(1+(-1)^n t^*)(H^{n-1}(M; Z)\otimes H^{n-1}(M; Z))$$

$$\times\begin{cases} [H^{n-1}(M; Z)]'\times H^{n-2}(M; Z) & \text{for } n\equiv 0\,(2), \\ H^{n-2}(M; Z_2) & \text{for } n\equiv 1\,(2). \end{cases}$$

REMARK. It has been shown by Thomas [17, § 4] that if M is orientable and if either $w_3(M)\neq 0$, or $w_2(M)\neq 0$ and $H_1(M; Z)$ has no 2-torsion, then $Sq^2\rho_2 H^{n-2}(M; Z)\neq 0$.

PROPOSITION 4. *Let n be even and assume that an unorientable n-manifold* M ($n\geqq 6$) *is embedded in* R^{2n-1}. *Then*

$$[M\subset R^{2n-1}]=(1+(-1)^n t^*)(H^{n-1}(M; Z)\otimes H^{n-1}(M; Z))$$

$$\times[H^{n-1}(M; Z)]'\times H^{n-2}(M; Z_2),$$

if one of the conditions (i), (ii) *and* (iii) *is satisfied:*

(i) $Sq^2\rho_2 H^{n-3}(M; Z)=H^{n-1}(M; Z_2)$,

(ii) $Sq^2 H^{n-3}(M; Z_2)=H^{n-1}(M; Z_2)$ *and either* $w_1(M)^3=w_3(M)$ *or* $Sq^2\rho_2 H^{n-2}(M; Z)=0$,

(iii) $n\equiv 2\,(4)$ *and* $w_1(M)\rho_2 H^{n-2}(M; Z)=H^{n-1}(M; Z_2)$.

As an example, we shall consider $[P(m, n)\subset R^{2m+4n-1}]$ for the Dold manifold $P(m, n)$ of type (m, n) of dimension $m+2n$.

PROPOSITION 5. *Let* $m, n\geqq 1$ *with* $m+2n\geqq 6$ *and assume that either* $m\equiv 0\,(2)$ *or* $n\equiv 0\,(2)$ *holds. Then*

$$\#[P(m, n)\subset R^{2m+4n-1}]=\begin{cases} 16 & \textit{if } n\equiv 3\,(4) \textit{ and either } m=2 \textit{ or } n\equiv 0\,(4),\\ 8 & \textit{if } m\equiv 0\,(2),\ n\equiv 1\,(4) \textit{ or}\\ & \quad \textit{if } m\geqq 4,\ m\equiv 2\,(4),\ n\equiv 3\,(4),\\ 4 & \textit{if } n\equiv 0\,(2),\ m\geqq 2,\\ 2 & \textit{if } n\equiv 0\,(2),\ m=1,\end{cases}$$

where $\#S$ *denotes the cardinality of the set* S.

REMARK. For all the other Dold manifolds $P(m, n)$ with $m+2n\geqq 6$, the cardinality of $[P(m, n)\subset R^{2m+4n-1}]$ are given as follows:

$$\#[P(m, n)\subset R^{2m+4n-1}]=\begin{cases} \infty & \text{if } m=0 \text{ or } n\equiv 1\,(2),\ m=1,\\ 8 & \text{if } n\equiv 1\,(2),\ m\geqq 3,\ m\equiv 1\,(2),\\ 4 & \text{if } n=0,\ m\equiv 3\,(4),\\ 2 & \text{if } n=0,\ m\not\equiv 3\,(4),\ m\neq 2^r\ (r\geqq 3).\end{cases}$$

In fact, $\#[P(m, n)\subset R^{2m+4n-1}]$ for $m\equiv 1\,(2)$, $n\equiv 1\,(2)$ is calculated in [**20**] and that for $n=0$ or $m=0$ is calculated by Bausum [**1**], Larmore and Rigdon [**9**], Yasui [**18**], and Haefliger and Hirsch [**6**], [**7**], because $P(m, 0)$ and $P(0, n)$ are the real and the complex projective spaces, respectively.

REMARK. The set $[M\subset R^{2n-1}]$ in case $n\equiv 1\,(2)$ and $w_1(M)\neq 0$ is not treated in this paper but this set has been studied in [**20**] under the condition that $H_2(M; Z)$ is isomorphic to a direct sum of some copies of Z_2.

The remainder of this paper is organized as follows: In §1, we state the method for computing $[M\subset R^{2n-1}]$. In §2, the action of Sq^i $(i=1, 2)$ on the mod 2 cohomology of M^* is studied. The morphism $i^*: H^*(\Lambda^2 M, \Delta M; Z_2)\to H^*(M^*; Z_2)$ is determined in §3. Here $(\Lambda^2 M, \Delta M)$ is the pair of quotient spaces $(M^2/Z_2, \Delta M/Z_2)$. The cohomology groups $H^{2n-2}(M^*; Z[v])$ and $\tilde{\rho}_2 H^{2n-3}(M^*; Z[v])$ are given in §4, whose proofs for even n and for odd n are given in §5 and §6, respectively. §7 is concerned with computing Coker Θ. In the last section, §8, we give the proofs of the results stated in the introduction.

§1. Method for computing $[M\subset R^{2n-1}]$.

We now recall Haefliger's theorem [**5**, Théorème 1′]. Let $S^\infty\to P^\infty$ be the universal double covering. Then the bundle $S^\infty\times_{Z_2}S^m\to P^\infty$ is homotopically equivalent to the natural inclusion $P^m\to P^\infty$. Therefore Haefliger's theorem can be restated as follows, where

$$[X, P^m; a]=[X, S^\infty\times_{Z_2}S^m; a] \qquad \text{for} \quad a: X\longrightarrow P^\infty$$

denotes the homotopy set of liftings of a to $S^\infty \times_{Z_2} S^m$:

THEOREM 1.1 (Haefliger). *If $2m>3(n+1)$, then for an n-manifold M, there is a bijection*

$$[M\subset R^m]=[M^*,\ P^{m-1};\ \xi].$$

In particular

$$[M\subset R^{2n-1}]=[M^*,\ P^{2n-2};\ \xi] \qquad \text{if} \quad n\geqq 6.$$

As for the right-hand side of this equation, we know

PROPOSITION 1.2 (Bausum [**1**], Larmore and Rigdon [**9**], Yasui [**18**] etc.). *Let $n\geqq 6$ and assume that there is a lifting of $\xi: M^*\to P^\infty$ to P^{2n-2}. Then, as a set,*

$$[M^*,\ P^{2n-2};\ \xi]=H^{2n-2}(M^*;\ Z[v])\times \operatorname{Coker}\Theta,$$

where

$$\Theta=\left(Sq^2+\binom{2n-1}{2}v^2\right)\tilde{\rho}_2:\ H^{2n-3}(M^*;\ Z[v])\longrightarrow H^{2n-1}(M^*;\ Z_2),$$

v is the first Stiefel-Whitney class of the double covering $M^2-\Delta M\to M^*$ and $Z[v]$ is the sheaf of coefficients over M^*, locally isomorphic to Z, twisted by v.

The mod 2 cohomology of M^* has been studied by Thomas [**17**], and Bausum [**1**]. Therefore it is important for our purpose to study the groups $H^{2n-2}(M^*;\ Z[v])$ and $\tilde{\rho}_2 H^{2n-3}(M^*;\ Z[v])$, and the action of Sq^i $(i=1, 2)$ on $H^*(M^*;\ Z_2)$. For a manifold M, let PM denote the projective bundle associated with the tangent bundle of M and let

$$j:\ PM\longrightarrow M^*$$

be the inclusion, which is given by the Z_2-equivariant map $\tilde{j}$ of the tangent sphere bundle SM to $M^2-\Delta M$ defined by $\tilde{j}(v)=(\exp(v),\ \exp(-v))$, the Z_2-action on SM being induced from the antipodal map on each fibre. The Z_2-action on M^2, which is defined by interchanging factors, determines the quotient spaces

$$\Lambda^2 M=M^2/Z_2 \quad \text{and} \quad \Delta M=(\Delta M)/Z_2.$$

Therefore, it follows that

$$M^*=\Lambda^2 M-\Delta M.$$

Let

$$\pi:\ (M^2,\ \Delta M)\longrightarrow(\Lambda^2 M,\ \Delta M),$$

$$i:\ M^*\longrightarrow(\Lambda^2 M,\ \Delta M)$$

be the natural projection and inclusion, respectively.

LEMMA 1.3 ([**19**, Lemma 1.4]). *For a manifold M and a cyclic group G, there is an exact sequence*

$$\cdots \longrightarrow H^{i-1}(PM;\ j^*G[v]) \xrightarrow{\delta} H^i(\Lambda^2 M,\ \Delta M;\ G[v]) \xrightarrow{i^*} H^i(M^*;\ G[v]) \xrightarrow{j^*} H^i(PM;\ j^*G[v]) \longrightarrow \cdots,$$

where $G[v]$ is a sheaf of coefficients over M^*, locally isomorphic to G, twisted by v.

This lemma will play a dominant role for studying $H^{2n-2}(M^*;\ Z[v])$ and $\tilde{\rho}_2 H^{2n-3}(M^*;\ Z[v])$, since the twisted integral cohomology groups of $(\Lambda^2 M,\ \Delta M)$ is investigated by Larmore [8] (cf. [**19**, §5]) and so is PM by Rigdon [**13**, §9] and since the mod 2 cohomology groups both of $(\Lambda^2 M,\ \Delta M)$ and M^* are investigated in [8], [**12**] and [**1**], [**17**], respectively, and that of PM is well-known, and moreover the morphism j^* and δ are given by [**17**, §2] and [**19**, Lemma 1.5]. From now on, we shall make much use of notations introduced in [**17**] and [8] to represent elements of the mod 2 cohomology of M^* and of the cohomology of $(\Lambda^2 M,\ \Delta M)$, respectively. For example, let

$$j^*v = v \in H^1(PM;\ Z_2).$$

Then there are the following relations in the mod 2 cohomology:

(1.4) $$j^*\rho(u^i \otimes x^2) = \sum_{s=0}^{r} v^{i+r-s} Sq^s x \qquad \text{for} \quad x \in H^r(M;\ Z_2),$$

(1.5) $$\delta(v^i x) = v^{i+1} \Lambda x \qquad \text{for} \quad x \in H^*(M;\ Z_2).$$

§2. Actions of Sq^i ($i=1, 2$) on the mod 2 cohomology of M^*.

First, we shall study, along the lines of Thomas [**17**], the mod 2 cohomology of M^* more exactly than [**17**] and so we assume that the reader is familiar with [**17**]. Moreover we shall quote the properties referring to M^* and to spaces related to M^* mainly from [**17**] rather than [**4**].

THEOREM 2.1 (Haefliger). *For an n-manifold M, there is a commutative diagram, in which each row is exact ($i \geqq 0$):*

$$\begin{array}{ccccccccc}
0 \to & H^{i-n}(M;\ Z_2) & \xrightarrow{\phi_1} & H^i(M^2;\ Z_2) & \longrightarrow & H^i(M^2-\Delta M;\ Z_2) & \to 0 & \\
& \uparrow r^* & & \uparrow q^* & & \uparrow p^* & & \\
0 \to & H^{i-n}(P^\infty \times M;\ Z_2) & \xrightarrow{\phi} & H^i(\Gamma M;\ Z_2) & \xrightarrow{\rho} & H^i(M^*;\ Z_2) & \to 0 & (\Gamma M = S^\infty \times_{Z_2} M^2) \\
& \| & & \downarrow k^* & & \downarrow j^* & & \\
0 \to & H^{i-n}(P^\infty \times M;\ Z_2) & \xrightarrow{\phi_2} & H^i(P^\infty \times M;\ Z_2) & \to & H^i(PM;\ Z_2) & \to 0. &
\end{array}$$

Here, the maps r, q, k, j are inclusions and p is a projection. ϕ_1, ϕ_2 can be thought of as Gysin maps. ρ is an $H^*(P^\infty; Z_2)$-module homomorphism defined by

$$\rho = p'^{*-1} i'^* \tag{2.2}$$

where

$$p' : S^\infty \times_{Z_2} (M^2 - \Delta M) \longrightarrow M^* \quad \text{and} \quad i' : S^\infty \times_{Z_2} (M^2 - \Delta M) \longrightarrow S^\infty \times_{Z_2} M^2 = \Gamma M$$

are the natural projection and inclusion, respectively. r^* is the obvious projection. Moreover, the morphisms ϕ_1, ϕ_2, q^*, k^* and the mod 2 cohomology of ΓM are given in [**17**, §2]. To determine $\operatorname{Ker} \rho = \operatorname{Im} \phi$, consider the certain operations Q^i ($i \geqq 0$) introduced by Yo [**21**, p. 1481].

PROPOSITION 2.3 (Yo). *For an n-manifold M, there exist operations*

$$Q^i : H^q(M; Z_2) \longrightarrow H^{q+i}(M; Z_2) \qquad \text{for} \quad i \geqq 0,$$

with the following properties:

(i) *if* $x \in H^q(M; Z_2)$, *then* $Q^i x = 0$ *for* $i > (n-q)/2$,

(ii) Q^0 *is equal to the identity,*

(iii) *for any* $z \in H_*(M; Z_2)$ *and* $x \in H^*(M; Z_2)$, $\langle Sq^i x, z \rangle = \langle x Q^i(\text{P.D.} z), [M] \rangle$, *where* $\langle , \rangle$ *denotes the Kronecker pairing,* P.D. *is the Poincaré duality and* $[M] \in H_n(M; Z_2)$ *is the generator.*

These operations Q^i ($i \geqq 0$) are related to the squaring operations Sq^j ($j \geqq 0$) and the Stiefel-Whitney classes $w_k(M)$ of M ($k \geqq 0$) by the equation ([**21**, Corollary 4, p. 1485])

$$\sum_{i+j=k} Sq^i Q^j(x) = x w_k, \qquad (w_k = w_k(M)). \tag{2.4}$$

(From now on, w_k will stand for $w_k(M)$.) From (2.4), it follows that

$$\begin{aligned} &Q^1 = Sq^1 + w_1, && Q^2 = Sq^2 + w_1^2 + w_2 + w_1 Sq^1, \\ &Q^1 = 0 \text{ on } H^{n-1}(M; Z_2), && Q^2 = 0 \text{ on } H^i(M; Z_2) \text{ for } i \geqq n-3. \end{aligned} \tag{2.5}$$

PROPOSITION 2.6. *Let* $x \in H^r(M; Z_2)$ *and let* $U \in H^n(M^2; Z_2)$ *is the* mod 2 *Thom class of M. Then the following relations hold:*

(1) $$\phi(1 \otimes x) = \sum_{i=0}^{[(n-r-1)/2]} u^{n-r-2i} \otimes (Q^i x)^2 + U(1 \otimes x),$$

$$\begin{aligned} &U(1 \otimes x) \in I^* \ (= (1+t^*) H^*(M^2; Z_2)) && \text{if} \quad n-r \equiv 1\ (2), \\ &U(1 \otimes x) + 1 \otimes (Q^{(n-r)/2} x)^2 \in I^* && \text{if} \quad n-r \equiv 0\ (2); \end{aligned}$$

(2) $$\phi(u^j \otimes x) = \sum_{i=0}^{[(n-r)/2]} u^{j+n-r-2i} \otimes (Q^i x)^2 \qquad \text{for} \quad j > 0.$$

PROOF. By [**17**, Theorem 2.4], $\phi(u^j \otimes x)$ is of the form

$$\phi(u^j \otimes x) = X_1 + X_2, \qquad X_1 \in H^*(P^\infty;\ Z_2) \otimes \bar{K}^*, \qquad X_2 \in I^*,$$

where $\bar{K}^* = (\mathrm{Ker}\,(1+t^*))/I^*$. From (2.4) and [**17**, Proposition 2.5 (iii)] it follows that

$$k^*\Big(\sum_{i=0}^{[(n-r)/2]} u^{j+n-r-2i} \otimes (Q^i x)^2\Big) = \sum_{m \geq 0} u^{j+n-m} \otimes x w_m,$$

and from the fact that $k^*\phi = \phi_2$ and [**17**, Proposition 2.5 (v) and (2.3)], it follows that

$$k^*(X_1) = \sum_{m \geq 0} u^{j+n-m} \otimes x w_m.$$

Hence we have

$$X_1 = \sum_{i=0}^{[(n-r)/2]} u^{j+n-r-2i} \otimes (Q^i x)^2,$$

since $k^* | (H^*(P^\infty;\ Z_2) \otimes \bar{K}^*)$ is injective by [**17**, Proposition 2.5]. If $j > 0$, then $\phi_1 r^*(u^j \otimes x) = 0$ and so $X_2 = 0$ follows from [**17**, Proposition 2.5]. On the other hand, if $j = 0$, then

$$\phi_1 r^*(1 \otimes x) = U(1 \otimes x) \qquad \text{in} \quad H^*(M^2;\ Z_2)$$

by [**17**, (2.2)] and therefore

$$U(1 \otimes x) = \begin{cases} X_2 & \text{for} \quad n - r \equiv 1\ (2), \\ X_2 + 1 \otimes (Q^{(n-r)/2} x)^2 & \text{for} \quad n - r \equiv 0\ (2). \end{cases}$$

This completes the proof.

As a corollary to the proposition, the property of [**17**, (7.2)] holds and hence we have

PROPOSITION 2.7 (Thomas). *Let B^k be the Z_2-vector subspace of $H^k(\Gamma M;\ Z_2)$ generated by all the elements of the form $u^j \otimes x^2$ with $2(\dim x) + j = k$ and $j + \dim x < n = \dim M$. Then*

(1) $\mathrm{Ker}\, j^* = \rho(I^*)$,

(2) $j^*\rho : B^* \to \mathrm{Im}\, j^*$ *is an isomorphism,*

(3) $\rho : B^* + I^* \to H^*(M^*;\ Z_2)$ *is an isomorphism,*

(4) $v\rho(I^*) = 0$.

In consequence of (2.5), Propositions 2.6 and 2.7, we have

LEMMA 2.8. *Let $x_r \in H^r(M;\ Z_2)$. Then the following relations hold:*

(1) $\rho(u \otimes (x_{n-1})^2) = \rho(U(1 \otimes x_{n-1}))$,

(2) $\rho(u^3 \otimes (x_{n-2})^2) = \rho(U(1 \otimes (Sq^1 + w_1) x_{n-2}))$,

(3) $\rho(u^2 \otimes (x_{n-2})^2) = \rho(U(1 \otimes x_{n-2}))$,

(4) $\rho(u^3 \otimes (x_{n-3})^2) = \rho(u \otimes ((Sq^1 + w_1)x_{n-3})^2 + U(1 \otimes x_{n-3}))$.

Lastly, we study the actions of Sq^i ($i=1, 2$) on $H^*(M^*; Z_2)$.

LEMMA 2.9. *Let $x_r \in H^r(M; Z_2)$. Then the following relations hold:*

(1) $Sq^1 \rho(1 \otimes (x_{n-1})^2) = \rho(Sq^1 x_{n-1} \otimes x_{n-1} + x_{n-1} \otimes Sq^1 x_{n-1})$ *if* $n \equiv 1\ (2)$,

(2) $Sq^1 \rho(u \otimes (x_{n-2})^2) = (n-1)\rho(U(1 \otimes x_{n-2}))$,

(3) $$Sq^2 \rho(u \otimes (x_{n-2})^2) = \begin{cases} \rho(U(1 \otimes w_1 x_{n-2})) & \text{if } n \equiv 0, 3\ (4), \\ \rho(U(1 \otimes Sq^1 x_{n-2})) & \text{if } n \equiv 1, 2\ (4), \end{cases}$$

(4) $Sq^1 \rho(1 \otimes (x_{n-2})^2) = n\rho(u \otimes (x_{n-2})^2) + \rho(Sq^1 x_{n-2} \otimes x_{n-2} + x_{n-2} \otimes Sq^1 x_{n-2})$,

(5) $Sq^1 \rho(u^2 \otimes (x_{n-3})^2) = (n-1)\rho(u \otimes ((Sq^1 + w_1)x_{n-3})^2 + U(1 \otimes x_{n-3}))$.

PROOF. The relations (1)-(5) are seen to be immediate consequences of Lemma 2.8 and the following fact, which is shown by Bausum [**1**, Lemmas 11 and 24] : if $x \in H^r(M; Z_2)$, then

$$Sq^1(u^i \otimes x^2) = \begin{cases} (i+r)u^{i+1} \otimes x^2 & \text{for } i>0, \\ ru \otimes x^2 + Sq^1 x \otimes x + x \otimes Sq^1 x & \text{for } i=0, \end{cases}$$

$$Sq^2(u^i \otimes x^2) = \begin{cases} \binom{i+r}{2} u^{i+2} \otimes x^2 + u^i \otimes (Sq^1 x)^2 & \text{for } i>0, \\ \binom{r}{2} u^2 \otimes x^2 + 1 \otimes (Sq^1 x)^2 + Sq^2 x \otimes x + x \otimes Sq^2 x & \text{for } i=0. \end{cases}$$

REMARK. This lemma, essentially, overlaps with the results of Bausum [**1**]. But his relations are not in $\rho H^*(\Gamma M; Z_2) = H^*(M^*; Z_2)$ and we would like to describe the actions of Sq^i in terms of the elements in $\rho H^*(\Gamma M; Z_2)$.

§ 3. On the morphism $i^*: H^*(A^2 M, \Delta M; Z_2) \to H^*(M^*; Z_2)$.

By [8, Theorem 11], as an algebra over $Z_2[v]$, $H^*(A^2 M, \Delta M; Z_2)$ is generated by Λx for all $x \in H^k(M; Z_2)$ ($k>0$), where Λx satisfies the condition

$$\pi^*(\Lambda x) = x \otimes 1 + 1 \otimes x \quad \text{in } H^*(M^2, \Delta M; Z_2). \tag{3.1}$$

The purpose of this section is to determine the morphism $i^*: H^*(A^2 M, \Delta M; Z_2) \to H^*(M^*; Z_2)$. Since

$$i^*(v^i \Lambda x) = 0 \quad \text{for } i>0 \tag{3.2}$$

by Lemma 1.3 and (1.5), we prove the following

LEMMA 3.3. *Let M be an n-manifold.*

(1) $i^*(\Lambda x)=\begin{cases} \rho(x\otimes 1+1\otimes x) & \text{if } \dim x<n, \\ \rho(x\otimes 1+1\otimes x+\lambda U) \quad (\lambda=0 \text{ or } 1) & \text{if } \dim x=n. \end{cases}$

(2) *If* $\dim x+\dim y<2n$, *then*

$$i^*(\Lambda x \Lambda y)=\rho(x\otimes y+y\otimes x+xy\otimes 1+1\otimes xy).$$

PROOF. Let

$$q' : \Gamma M=S^\infty\times_{Z_2}M^2 \longrightarrow \Lambda^2 M=M^2/Z_2\subset(\Lambda^2 M, \Delta M)$$

be the composition of the natural projection and inclusion. Then the following diagram is commutative:

(3.4)
$$\begin{array}{ccccccc}
(M^2, \Delta M) & \supset & M^2 & & \supset & & M^2-\Delta M \\
\downarrow \pi & & \downarrow q & & & \supset & \downarrow p \\
 & \overset{q'}{\swarrow} & \Gamma M & \overset{i'}{\longleftarrow} & S^\infty\times_{Z_2}(M^2-\Delta M) & \overset{p'}{\searrow} & \\
(\Lambda^2 M, \Delta M) & & & \overset{i}{\longleftarrow} & & & M^*.
\end{array}$$

Since $\rho=p'^{*-1}i'^*$ by (2.2), it follows that

$$i^*=\rho q'^*.$$

Moreover it follows that

(3.5)
$$\rho q'^*(\Lambda x)\in\rho(I^*),$$

because $\mathrm{Im}\, i^*=\mathrm{Ker}\, j^*=\rho(I^*)$ by Lemma 1.3 and Proposition 2.7. If $k<n$, then $\rho : H^k(\Gamma M; Z_2)\to H^k(M^*; Z_2)$ is an isomorphism by Theorem 2.1, and so

$$q'^*(\Lambda x)\in I^* \qquad \text{if} \quad \dim x<n.$$

Now, the relation $q^*q'^*(\Lambda x)=q^*(x\otimes 1+1\otimes x)$ holds by (3.1), (3.4) and [**17**, Proposition 2.5] and q^* is an injection on I^* by [**17**]. Therefore we have

$$q'^*(\Lambda x)=x\otimes 1+1\otimes x \qquad \text{if} \quad \dim x<n$$

and so

$$i^*(\Lambda x)=\rho q'^*(\Lambda x)=\rho(x\otimes 1+1\otimes x) \qquad \text{if} \quad \dim x<n.$$

If $x\in H^n(M; Z_2)$ is the generator, then there is an element $z\in I^*$ such that $\rho q'^*(\Lambda x)=\rho z$ by (3.5). Since $\mathrm{Ker}\,\rho=\mathrm{Im}\,\phi$ and $\phi H^0(P^\infty\times M; Z_2)=Z_2$ generated by $\phi(1\otimes 1)$, $q'^*(\Lambda x)$ is of the form

$$q'^*(\Lambda x)=z+\lambda\phi(1\otimes 1) \qquad (\lambda=0 \text{ or } 1).$$

Hence

$$x\otimes 1+1\otimes x=q^*q'^*(\Lambda x)=z+\lambda U \qquad \text{in} \quad H^n(M^2; Z_2)$$

by (3.1), (3.4) and [**17**], and so

$$z=x\otimes 1+1\otimes x+\lambda U \qquad (\lambda=0 \text{ or } 1).$$

Thus we have

$$i^*(\Lambda x)=\rho z=\rho(x\otimes 1+1\otimes x+\lambda U) \qquad (\lambda=0 \text{ or } 1)$$

and the proof of (1) is complete. The proof of (2) follows immediately from (1) and the relation $U(x\otimes 1)=U(1\otimes x)$ (cf. [**11**, Lemma 11.8]).

§4. Groups $H^{2n-2}(M^*; Z[v])$ and $\tilde{\rho}_2 H^{2n-3}(M^*; Z[v])$.

We begin this section by explaining notations.

$Z_r\langle a\rangle$ denotes the cyclic group of order r $(r\leqq\infty)$ generated by a.

For $v\in H^1(X; Z_2)$, $Z_r[v]$ denotes the sheaf of coefficients over X, locally isomorphic to Z_r, twisted by v $(r\leqq\infty)$, and

$$\tilde{\rho}_r : H^i(X; Z_s[v]) \longrightarrow H^i(X; Z_r[v]) \qquad (s\equiv 0\,(r) \text{ or } s=\infty)$$

and

$$\tilde{\beta}_r : H^{i-1}(X; Z_r[v]) \longrightarrow H^i(X; Z[v]) \qquad (r<\infty)$$

denote the reduction mod r and the Bockstein operator, respectively, twisted by v. Then $\tilde{\rho}_r$ and $\tilde{\beta}_r$ for $v=0$ are the ordinary ones ρ_r and β_r, respectively. Moreover, the following relations are well-known ([**3**] and [**14**]):

$$\rho_2\beta_2=Sq^1, \qquad \tilde{\rho}_2\tilde{\beta}_2=Sq^1+v. \tag{4.1}$$

Assume that the integral cohomology groups $H^i(M; Z)$ for an n-manifold M are of the form

$$H^n(M; Z)=\begin{cases} Z\langle M\rangle & (\rho_2 M=M) \quad \text{if } M \text{ is orientable,} \\ Z_2\langle\beta_2 M'\rangle & (Sq^1 M'=M) \quad \text{if } M \text{ is unorientable,}\end{cases}$$

(4.2)

$$H^m(M; Z)=\sum_{i=1}^{\gamma(m)} Z_{r(m,i)}\langle x_{m,i}\rangle \quad \text{(direct sum)} \qquad \text{for } m\leqq n-1,$$

$$x_{m,i}=\beta_{r(m,i)}y_{m,i} \quad (y_{m,i}\in H^{m-1}(M; Z_{r(m,i)})) \qquad \text{for } \alpha(m)<i\leqq\gamma(m),$$

where the order $r(m, i)$ is infinite for $1\leqq i\leqq\alpha(m)$, a power of 2 for $\alpha(m)<i\leqq\beta(m)$ and a power of an odd prime for $\beta(m)<i\leqq\gamma(m)$, and if $\alpha(m)<i<j\leqq\gamma(m)$ then either $(r(m, i), r(m, j))=1$ or $r(m, i)\,|\,r(m, j)$ holds.

For brevity,

(4.2)′ denote $\alpha(m)$, $\beta(m)$, $\gamma(m)$, $r(m, i)$, $x_{m,i}$ and $y_{m,i}$ in (4.2), respectively, by

$$\alpha,\ \beta,\ \gamma,\ r(i),\ x_i \text{ and } y_i \qquad \text{when } m=n-1,$$

$$\alpha',\ \beta',\ \gamma',\ r'(i),\ x'_i \text{ and } y'_i \qquad \text{when } m=n-2.$$

Using the above notations and the symbols Λx and $\Delta(x, y)$ introduced in [**8**], we have the theorems, postponing the proofs till §§ 5-6.

THEOREM 4.3. *Let M be an n-manifold $(n\geqq 4)$ and assume that M is orientable or n is even. Then*

$$H^{2n-2}(M^*;\ Z[v])\cong(1+(-1)^n t^*)(H^{n-1}(M;\ Z)\otimes H^{n-1}(M;\ Z))$$

$$+\begin{cases}[H^{n-1}(M;\ Z)]'+H^{n-2}(M;\ Z) & if\ \ n\equiv 0\,(2),\ w_1=0,\\ [H^{n-1}(M;\ Z)]'+H^{n-2}(M;\ Z_2) & if\ \ n\equiv 0\,(2),\ w_1\neq 0,\\ H^{n-2}(M;\ Z_2) & if\ \ n\equiv 1\,(2),\ w_1=0,\end{cases}$$

$$\cong G_1+G_3+\begin{cases}G_0'+G_6 & if\ \ n\equiv 0\,(2),\ w_1=0,\\ G_0'+G_7 & if\ \ n\equiv 0\,(2),\ w_1\neq 0,\\ G_8 & if\ \ n\equiv 1\,(2),\ w_1=0,\end{cases}$$

where

$$G_0'=\sum_{1\leq i\leq\alpha}Z\langle(1/2)i^*\Delta(x_i,\ x_i)\rangle$$

$$+\sum_{\alpha<i\leq\beta}Z_{2r(i)}\langle(1/2)i^*\tilde{\beta}_{r(i)}\Delta(y_i,\ \rho_{r(i)}x_i)\rangle$$

$$+\sum_{\beta<i\leq\gamma}Z_{r(i)}\langle i^*\tilde{\beta}_{r(i)}\Delta(y_i,\ \rho_{r(i)}x_i)\rangle\quad(\cong[H^{n-1}(M;\ Z)]'),$$

$$G_1=\sum_{1\leq i<j\leq\alpha}Z\langle i^*\Delta(x_i,\ x_j)\rangle,$$

$$G_3=\Big(\sum_{1\leq i\leq\alpha<j\leq\gamma}+\sum_{\alpha<j<i\leq\gamma}\Big)Z_{r(j)}\langle i^*\tilde{\beta}_{r(j)}\Delta(y_j,\ \rho_{r(j)}x_i)\rangle,$$

$$G_6=\sum_{\alpha'<k\leq\gamma'}Z_{r'(k)}\langle i^*\tilde{\beta}_{r'(k)}\Delta(y_k',\ \rho_{r'(k)}M)\rangle+\sum_{1\leq k\leq\alpha'}Z\langle i^*\Delta(x_k',\ M)\rangle$$

$$(\cong H^{n-2}(M;\ Z)),$$

$$G_7=\{\tilde{\beta}_2\rho(x\otimes M'+M'\otimes x)\,|\,x\in H^{n-2}(M;\ Z_2)\}\quad(\cong H^{n-2}(M;\ Z_2)),$$

$$G_8=\{\tilde{\beta}_2\rho(u\otimes x^2)\,|\,x\in H^{n-2}(M;\ Z_2)\}\quad(\cong H^{n-2}(M;\ Z_2)),$$

and $t: M^2\to M^2$ is the map defined by interchanging factors. (The definition of G' for a group G is given in the introduction.)

THEOREM 4.4. *Under the same condition as in Theorem 4.3, there holds the following equation:*

$$\tilde{\rho}_2H^{2n-3}(M^*;\ Z[v])=\begin{cases}H_1+H_2+H_3+H_6 & if\ \ n\equiv 0\,(2),\ w_1=0,\\ H_1+H_2+H_3+H_4+H_6 & if\ \ n\equiv 0\,(2),\ w_1\neq 0,\\ H_2+H_3+H_5+H_7 & if\ \ n\equiv 1\,(2),\ w_1=0,\end{cases}$$

where

$$H_1=\{\rho\sigma(\rho_2 x\otimes M)\mid x\in H^{n-3}(M;\ Z)\},$$

$$H_2=\{\rho\sigma(\rho_2 x\otimes\rho_2 y)\mid x\in H^{n-2}(M;\ Z),\ y\in H^{n-1}(M;\ Z)\},$$

$$H_3=\sum_{\alpha<i<j\leq\beta} Z_2\langle\rho\sigma(\rho_2 x_i\otimes\rho_2 y_j)+(r(j)/r(i))\rho\sigma(\rho_2 y_i\otimes\rho_2 x_j)\rangle,$$

$$H_4=\sum_{\alpha'<k\leq\beta'} Z_2\langle\rho\sigma(\rho_2 y'_k\otimes M+Sq^1\rho_2 y'_k\otimes M')\rangle,$$

$$H_5=\sum_{\alpha<i\leq\beta} Z_2\langle\rho\sigma(\rho_2 y_i\otimes\rho_2 x_i)\rangle,$$

$$H_6=\{\rho(u\otimes x^2+\sigma(Sq^1x\otimes x))\mid x\in H^{n-2}(M;\ Z_2)\},$$

$$H_7=\{\rho(u\otimes(Sq^1x)^2+U(1\otimes x))\mid x\in H^{n-3}(M;\ Z_2)\},$$

and $\sigma=1+t^*$.

REMARK. In Theorems 4.3 and 4.4, G's and H's, except G_0, G_8, H_6 and H_7, are isomorphic, by i^*, to those in [**19**, §5].

§5. Proofs of Theorems 4.3 and 4.4 for even n.

Assume that n is even in this section and consider the exact sequence of Lemma 1.3 for $G[v]=Z[v]$, in which the twisted integral cohomology of $(\Lambda^2M, \Delta M)$ is given by [**19**, §5] (cf. [**8**]) and that of PM is given by Rigdon [**13**, §9] as follows:

(5.1) (Rigdon) If n is even, then there are isomorphisms

$$\theta\ :\ H^{n-1}(M;\ Z_2)\xrightarrow{\cong} H^{2n-2}(PM;\ Z[v]),\qquad \theta(x)=\tilde{\beta}_2(v^{n-2}x),$$

$$\theta'\ :\ H^{n-2}(M;\ Z_2)+H^n(M;\ Z_2)\xrightarrow{\cong} H^{2n-3}(PM;\ Z[v]),$$

$$\theta'(x,\ y)=\tilde{\beta}_2(v^{n-2}x+v^{n-4}y).$$

The morphism δ on $H^{2n-2}(PM;\ Z[v])$ has been studied in [**19**, Lemma 3.2] while δ on $H^{2n-3}(PM;\ Z[v])$ is given in the same way as before, i.e., by using (1.5), (5.1) and [**8**], as follows:

$$\delta\tilde{\beta}_2(v^{n-4}M)=\tilde{\beta}_2(v^{n-3}\Lambda M),$$

$$\delta\tilde{\beta}_2(v^{n-2}x)=\tilde{\beta}_2(v^{n-3}\Lambda Sq^2x)\qquad\text{for}\quad x\in H^{n-2}(M;\ Z_2).$$

Therefore, the exact sequence in Lemma 1.3 leads to the lemma.

LEMMA 5.2. *There are two exact sequences*

(1) $0\to H^{2n-2}(\Lambda^2M,\ \Delta M;\ Z[v])/\mathrm{Im}\,\delta\xrightarrow{i^*} H^{2n-2}(M^*;\ Z[v])\xrightarrow{j^*}(Z_2)^{\beta}\to 0,$

(2) $\cdots\to H^{2n-3}(\Lambda^2M,\ \Delta M;\ Z[v])\xrightarrow{i^*} H^{2n-3}(M^*;\ Z[v])\xrightarrow{j^*}(Z_2)^{\beta'+\beta-\alpha}\to 0,$

such that

(3) $\operatorname{Im}\delta = Z_2\langle\tilde{\beta}_2(v^{n-3}\Lambda M)\rangle$,

(4) $(Z_2)^{\beta} = \{\tilde{\beta}_2(v^{n-2}\rho_2 x) \mid x \in H^{n-1}(M;\, Z)\}$,

(5) $(Z_2)^{\beta'+\beta-\alpha} = \{\tilde{\beta}_2(v^{n-2}x + v^{n-4}Sq^2 x) \mid x \in H^{n-2}(M;\, Z_2)\}$.

LEMMA 5.3. *In the exact sequences in Lemma* 5.2, *the following properties hold:*

(1) *For any subgroup* Z_2 *of* $(Z_2)^{\beta}$ *in Lemma* 5.2 (1), *there does not exist a subgroup* Z_2 *of* $H^{2n-2}(M^*;\, Z[v])$ *such that* $j^* Z_2 = Z_2$.

(2) $j^*\tilde{\beta}_2\rho(1\otimes x^2) = \tilde{\beta}_2(v^{n-2}x + v^{n-4}Sq^2 x)$ *for* $x \in H^{n-2}(M;\, Z_2)$ *and so* j^* *in Lemma* 5.2 (2) *is a split epimorphism.*

PROOF. To prove (1), it is sufficient to show that $j^*\tilde{\beta}_2 H^{2n-3}(M^*;\, Z_2) = 0$. Now, it is shown, by Proposition 2.7, that

$$H^{2n-3}(M^*;\, Z_2) = \rho(B^{2n-3}) + \rho(I^{2n-3}),$$

$$B^{2n-3} = \{u\otimes x^2 \mid x \in H^{n-2}(M;\, Z_2)\}, \qquad j^*\rho(I^{2n-3}) = 0.$$

If $x \in H^{n-2}(M;\, Z_2)$, then

$$j^*\tilde{\beta}_2\rho(u\otimes x^2) = \tilde{\beta}_2\tilde{\rho}_2\tilde{\beta}_2(v^{n-2}x + v^{n-4}Sq^2 x) = 0$$

by (1.4) and (4.1), and hence (1) is established. On the other hand, (2) is obtained in the same way as above.

PROOF OF THEOREM 4.3 FOR EVEN n. Consider the group extension of the short exact sequence in Lemma 5.2 (1), in which the group $H^{2n-2}(\Lambda^2 M, \Delta M;\, Z[v])/\operatorname{Im}\delta$ is determined by Lemma 5.2 (3) and [**19**, Proposition 5.4]. By using the Gysin exact sequence of the double covering $M^2 - \Delta M \to M^*$ (cf. [**19**, (5.6)]), it is shown that

$$\tilde{\rho}_2\tilde{\beta}_{r(j)}\Delta(y_j,\, \rho_{r(j)}x_i) - \tilde{\rho}_2\Delta(x_j,\, x_i) \in \operatorname{Im}\delta \quad \text{for } 1 \leqq i \leqq \alpha < j \leqq \beta \text{ or } \alpha < j < i \leqq \beta.$$

Therefore, the $\tilde{\rho}_2 i^*$-images of the generators of $H^{2n-2}(\Lambda^2 M, \Delta M;\, Z[v])/\operatorname{Im}\delta$ are given, by using Lemma 3.3 and [**19**, Lemma 1.5], as follows:

$$\tilde{\rho}_2 i^*\Delta(x_i,\, x_i) = 0 \qquad \text{for } 1 \leqq i \leqq \alpha,$$

$$\tilde{\rho}_2 i^*\tilde{\beta}_{r(j)}\Delta(y_j,\, \rho_{r(j)}x_j) = 0 \qquad \text{for } \alpha < j \leqq \beta,$$

and the $\tilde{\rho}_2 i^*$-images of the other generators of $H^{2n-2}(\Lambda^2 M, \Delta M;\, Z[v])/\operatorname{Im}\delta$, mod odd torsion, are linearly independent in $\rho(I^{2n-2})$. Hence, from this result, Lemma 5.2 (1), (3) and Lemma 5.3 (1), Theorem 4.3 is established when n is even.

PROOF OF THEOREM 4.4 FOR EVEN n. By Lemmas 5.2 and 5.3, it is seen that

$$\tilde{\rho}_2 H^{2n-3}(M^*;\, Z[v]) = i^*\tilde{\rho}_2 H^{2n-3}(\Lambda^2 M, \Delta M;\, Z[v]) + \{\tilde{\rho}_2\tilde{\beta}_2\rho(1\otimes x^2) \mid x \in H^{n-2}(M;\, Z_2)\},$$

in which the relation

$$\tilde{\rho}_2\tilde{\beta}_2\rho(1\otimes x^2)=\rho(u\otimes x^2+\sigma(Sq^1x\otimes x)) \qquad \text{for} \quad x\in H^{n-2}(M;\ Z_2)$$

holds by Lemma 2.9 and (4.1). Therefore, Theorem 4.4 for even n is verified, since $i^*\tilde{\rho}_2H^{2n-3}(\Lambda^2M,\ \Delta M;\ Z[v])$ is determined by Lemma 3.3 and [**19**, Proposition 5.5].

§ 6. Proofs of Theorems 4.3 and 4.4 for odd n.

For an orientable n-manifold M, there is an exact sequence

$$(6.1) \qquad 0\to H^{i-n}(M;\ Z)\xrightarrow{\phi_1}H^i(M^2;\ Z)\xrightarrow{\tilde{i}^*}H^i(M^2-\Delta M;\ Z)\to 0,$$
$$(\tilde{i}:\ M^2-\Delta M\subset M^2),$$

where

$$(6.2) \qquad \phi_1(x)=U(1\otimes x) \qquad \text{for} \quad x\in H^{i-n}(M;Z)$$

and $U=\phi_1(1)$ is called the Thom class or the diagonal class of M, e.g., by Milnor [**11**]. Notice that

$$(6.3) \qquad t^*\phi_1(x)=(-1)^n\phi_1(x) \qquad \text{for} \quad x\in H^*(M;Z)$$

(e.g., [**16**, P. 305]) and that

$$(6.4) \quad U=\pm(M\otimes 1+(-1)^n 1\otimes M)$$
$$\bmod\Big(\sum_{i=1}^{n-1}H^{n-i}(M;Z)\otimes H^i(M;Z)+\sum_{i=1}^{n-2}H^{n-i}(M;Z)*H^{i+1}(M;Z)\Big).$$

In the rest of this section, assume that n is odd and that M is an orientable n-manifold.

LEMMA 6.5. *The odd torsion subgroup of $H^{2n-2}(M^*;Z[v])$ is equal to*

$$\Big(\sum_{\substack{1\le i\le\alpha\\ \beta<j\le\gamma}}+\sum_{\beta<j<i\le\gamma}\Big)Z_{r(j)}\langle\tilde{i}^*\tilde{\beta}_{r(j)}\Delta(y_j,\ \rho_{r(j)}x_i)\rangle\,.$$

PROOF. By the spectral sequence argument for $M^2-\Delta M\xrightarrow{p}M^*\xrightarrow{\xi}P^\infty$ (cf. [**15**, Theorem 2.9]), the odd torsion subgroup of $H^{2n-2}(M^*;Z[v])$ is isomorphic, by p^*, to that of $\{x\in H^{2n-2}(M^2-\Delta M;Z)\mid t^*x=-x\}$, which is easily shown to be equal to

$$\Big(\sum_{\substack{1\le i\le\alpha\\ \beta<j\le\gamma}}+\sum_{\beta<j<i\le\gamma}\Big)Z_{r(j)}\langle\tilde{i}^*(x_j\otimes x_i-x_i\otimes x_j)\rangle$$

by (6.1)-(6.4). Now, there hold the relations

$$p^*i^*\tilde{\beta}_{r(j)}\Delta(y_j,\ \rho_{r(j)}x_i)=\tilde{i}^*(x_j\otimes x_i-x_i\otimes x_j)$$
$$\text{for } 1\leqq i\leqq\alpha<j\leqq\gamma \text{ or } \alpha<j<i\leqq\gamma\,,$$

by [**19**, (5.9)] and the property of being $\beta_{r(j)}(y_j \otimes \rho_{r(j)} x_i) = x_j \otimes x_i$, and so the lemma is established.

Consider, next, the Bockstein exact sequence associated with $0 \to Z[v] \xrightarrow{\times 2} Z[v] \to Z_2 \to 0$. In Proposition 2.7, it is shown that

$$H^{2n-1}(M^*; Z_2) = \{\rho\sigma(M \otimes x)(=\rho(U(1 \otimes x))) \mid x \in H^{n-1}(M; Z_2)\},$$

$$H^{2n-2}(M^*; Z_2) = \{\rho(1 \otimes x^2) \mid x \in H^{n-1}(M; Z_2)\} + A_1,$$

$$A_1 = \{\rho(U(1 \otimes x)) \mid x \in H^{n-2}(M; Z_2)\} + \sum_{1 \leq j < i \leq \beta} Z_2 \langle \rho\sigma(\rho_2 x_j \otimes \rho_2 x_i) \rangle$$

since for $x \in H^{n-2}(M; Z_2)$,

$$U(1 \otimes x) \equiv 1 \otimes (Sq^1 x)^2 + \sigma(M \otimes x) \mod \sigma(H^{n-1}(M; Z_2) \otimes H^{n-1}(M; Z_2)),$$

by (2.5), Proposition 2.6 and (6.4). Moreover, it is easily verified by Proposition 2.7 (4), Lemmas 2.8 and 2.9 and (4.1) that

$$\tilde{\rho}_2 \tilde{\beta}_2 \rho(1 \otimes x^2) = \rho\sigma(M \otimes x) \qquad \text{for} \quad x \in H^{n-1}(M; Z_2),$$

$$\tilde{\rho}_2 \tilde{\beta}_2 \rho(u \otimes x^2) = \rho(U(1 \otimes x)) \qquad \text{for} \quad x \in H^{n-2}(M; Z_2),$$

$$\tilde{\rho}_2 \tilde{\beta}_2 \rho\sigma(\rho_2 x_j \otimes \rho_2 x_i) = 0 \qquad \text{for} \quad 1 \leqq j < i \leqq \beta.$$

Therefore, it follows that

$$\tilde{\rho}_2 H^{2n-2}(M^*; Z[v]) = A_1.$$

On the other hand, it is shown in the same way as in proving Lemma 6.5 that

$$p^* i^* \Delta(x_j, x_i) = \tilde{i}^*(x_j \otimes x_i - x_i \otimes x_j) \qquad \text{for} \quad 1 \leqq j < i \leqq \alpha,$$

$$p^* i^* \tilde{\beta}_{r(j)} \Delta(y_j, \rho_{r(j)} x_i) = \tilde{i}^*(x_j \otimes x_i - x_i \otimes x_j)$$

$$\text{for } 1 \leqq i \leqq \alpha < j \leqq \beta \text{ or } \alpha < j < i \leqq \beta,$$

and by the argument similar to that used in proving Theorem 4.3 for even n, it is shown that

$$\tilde{\rho}_2 i^* \Delta(x_j, x_i) = \rho\sigma(\rho_2 x_j \otimes \rho_2 x_i) \qquad \text{for} \quad 1 \leqq j < i \leqq \alpha,$$

$$\tilde{\rho}_2 i^* \tilde{\beta}_{r(j)} \Delta(y_j, \rho_{r(j)} x_i) = \rho\sigma(\rho_2 x_j \otimes \rho_2 x_i)$$

$$\text{for } 1 \leqq i \leqq \alpha < j \leqq \beta \text{ or } \alpha < j < i \leqq \beta.$$

Therefore, $i^* \Delta(x_j, x_i)$ is of infinite order for $1 \leqq j < i \leqq \alpha$ and $i^* \tilde{\beta}_{r(j)} \Delta(y_j, \rho_{r(j)} x_i)$ is of order $r(j)$ for $1 \leqq i \leqq \alpha < j \leqq \beta$ or $\alpha < j < i \leqq \beta$, since the same is true of $i^*(x_j \otimes x_i - x_i \otimes x_j)$ for $1 \leqq i < j \leqq \beta$ by (6.1)-(6.4). From the argument made above, we have the following lemma:

LEMMA 6.6. *There holds the following congruence* mod *odd torsion:*

$$H^{2n-2}(M^*;Z[v]) \equiv \sum_{1\le i<j\le\alpha} Z\langle i^*\Delta(x_i,\ x_j)\rangle$$

$$+\Big(\sum_{1\le i\le\alpha<j\le\beta}+\sum_{\alpha<j<i\le\beta}\Big)Z_{r(j)}\langle i^*\tilde{\beta}_{r(j)}\Delta(y_j,\ \rho_{r(j)}x_i)\rangle$$

$$+\{\tilde{\beta}_2\rho(u\otimes x^2)\,|\,x\in H^{n-2}(M;Z_2)\} \quad \text{mod } \textit{odd torsion}.$$

Thus Theorem 4.3 for odd n is given by Lemmas 6.5 and 6.6.

We continue to prove Theorem 4.4 for odd n. Lemma 6.6 at once leads to an isomorphism

$$\tilde{\beta}_2 : A_2 \xrightarrow{\cong} \tilde{\beta}_2 H^{2n-3}(M^*;Z_2),$$

(6.7) $$A_2 = \{\rho(u\otimes x^2)\,|\,x\in H^{n-2}(M;Z_2)\}$$

$$+\Big(\sum_{1\le i\le\alpha<j\le\beta}+\sum_{\alpha<j<i\le\beta}\Big)Z_2\langle\rho\sigma(\rho_2 y_j\otimes\rho_2 x_i)\rangle,$$

since

$$(r(j)/2)i^*\tilde{\beta}_{r(j)}\Delta(y_j,\ \rho_{r(j)}x_i) = \tilde{\beta}_2 i^*\tilde{\rho}_2\Delta(y_j,\ \rho_{r(j)}x_i)$$

$$= \tilde{\beta}_2\rho\sigma(\rho_2 y_j\otimes\rho_2 x_i)$$

by Lemma 3.3 and [**19**, Lemma 1.5]. On the other hand, by Proposition 2.7, $H^{2n-3}(M^*;Z_2)$ can be described as

$$H^{2n-3}(M^*;Z_2) = A_2 + A_3,$$

$$A_3 = \{\rho(U(1\otimes x)+u\otimes(Sq^1x)^2)\ |\ x\in H^{n-3}(M;Z_2)\}$$

$$+\{\rho\sigma(\rho_2 x\otimes\rho_2 y)\ |\ x\in H^{n-2}(M;Z),\ y\in H^{n-1}(M;Z)\}$$

$$+\sum_{\alpha<i<j\le\beta} Z_2\langle\rho\sigma(\rho_2 x_i\otimes\rho_2 y_j + (r(j)/r(i))(\rho_2 y_i\otimes\rho_2 x_j))\rangle$$

$$+\sum_{\alpha<i\le\beta} Z_2\langle\rho\sigma(\rho_2 y_i\otimes\rho_2 x_i)\rangle.$$

By Lemmas 2.8, 2.9 and 3.3 and [**19**, Proposition 5.5], it is easy to see that $A_3\subset\tilde{\rho}_2 H^{2n-3}(M^*;Z[v])$ and hence

$$A_3 = \tilde{\rho}_2 H^{2n-3}(M^*;Z[v])$$

by (6.7). This completes the proof of Theorem 4.4 for odd n.

§7. Coker Θ.

The purpose of this section is to study the condition for computing Coker Θ, where

$$\Theta = \Big(Sq^2+\binom{2n-1}{2}v^2\Big)\tilde{\rho}_2 :\ H^{2n-3}(M^*;Z[v]) \longrightarrow H^{2n-1}(M^*;Z_2).$$

Here $\tilde{\rho}_2 H^{2n-3}(M^*; Z[v])$ is given by Theorem 4.4, and $H^{2n-1}(M^*; Z_2)$ is given by Thomas [**17**, §2] as follows:

LEMMA 7.1 (Thomas). *There exists an isomorphism*

$$\psi : H^{n-1}(M; Z_2) \xrightarrow{\cong} H^{2n-1}(M^*; Z_2)$$

defined by

$$\psi(x) = \rho\sigma(x \otimes M) = \rho(x \otimes M + M \otimes x).$$

LEMMA 7.2. *Let M be an n-manifold ($n \geqq 4$) and assume that n is even or M is orientable. Then* $\mathrm{Im}\,\Theta$ *is a Z_2-vector space generated by the elements listed below:*

(1) $\rho\sigma(Sq^2\rho_2 x \otimes M)$ *for* $x \in H^{n-3}(M; Z)$ *if* $n \equiv 0\,(2)$,

(2) $\rho\sigma(Sq^2\rho_2 x \otimes \rho_2 y)$ *for* $x \in H^{n-2}(M; Z)$, $y \in H^{n-1}(M; Z)$,

(3) $\rho\sigma(\rho_2 x_i \otimes Sq^2\rho_2 y_j + (r(j)/r(i))(Sq^2\rho_2 y_i \otimes \rho_2 x_j)$ *for* $\alpha < i < j \leqq \beta$,

(4) $\rho\sigma(Sq^2\rho_2 y'_k \otimes M + Sq^2 Sq^1 \rho_2 y'_k \otimes M')$ *for* $\alpha' < k \leqq \beta'$ *if* $n \equiv 0\,(2)$ *and* $w_1 \neq 0$,

(5) $\rho\sigma(Sq^2\rho_2 y_i \otimes \rho_2 x_i)$ *for* $\alpha < i \leqq \beta$ *if* $n \equiv 1\,(2)$,

(6) $\begin{cases} \rho(\sigma(Sq^1 x \otimes Sq^2 x) + U(1 \otimes Sq^1 x)) & \text{for } x \in H^{n-2}(M; Z_2) \text{ if } n \equiv 0\,(4), \\ \rho(\sigma(Sq^1 x \otimes Sq^2 x) + U(1 \otimes w_1 x)) & \text{for } x \in H^{n-2}(M; Z_2) \text{ if } n \equiv 2\,(4). \end{cases}$

PROOF. On calculating $\left(Sq^2 + \binom{2n-1}{2} v^2\right) H_i$ with the help of Lemmas 2.8 and 2.9, the element in (i) in the lemma appears for $1 \leqq i \leqq 6$, while $Sq^2 H_7 = 0$ is verified by using Proposition 2.6 (1), Lemma 2.9 and the equation (cf. [**11**])

$$Sq^i U = U(1 \otimes w_i) \quad \text{in} \quad H^*(M^2; Z_2).$$

REMARK. The elements of (1) for odd n (not necessarily $w_1 = 0$) always belong to $\mathrm{Im}\,\Theta$ and so do those of (4) and (5) for odd n and $w_1 \neq 0$. For these elements belong to $Sq^2 i^* \tilde{\rho}_2 H^{2n-3}(\Lambda^2 M, \Delta M; Z[v])$ by Lemma 3.2 and [**19**, Proposition 5.5].

COROLLARY 7.3. *Assume that the integral cohomology groups $H^i(M; Z)$ for $n-3 \leqq i \leqq n$ are given as (4.2). Then, with the assumption of Lemma 7.2, the following information suffices to determine* $\mathrm{Coker}\,\Theta$:

(i) *The action of Sq^2 on $H^i(M; Z_2)$ for $i = n-2$, $n-3$,*

(ii) *The action of w_1 on $H^{n-2}(M; Z_2)$ for $n \equiv 2\,(4)$.*

PROOF. This is an immediate consequence of Lemmas 7.1 and 7.2, because the action of Sq^1 on $H^*(M; Z_2)$ is determined by the structure of the integral cohomology group of M.

REMARK. The action of Sq^2 on $H^i(M; Z_2)$ $(i=n-2, n-3)$ is given by (2.5) above as follows:

$$Sq^2 x = \begin{cases} (w_2+w_1^2)x & \text{for} \quad x \in H^{n-2}(M; Z_2), \\ (w_2+w_1^2+w_1 Sq^1)x & \text{for} \quad x \in H^{n-3}(M; Z_2). \end{cases}$$

COROLLARY 7.4. *Let M be an n-manifold $(n \geqq 4)$. If one of the conditions* (i), $\cdots$, (iv) *below is satisfied, then* $\mathrm{Coker}\,\Theta=0$:

(i) $Sq^2 \rho_2 H^{n-3}(M; Z) = H^{n-1}(M; Z_2)$,

(ii) $w_1=0$ *and* $Sq^2 \rho_2 H^{n-2}(M; Z) \neq 0$,

(iii) $w_1 \neq 0$, $Sq^2 H^{n-3}(M; Z_2) = H^{n-1}(M; Z_2)$

and either $w_1^3 = w_3$ *or* $Sq^2 \rho_2 H^{n-2}(M; Z) = 0$,

(iv) $n \equiv 2\,(4)$, $w_1 \neq 0$ *and* $w_1 \rho_2 H^{n-2}(M; Z) = H^{n-1}(M; Z_2)$.

PROOF. By Lemma 7.1, the conditions (i), $\cdots$, (iv) imply that Θ is surjective by the elements in (1), (2), (1) and (4), and (6) of Lemma 7.2, respectively.

REMARK. By Lemma 7.1 and the remark following Lemma 7.2, this corollary is valid for the case when $n \equiv 1\,(2)$ and $w_1 \neq 0$.

§8. Proofs of results in the introduction.

Let $n \geqq 6$ and let M be an n-manifold such that $n \equiv 0\,(2)$ or $w_1=0$. Assume, further, that there exists an embedding of M in Euclidean $(2n-1)$-space R^{2n-1}. As is stated in §1, there is a bijection, as a set,

$$[M \subset R^{2n-1}] \cong H^{2n-2}(M^*; Z[v]) \times \mathrm{Coker}\,\Theta.$$

Therefore, Theorem 1 follows from Theorem 4.3 and Corollary 7.3. If M is a spin manifold, then $w_1=w_2=0$ and so $Sq^2=0$ on $H^i(M; Z_2)$ for $i=n-2$ and $n-3$ by the remark following Corollary 7.3. Therefore by Lemmas 7.1 and 7.2, it is shown that

$$\mathrm{Im}\,\Theta = \begin{cases} \{\rho\sigma(Sq^1 x \otimes M) \mid x \in H^{n-2}(M; Z_2)\} & \text{if} \quad n \equiv 0\,(4), \\ 0 & \text{otherwise}, \end{cases}$$

and hence that

$$\mathrm{Coker}\,\Theta \cong \begin{cases} H^{n-1}(M; Z_2)/Sq^1 H^{n-2}(M; Z_2) & \text{if} \quad n \equiv 0\,(4), \\ H^{n-1}(M; Z_2) & \text{otherwise}. \end{cases}$$

This and Theorem 4.3 lead to Proposition 2. Propositions 3 and 4 are obtained from Theorem 4.3 and Corollary 7.4.

We conclude this paper with proving Proposition 5. The Dold manifold $P(m, n)$ of type (m, n) of dimension $m+2n$ is obtained from $S^m \times CP^n$ by identifying (x, z) with $(-x, \bar{z})$, S^m and CP^n being the m-sphere and the complex projective space of complex dimension n, respectively. In particular, $P(m, 0)$ and $P(0, n)$ are the real and the complex projective spaces. Dold [2] has stated that

(8.1) $P(m, n)$ is given a cell decomposition with k-cell (C_i, D_j) for every pair (i, j) $(i, j \geqq 0)$, for which $i+2j=k \leqq m+2n$, and the boundary operator satisfies

$$\partial(C_i, D_j)=\begin{cases} (1+(-1)^{i+j})(C_{i-1}, D_j) & \text{for } i>0, \\ 0 & \text{for } i=0. \end{cases}$$

Let C^iD^j denote the cochain which assigns 1 to (C_i, D_j) and 0 to all the other $(i+2j)$-cells or its integral cohomology class if it is a cocycle, and let c^id^j denote the mod 2 cohomology class defined by C^iD^j. Then, in [2], it is shown that

$$\begin{aligned} &H^*(P(m, n); Z_2)=Z_2[c]/(c^{m+1})\otimes Z_2[d]/(d^{n+1}), \\ &Sq^1d=cd, \quad w_1(P(m, n))=(m+n+1)c, \end{aligned} \tag{8.2}$$

where $c=c^1d^0$ and $d=c^0d^1$.

The integral cohomology group of $P(m, n)$ is determined, by using (8.1), as follows:

(8.3) If $m, n>0$, then

$$H^{m+2n-1}(P(m, n); Z)=\begin{cases} Z_2\langle\beta_2(c^{m-2}d^n)\rangle & \text{for } m+n\equiv 1\,(2),\ m\geqq 2, \\ Z\langle C^{m-1}D^n\rangle & \text{for } m+n\equiv 1\,(2),\ m=1, \\ 0 & \text{for } m+n\equiv 0\,(2); \end{cases}$$

$$H^{m+2n-2}(P(m, n); Z)=\begin{cases} Z_2\langle\beta_2(c^{m-1}d^{n-1})\rangle & \text{for } m+n\equiv 1\,(2), \\ Z\langle C^mD^{n-1}\rangle+Z_2\langle\beta_2(c^{m-3}d^n)\rangle & \text{for } m+n\equiv 0\,(2),\ m\geqq 3, \\ Z\langle C^mD^{n-1}\rangle+Z\langle C^{m-2}D^n\rangle & \text{for } m+n\equiv 0\,(2),\ m=2, \\ Z\langle C^mD^{n-1}\rangle & \text{for } m+n\equiv 0\,(2),\ m=1; \end{cases}$$

$$\rho_2H^{m+2n-3}(P(m, n); Z)=\begin{cases} Z_2\langle c^{m-3}d^n\rangle & \text{for } m+n\equiv 1\,(2),\ m\geqq 3, \\ 0 & \text{for } m+n\equiv 1\,(2),\ m=1, 2 \\ Z_2\langle c^{m-1}d^{n-1}\rangle & \text{for } m+n\equiv 0\,(2). \end{cases}$$

From Theorem 4.4 and (8.3), we see easily that, under the condition

(*) $m \geqq 1$, $n \geqq 1$ and either $m+n \equiv 1\,(2)$ or $m \equiv 0\,(2)$,

there hold the relations

$$(8.4)\quad \tilde{\rho}_2 H^{2m+4n-3}(P(m,n)^*; Z[v])$$
$$= Z_2\langle \rho\sigma(c^{m-1}d^n \otimes c^m d^{n-1})\rangle + Z_2\langle \rho(u \otimes (c^m d^{n-1})^2)\rangle$$
$$+ Z_2\langle \rho(u \otimes (c^{m-2}d^n)^2 + \sigma(c^{m-1}d^n \otimes c^{m-2}d^n))\rangle$$
$$+ \begin{cases} Z_2\langle \rho\sigma(c^{m-3}d^n \otimes c^m d^n)\rangle & \text{for } m \equiv 0\,(2),\ m \geqq 4,\ n \equiv 1\,(2), \\ 0 & \text{for } m=2,\ n \equiv 1\,(2), \end{cases}$$
$$\ni \rho\sigma(c^{m-1}d^{n-1} \otimes c^m d^n) \quad \text{for } m \equiv 0\,(2),\ n \equiv 0\,(2),$$
$$\ni \rho\sigma(c^{m-1}d^n \otimes c^m d^{n-1}) \quad \text{for } m \equiv 1\,(2),\ n \equiv 0\,(2).$$

By Lemmas 2.8, 2.9 and 7.1, (8.2) and (8.4), it is easy to see that if the condition (*) is satisfied then

$$(8.5)\qquad \operatorname{Coker}\Theta = \begin{cases} Z_2 & \text{for } n \equiv 3\,(4) \text{ and either } m \equiv 0\,(4) \text{ or } m=2, \\ 0 & \text{otherwise}. \end{cases}$$

Therefore, Proposition 5 follows from Theorems 1 and 4.3, (8.3) and (8.5).

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