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ON CONDITIONS THAT A MAP IS COBORDANT TO AN EMBEDDING

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Abstract

In this paper, we try to get simple expressions of conditions that a map $f : M^n \to N^{2n-k}$ between closed manifolds $(3 \le k \le 8)$ is cobordant to an embedding in the sense of Stong, while using the theorem of Brown along the lines of Aguilar and Pastor.

1. Introduction and Statement of Results

Throughout this article, n-manifolds mean compact differentiable manifolds of dimension n. The (co)-homology is understood to have Z_2 for coefficients.

For a map $f : M^n \to N^{2n-k}$ between compact manifolds without boundary, let $w_i(f)$ be the i-th Stiefel-Whitney class of f and $f_! : H^i(M) \to H^{i+n-k}(N)$ the transfer homomorphism (or Umkehr homomorphism) of f. Further, let

$$\theta(f) = f^* f_!(1) - w_{n-k}(f).$$

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Then by [6, Lemma 2], $M \times \theta(f)$ is the $H^n(M) \times H^{n-k}(M)$-component of $U_M(1 \times w_{n-k}(f)) + (f \times f)^* U_N$, where $U_V \in H^{\dim V}(V \times V)$ denotes the Z_2-Thom class (or the Z_2-diagonal class) of a manifold V. Therefore, A. Heafliger [5, Theorem 5.2] implies that

Theorem (Heafliger). *If f is homotopic to an embedding, then*

$$\theta(f) = 0 \quad \text{and} \quad w_{n-i}(f) = 0 \quad \text{for } i < k. \tag{1.1}$$

The inverse of this theorem may be hard to study. So we will study whether f is cobordant to an embedding in the sense of Stong [9] if the condition (1.1) in the above theorem is satisfied. Here a map $f_1 : M_1^n \to N_1^{n+k}$ is said to be *cobordant* to $f_2 : M_2^n \to N_2^{n+k}$ if there exist two cobordisms (W, M_1^n, M_2^n), $(V, N_1^{n+k}, N_2^{n+k})$ and a map $F : W \to V$ such that $F | M_i = f_i$ $(i = 1, 2)$. Aguilar and Pastor [1] determined the necessary and sufficient condition that a map $f : M^n \to N^{2n-k}$, $(k = 1, 2)$ is cobordant to an embedding. Hence we consider cases when $k \geq 3$.

Theorem 1.1. *Let $n \geq 7$. A map $f : M^n \to N^{2n-3}$ is cobordant to an embedding if the following conditions are satisfied:*

$$w_1(M) w_{n-1}(f) = 0, \quad w_2(M) w_{n-2}(f) = 0, \quad \text{and} \quad \theta(f) = 0.$$

Theorem 1.2. *Let $n \geq 9$. A map $f : M^n \to N^{2n-4}$ is cobordant to an embedding if the following conditions are satisfied:*

$$w_1(M) w_{n-1}(f) = 0, \quad w_2(M) w_{n-2}(f) = 0,$$

$$w_1^3(M) w_{n-3}(f) = w_2(M) w_1(M) w_{n-3}(f) = w_3(M) w_{n-3}(f) = 0,$$

$$(n-1) w_1(M) f^*(w_1^2(N)) w_{n-3}(f) = 0,$$

and

$$\theta(f) = 0.$$

As a consequence of Theorems 1.1 and 1.2, we have the following:

Corollary 1.3. *Let* $f : M^n \to N^{2n-k}$, $(k = 3, 4)$ *be a map. If* $w_{n-i}(f) = 0$ *for* $0 < i < k$ *and* $\theta(f) = 0$, *then* f *is cobordant to an embedding.*

This article is organized as follows: In Section 2, we recall the Stiefel-Whitney class $w(f)$ and the transfer homomorphism $f_!$ of a map $f : M^n \to N^{2n-k}$, and study relations among $f_!$, $w_i(f)$'s, and the Steenrod squaring operations Sq^j's. Theorems 1.1 and 1.2 are proved in Sections 3 and 4, respectively. Furthermore, the necessary and sufficient conditions that $f : M^n \to N^{2n-k}$, $(k = 3, 4)$ are cobordant to an embedding are given in Sections 3 and 4, respectively. In Section 5, we will give some sufficient conditions that a map $f : M^n \to N^{2n-k}$, $(k > 4)$ is cobordant to an embedding. Some examples are given in Section 6.

2. Preliminaries

For a manifold V, we denote by $w(V)$ and $\overline{w}(V) = w(V)^{-1}$ the total Stiefel-Whitney class and the total normal Stiefel-Whitney class of V, respectively. For a map $f : M^n \to N^{2n-k}$, the total Stiefel-Whitney class of f, $w(f) = \sum_{i \geq 0} w_i(f)$, is defined by the equation

$$w(f) = \overline{w}(M) f^*(w(N)), \tag{2.1}$$

and the transfer homomorphism $f_! : H^i(M) \to H^{i+n-k}(N)$ is defined by

$$f_!(x) = D_N f_*(x \cap [M]),$$

where D_N is the Poincaré duality and $[M] \in H_n(M)$ denotes the fundamental class of M. For $\mu = (i_1, i_2, \ldots, i_p)$, let $w_\mu(V) = w_{i_1}(V) w_{i_2}(V) \cdots w_{i_p}(V)$ and $|\mu| = \sum_{1 \leq j \leq p} i_j$. Then R. L. W. Brown's theorem [2, p. 247] implies that

Theorem (Brown). *Let* $n > 2k > 0$. *Then a map* $f : M^n \to N^{2n-k}$ *is cobordant to an embedding if and only if the following conditions* (1) *and*

(2) *are satisfied*:

(1) $\langle w_\mu(M)w_{n-i}(f), [M]\rangle = 0$ *for all* μ *and* i *with* $|\mu| = i < k$, *and*

(2) $\langle f^*(w_\lambda(N))w_\mu(M)f^*f_!(w_\nu(M)) - f^*(w_\lambda(N))w_\mu(M)w_\nu(M)w_{n-k}(f),$ $[M]\rangle = 0$ *for all* λ, μ *and* ν *with* $|\lambda| + |\mu| + |\nu| = k$.

We denote by $v(M) = \sum_{i\geq 0} v_i(M)$ the total Wu class of M. The following relations are well-known:

$$Sq(v(M)) = w(M), \tag{2.2}$$

$$Sq^i x_{n-i} = v_i(M)x_{n-i} \quad \text{for all } x_{n-i} \in H^{n-i}(M), \tag{2.3}$$

$$Sq^i w_j(\xi) = \sum_{0\leq t\leq i} \binom{j-i+t-1}{t} w_{i-t}(\xi)w_{j+t}(\xi). \tag{2.4}$$

In the following lemma, we list some relations among $f_!$, the Steenrod operations Sq^i and the Stiefel-Whitney classes, the first two of which are seen in, e.g., [3] (cf. [1]), while the last two follow from the definition of $f_!$, (cf. [2]):

Lemma 2.1. *For a map* $f : M^n \to N^{2n-k}$, *there are relations*

(1) $f_!(f^*(x)y) = xf_!(y)$ *for* $x \in H^*(N)$, $y \in H^*(M)$,

(2) $Sq\, f_!(x) = f_!(Sq(x)w(f))$,

(3) $\langle x\, f_!(y), [N]\rangle = \langle f^*(x)y, [M]\rangle$ *if* $\dim x + \dim y = n$,

(4) $\langle f^*(x)\, yf^*f_!(z), [M]\rangle = \langle f^*(x)\, zf^*f_!(y), [M]\rangle$ *if* $\dim x + \dim y + \dim z = k$. *In particular*, $\langle f^*(x)f^*f_!(y), [M]\rangle = \langle f^*(x)\, yf^*f_!(1), [M]\rangle$.

Further, we have the following:

Lemma 2.2. *Let* $f : M^n \to N^{2n-k}$ *be a map. Then*

(1) $f^*(x_i)w_{n-i}(f) = 0$ *for* $x_i \in H^i(N)$, $(0 \leq i < k)$.

(2) $f^*(y)xf^*f_!(x) = f^*(y)\sum_{t=0}^{i-1} Sq^t(x)w_{n-k+i-t}(f) + f^*(y)x^2 w_{n-k}(f)$

for $x \in H^i(M)$, $y \in H^{k-2i}(N)$, $(0 \le 2i \le k)$.

(3) *In particular,* $f^*(y)f^*f_!(1) - f^*(y)w_{n-k}(f) = 0$ *for* $y \in H^k(N)$.

Proof. One can see easily that

$$\begin{aligned}\langle f^*(x_i)w_{n-i}(f), [M]\rangle &= \langle x_i f_!(w_{n-i}(f)), [N]\rangle \quad \text{by Lemma 2.1(3)}\\ &= \langle x_i Sq^{n-i} f_!(1), [N]\rangle \quad \text{by Lemma 2.1(2)}\\ &= 0 \quad \text{because } n-i > n-k.\end{aligned}$$

Thus the first relation (1) is obtained. Let $x \in H^i(M)$. Then

$$\begin{aligned}&\langle f^*(y)xf^*f_!(x), [M]\rangle\\ &= \langle yf_!(x)f_!(x), [N]\rangle \quad \text{by Lemma 2.1(3)}\\ &= \langle ySq^{n-k+i}f_!(x), [N]\rangle\\ &= \left\langle y\sum_{t\ge 0} f_!(Sq^t(x)w_{n-k+i-t}(f)), [N]\right\rangle \quad \text{by Lemma 2.1(2)}\\ &= \left\langle f^*(y)\sum_{0\le t\le i} Sq^t(x)w_{n-k+i-t}(f), [M]\right\rangle \quad \text{by Lemma 2.1(3).}\end{aligned}$$

Thus we have the relation (2).

Lemma 2.3. *Let* $f : M^n \to N^{2n-k}$ *be a map. Then*

(1) $w_1(M)f^*(w_1(N))w_{n-2}(f) = nw_1(M)w_{n-1}(f)$ *if* $k \ge 3$,

(2) $w_1^2(M)w_{n-2}(f) = w_1(M)w_{n-1}(f)$ *if* $k \ge 3$,

(3) $w_1^2(M)f^*(w_1(N))w_{n-3}(f) = nw_1(M)w_{n-1}(f)$ *if* $k \ge 3$,

(4) $(w_1(M)f^*(w_2(N)) + w_1^3(M))w_{n-3}(f) = \left(1 + \binom{n}{2}\right)w_1(M)w_{n-1}(f)$ *if* $k \ge 3$,

(5) $w_2(M)\, f^*(w_1(N))\, w_{n-3}(f) = (w_3(M) + w_2(M)\, w_1(M))\, w_{n-3}(f) + nw_2(M)\, w_{n-2}(f)$ *if* $k \geq 4$.

Proof. We have

$$\begin{aligned}
&\overline{w}_2(M)\, w_{n-2}(f)\\
&= v_2(M)\, w_{n-2}(f) = Sq^2 w_{n-2}(f) \qquad \text{by (2.3)}\\
&= w_2(f)\, w_{n-2}(f) + (n-4)\, w_1(f)\, w_{n-1}(f) + \binom{n-3}{2} w_n(f) \qquad \text{by (2.4)}\\
&= \overline{w}_2(M)\, w_{n-2}(f) + w_1(M)\, f^*(w_1(N))\, w_{n-2}(f) + nw_1(M)\, w_{n-1}(f)\\
&\qquad\qquad \text{by (2.1) and Lemma 2.2(1).}
\end{aligned}$$

This leads the relation (1). The relation (2) follows from the relations below:

$$\begin{aligned}
w_1^2(M)\, w_{n-2}(f) &= v_1(M)\, w_1(M)\, w_{n-2}(f) = Sq^1(w_1(M)\, w_{n-2}(f)) \quad \text{by (2.3)}\\
&= w_1(M)\, f^*(w_1(N))\, w_{n-2}(f) + (n-3)\, w_1(M)\, w_{n-1}(f)\\
&\qquad\qquad \text{by (2.1) and (2.4)}\\
&= w_1(M)\, w_{n-1}(f) \qquad \text{by (1) of this lemma;}
\end{aligned}$$

while (3) follows from

$$\begin{aligned}
w_1^2(M)\, f^*(w_1(N))\, w_{n-3}(f) &= Sq^1(w_1(M)\, f^*(w_1(N))\, w_{n-3}(f))\\
&= nw_1(M)\, f^*(w_1(N))\, w_{n-2}(f)\\
&= nw_1(M)\, w_{n-1}(f).
\end{aligned}$$

By calculating $w_2(M)\, w_1(M)\, w_{n-3}(f) = Sq^3 w_{n-3}(f) = Sq^1 Sq^2 w_{n-3}(f)$, we get

$$\begin{aligned}
&w_2(M)\, w_1(M)\, w_{n-3}(f)\\
&= w_1(M)\,(\overline{w}_2(M) + w_1(M)\, f^*(w_1(N)) + f^*(w_2(N)))\, w_{n-3}(f)\\
&\quad + (n-5)\, w_1(M)\,(w_1(M) + f^*(w_1(N)))\, w_{n-2}(f) + \binom{n-4}{2} w_1(M)\, w_{n-1}(f).
\end{aligned}$$

Hence we have (4). In the same way as the above, we calculate

$$\begin{aligned} w_2(M)w_1(M)w_{n-3}(f) &= Sq^3 w_{n-3}(f) \\ &= w_3(f)w_{n-3}(f) + (n-6)w_2(f)w_{n-2}(f) \\ &\quad + \binom{n-5}{2} w_1(f)w_{n-1}(f) + \binom{n-4}{3} w_n(f). \end{aligned}$$

Then we obtain (5) easily.

3. Proof of Theorem 1.1

In this section, we assume that $n \geq 7$. For a map $f : M^n \to N^{2n-3}$, let

$$\theta(f) = f^* f_!(1) - w_{n-3}(f)$$

and let

$$\phi_1 = w_1(M)w_{n-1}(f), \qquad \phi_2 = w_2(M)w_{n-2}(f),$$

$$\psi_1 = w_3(M)\theta(f), \qquad \psi_2 = w_2(M)w_1(M)\theta(f),$$

$$\psi_3 = w_1^3(M)\theta(f), \qquad \psi_4 = w_1^2(M)f^*(w_1(N))\theta(f),$$

$$\psi_5 = w_2(M)f^*(w_1(N))\theta(f), \qquad \psi_6 = w_1(M)f^*(w_1^2(N))\theta(f),$$

$$\psi_7 = w_1(M)f^*(w_2(N))\theta(f),$$

$$\psi_8 = w_1(M)f^*f_!(w_1^2(M)) - w_1^3(M)w_{n-3}(f),$$

$$\psi_9 = w_2(M)f^*f_!(w_1(M)) - w_2(M)w_1(M)w_{n-3}(f),$$

$$\psi_{10} = w_1(M)f^*(w_1(N))f^*f_!(w_1(M)) - w_1^2(M)f^*(w_1(N))w_{n-3}(f).$$

By Brown's theorem, Lemma 2.1(4), Lemma 2.2(3) and Lemma 2.3(2), we see easily that

Assertion 1. *$f : M^n \to N^{2n-3}$, $(n \geq 7)$ is cobordant to an embedding if and only if $\phi_i = 0$, $(i = 1, 2)$ and $\psi_j = 0$, $(1 \leq j \leq 10)$.*

Thus, to prove Theorem 1.1, it is sufficient to show the first three of the following four relations below: For $f : M^n \to N^{2n-3}$,

$$\psi_{10} = n\phi_1, \tag{3.1}$$

$$\psi_8 = \psi_3 + \psi_4 + n\phi_1, \tag{3.2}$$

$$\psi_8 + \psi_9 = \psi_2 + \psi_7 + \left(1 + \binom{n}{2}\right)\phi_1, \tag{3.3}$$

$$\psi_9 = \psi_1 + \psi_5 + n\phi_2. \tag{3.4}$$

The relation (3.1) follows from Lemmas 2.2(2) and 2.3(1) immediately.

Proof of (3.2).

$$\begin{aligned}
\psi_8 &= Sq^1(f^*f_!(w_1^2(M)) - w_1^2(M)\,w_{n-3}(f)) \\
&= f^*f_!(w_1^2(M)\,w_1(f)) - w_1^2(M)\,(w_1(f)\,w_{n-3}(f) + (n-4)\,w_{n-2}(f)) \\
&\qquad\qquad \text{by (2.3)-(2.4) and Lemma 2.1(2)} \\
&= (w_1^3(M) + w_1^2(M)\,f^*(w_1(N)))\,f^*f_!(1) \\
&\quad - (w_1^3(M) + w_1^2(M)\,f^*(w_1(N)))\,w_{n-3}(f) \\
&\quad + (n-4)\,w_1^2(M)\,w_{n-2}(f) \qquad \text{by Lemma 2.1(1) and (4)} \\
&= \psi_3 + \psi_4 + n\phi_1 \qquad \text{by Lemma 2.3(2).}
\end{aligned}$$

Proof of (3.3).

$$\begin{aligned}
\psi_8 + \psi_9 &= Sq^2 f^*f_!(w_1(M)) + Sq^2(w_1(M)\,w_{n-3}(f)) \\
&= f^*f_!(w_1^2(M)\,w_1(f) + w_1(M)\,w_2(f)) \\
&\quad + (w_1^2(M)\,w_1(f) + w_1(M)\,w_2(f))\,w_{n-3}(f) + \left(1 + \binom{n}{2}\right) w_1(M)\,w_{n-1}(f) \\
&= (w_2(M)\,w_1(M) + w_1(M)\,f^*(w_2(N)))\,\theta(f) + \left(1 + \binom{n}{2}\right) w_1(M)\,w_{n-1}(f). \\
&= \psi_2 + \psi_7 + \left(1 + \binom{n}{2}\right)\phi_1 .
\end{aligned}$$

Proof of (3.4).

$$\begin{aligned}\psi_9 &= Sq^1(f^*f_!(w_2(M)) + w_2(M)w_{n-3}(f)) \\ &= f^*f_!(w_3(M) + w_2(M)w_1(M) + w_2(M)w_1(f)) \\ &\quad + (w_3(M) + w_2(M)w_1(M) + w_2(M)w_1(f))w_{n-3}(f) \\ &\quad + (n-4)w_2(M)w_{n-2}(f) \\ &= \psi_1 + \psi_5 + n\phi_2.\end{aligned}$$

By virtue of the relations (3.1)-(3.4) and Assertion 1, we have the following:

Proposition 3.1. *Let* $n \geq 7$. *Then* $f : M^n \to N^{2n-3}$ *is cobordant to an embedding if and only if* $\phi_i = 0$, $(i = 1, 2)$ *and* $\psi_j = 0$, $(1 \leq j \leq 6)$.

4. Proof of Theorem 1.2

In this section, we assume that $n \geq 9$. For a map $f : M^n \to N^{2n-4}$, let

$$\theta(f) = f^*f_!(1) - w_{n-4}(f)$$

and let

$$\begin{aligned}
\phi_1 &= w_1(M)w_{n-1}(f), & \phi_2 &= w_2(M)w_{n-2}(f), \\
\phi_3 &= w_3(M)w_{n-3}(f), & \phi_4 &= w_2(M)w_1(M)w_{n-3}(f), \\
\phi_5 &= w_1^3(M)w_{n-3}(f), & \psi_1 &= w_4(M)\theta(f), \\
\psi_2 &= w_3(M)w_1(M)\theta(f), & \psi_3 &= w_2^2(M)\theta(f), \\
\psi_4 &= w_2(M)w_1^2(M)\theta(f), & \psi_5 &= w_1^4(M)\theta(f), \\
\psi_6 &= w_3(M)f^*(w_1(N))\theta(f), & \psi_7 &= w_2(M)w_1(M)f^*(w_1(N))\theta(f), \\
\psi_8 &= w_1^3(M)f^*(w_1(N))\theta(f), & \psi_9 &= w_2(M)f^*(w_1^2(N))\theta(f),
\end{aligned}$$

$$\psi_{10} = w_2(M)\,f^*(w_2(N))\,\theta(f), \qquad \psi_{11} = w_1^2(M)\,f^*(w_1^2(N))\,\theta(f),$$

$$\psi_{12} = w_1^2(M)\,f^*(w_2(N))\,\theta(f), \qquad \psi_{13} = w_1(M)\,f^*(w_1^3(N))\,\theta(f),$$

$$\psi_{14} = w_1(M)\,f^*(w_2(N)\,w_1(N))\,\theta(f), \qquad \psi_{15} = w_1(M)\,f^*(w_3(N))\,\theta(f),$$

$$\psi_{16} = w_1(M)\,f^*f_!(w_3(M)) - w_1(M)\,w_3(M)\,w_{n-4}(f),$$

$$\psi_{17} = w_2(M)\,w_1(M)\,f^*f_!(w_1(M)) - w_2(M)\,w_1^2(M)\,w_{n-4}(f),$$

$$\psi_{18} = w_1(M)\,f^*f_!(w_1^3(M)) - w_1^4(M)\,w_{n-4}(f),$$

$$\psi_{19} = w_2(M)\,f^*f_!(w_2(M)) - w_2^2(M)\,w_{n-4}(f),$$

$$\psi_{20} = w_2(M)\,f^*f_!(w_1^2(M)) - w_2(M)\,w_1^2(M)\,w_{n-4}(f),$$

$$\psi_{21} = w_1^2(M)\,f^*f_!(w_1^2(M)) - w_1^4(M)\,w_{n-4}(f),$$

$$\psi_{22} = w_1(M)\,f^*(w_1(N))\,f^*f_!(w_2(M))$$

$$- w_1(M)\,w_2(M)\,f^*(w_1(N))\,w_{n-4}(f),$$

$$\psi_{23} = w_1(M)\,f^*(w_1(N))\,f^*f_!(w_1^2(M)) - w_1^3(M)\,f^*(w_1(N))\,w_{n-4}(f),$$

$$\psi_{24} = w_1(M)\,f^*(w_2(N))\,f^*f_!(w_1(M)) - w_1^2(M)\,f^*(w_2(N))\,w_{n-4}(f),$$

$$\psi_{25} = w_1(M)\,f^*(w_1^2(N))\,f^*f_!(w_1(M)) - w_1^2(M)\,f^*(w_1^2(N))\,w_{n-4}(f).$$

By Brown's theorem, Lemma 2.1(4) and Lemma 2.2(3), we have

Assertion 2. *A map $f : M^n \to N^{2n-4}$, $(n \geq 9)$, is cobordant to an embedding if and only if the relations $\phi_i = 0$, $(1 \leq i \leq 5)$ and $\psi_j = 0$, $(1 \leq j \leq 25)$.*

Let H be a subgroup of $H^n(M)$ generated by ϕ_i, $(1 \leq i \leq 5)$. Then, there are relations below:

$$\psi_{19} \equiv 0 \bmod H, \quad \psi_{21} \equiv 0 \bmod H, \quad \psi_{24} \equiv 0 \bmod H, \tag{4.1}$$

$$\psi_{20} \equiv \psi_4 + \psi_8 + \psi_{12} \bmod H, \quad \psi_{16} \equiv \psi_{22} \bmod H, \tag{4.2}$$

$$\psi_{16} \equiv \psi_6 \bmod H, \quad \psi_{18} \equiv \psi_8 \bmod H, \quad \psi_{23} \equiv \psi_8 \bmod H, \tag{4.3}$$

$$\psi_{25} = \psi_{13} + (n-1)w_1(M)f^*(w_1^2(N))w_{n-3}(f), \tag{4.4}$$

$$\psi_{17} \equiv \psi_2 + \psi_4 + \psi_7 + \psi_{15} \bmod H. \tag{4.5}$$

Therefore by Assertion 2, if $\phi_i = 0, (1 \le i \le 5)$, $\theta(f) = 0$ and $(n-1)w_1(M)f^*(w_1^2(N)) = 0$, then f is cobordant to an embedding. This proves Theorem 1.2.

For $f : M^n \to N^{2n-4}$, there exist some other relations.

$$\psi_{25} = w_1(M)f^*(w_1^2(N))w_{n-3}(f), \tag{4.6}$$

$$\psi_{13} = nw_1(M)f^*(w_1^2(N))w_{n-3}(f), \tag{4.7}$$

$$\psi_{15} \equiv 0 \bmod H, \tag{4.8}$$

$$\psi_2 + \psi_4 + \psi_6 + \psi_8 + \psi_{10} + \psi_{12} + \psi_{15} \equiv 0 \bmod H, \tag{4.9}$$

$$\psi_6 + \psi_7 + \psi_8 + \psi_{13} + \psi_{14} \equiv w_1(M)f^*(w_1^2(N))w_{n-3}(f) \bmod H. \tag{4.10}$$

Thus, we have the following:

Proposition 4.1. *A map $f : M^n \to N^{2n-4}$, $(n \ge 9)$, is cobordant to an embedding if and only if $\phi_i = 0, (1 \le i \le 5)$, $\psi_j = 0, (1 \le j \le 11)$ and $w_1(M)f^*(w_1^2(N))w_{n-3}(f) = 0$.*

In the rest of this section, we prove relations (4.1)-(4.10). For simplicity's sake, we write $w_i(M) = w_i$ and $\overline{w}_i(M) = \overline{w}_i$ in the proofs of (4.1)-(4.10).

The relations (4.1) and (4.6) follow immediately from Lemma 2.2(2) and Lemma 2.3; and (4.7) follows from (4.4) and (4.6).

The proof of (4.2) is given below:

$$\begin{aligned}\psi_{20} &= (w_2 + w_1^2)(f^*f_!(w_1^2) + w_1^2 w_{n-4}(f)) - w_1^2 f^* f_!(w_1^2) + w_1^4 w_{n-4}(f) \\ &= Sq^2(f^*f_!(w_1^2) + w_1^2 w_{n-4}(f)) - \phi_1 \qquad \text{by (2.3) and (4.1)} \\ &\equiv (w_2 w_1^2 + w_1^3 f^*(w_1(N)) + w_1^2 f^*(w_2(N)))\,\theta(f) \bmod H \\ &\qquad\qquad \text{by Lemma 2.1(1), (2) and (2.4)} \\ &\equiv \psi_4 + \psi_8 + \psi_{12} \bmod H \qquad \text{by Lemma 2.1;}\end{aligned}$$

$$\begin{aligned}\psi_{16} + \psi_{22} &= Sq^1(f^*f_!(w_3 + w_2 f^*(w_1(N))) + (w_3 + w_2 f^*(w_1(N)))\, w_{n-4}(f)) \\ &= Sq^1(f^*f_!(Sq^1 w_2 + w_2 w_1(f)) + (Sq^1 w_2 + w_2 w_1(f))\, w_{n-4}(f)) \\ &\equiv Sq^1(Sq^1 f^* f_!(w_2) + Sq^1(w_2 w_{n-4}(f))) \bmod H \\ &\equiv 0 \bmod H.\end{aligned}$$

We have (4.3) and (4.4) by calculating $Sq^1(f^*f_!(x_3) - x_3 w_{n-4}(f))$ for $x_3 \in H^3(M)$, while using Lemma 2.1 and (2.4). On the other hand, we get (4.5) by calculating $Sq^3(f^*f_!(w_1) - w_1 w_{n-4}(f))$. The relations (4.8)-(4.10) are obtained by the equations

$$w_1 f^*(w_3(N))\,\theta(f) = w_1(f^*f_!(Sq^1 w_2(N)) - f^*(w_2(N)\, w_1(N)))\,\theta(f),$$

$$v_4(M)\,\theta(f) = Sq^4\theta(f),$$

$$w_2 w_1 f^*(w_1(N))\,\theta(f) = Sq^3(f^*(w_1(N))\,\theta(f)).$$

5. A Generalization of Theorems

It may be difficult, though not impossible, and less valuable to give a similar description of the necessary and sufficient condition that a map $f : M^n \to N^{2n-k}$, $(k \geq 5)$, is cobordant to an embedding. For the description is expected to be complicated. So we add some assumptions on M or f to get simple sufficient conditions.

Theorem 5.1. *Let* $f : M^n \to N^{2n-k}$ *be a map and let* $k \leq 8$, $n > 2k > 0$. *If either* M^n *is orientable or* f *is orientable, i.e.,* $w_1(f) = 0$, *and if* $w_{n-i}(f) = 0$, $(0 < i < k)$ *and* $f^*f_!(1) - w_{n-k}(f) = 0$, *then* f *is cobordant to an embedding.*

By a tedious calculation, we can generalize this theorem as follows:

Theorem 5.1′. *Let* $n > 2k > 0$, *and let* $f : M^n \to N^{2n-k}$ *be a map. If* $w_i(M) \in f^*H^i(N)$, $(4i < k)$, $w_{n-i}(f) = 0$, $(i < k)$, *and* $f^*f_!(1) - w_{n-k}(f) = 0$, *then* f *is cobordant to an embedding.*

Proof of Theorem 5.1. We prove the theorem only when $w_1(M) = 0$ for $k = 8$. Other cases can be settled similarly. By Brown's theorem, Lemmas 2.1 and 2.2, and the assumption, it is sufficient to show that ψ_i, $(1 \leq i \leq 5)$ below vanish:

$$\psi_1 = x_6 f^*f_!(w_2) - x_6 w_2 w_{n-8}(f) \quad \text{for } x_6 \in H^6(M),$$

$$\psi_2 = x_5 f^*f_!(w_3) - x_5 w_3 w_{n-8}(f) \quad \text{for } x_5 \in H^5(M),$$

$$\psi_3 = w_4 f^*f_!(w_4) - w_4^2 w_{n-8}(f),$$

$$\psi_4 = w_2^2 f^*f_!(w_2^2) - w_2^4 w_{n-8}(f),$$

$$\psi_5 = w_4\, f^*f_!(w_2^2) - w_4 w_2^2 w_{n-8}(f).$$

As for ψ_1, we have

$$\begin{aligned}
\psi_1 &= w_2(f^*f_!(x_6) - x_6 w_{n-8}(f)) \\
&= Sq^2(f^*f_!(x_6) - x_6 w_{n-8}(f)) \\
&= f^*f_!(Sq^2 x_6 + Sq^1 x_6 w_1(f) + x_6 w_2(f)) \\
&\quad + (Sq^2 x_6 w_{n-8}(f) + Sq^1 x_6 Sq^1 w_{n-8}(f) + x_6 Sq^2 w_{n-8}(f)) \\
&= (Sq^2 x_6 + Sq^1 x_6 w_1(f) + x_6 w_2(f))(f^*f_!(1) - w_{n-8}(f)) = 0;
\end{aligned}$$

while

$$\begin{aligned}\psi_2 &= x_5(f^*f_!(Sq^1w_2) - Sq^1w_2w_{n-8}(f)) \\ &= x_5(Sq^1f^*f_!(w_2) + f^*(w_1(N)))(f^*f_!(w_2) \\ &\quad + Sq^1(w_2w_{n-8}(f)) + f^*(w_1(N))w_2w_{n-8}(f)) \\ &= (Sq^1x_5 + x_5f^*(w_1(N)))(f^*f_!(w_2) - w_2w_{n-8}(f)) = 0.\end{aligned}$$

The relation $\psi_3 = \psi_4 = 0$ follows from the assumption and Lemma 2.2(2).

$$\begin{aligned}\psi_4 + \psi_5 &= (w_4 + w_2^2)(f^*f_!(w_2^2) - w_2^2w_{n-8}(f)) \\ &= Sq^4(f^*f_!(w_2^2) - w_2^2w_{n-8}(f)) \\ &= f^*f_!\left(\sum_{0\le i\le 4} Sq^iw_2^2w_{4-i}(f)\right) + \sum_{0\le i\le 4} Sq^iw_2^2Sq^{4-i}w_{n-8}(f) \\ &= \left(\sum_{0\le i\le 4} Sq^iw_2^2w_{4-i}(f)\right)(f^*f_!(1) - w_{n-8}(f)) = 0.\end{aligned}$$

Thus we have $\psi_i = 0$, $(1 \le i \le 5)$.

Sketch of the proof of Theorem 5.1′. The proof is essentially similar to that of Theorem 5.1. The condition (1) of Brown's theorem is fulfilled by the assumption, while the condition (2)

$$f^*(w_\lambda(N))\,w_\mu(M)\,f^*f_!(w_\nu(M)) = f^*(w_\lambda(N))\,w_\mu(M)\,w_\nu(M)\,w_{n-k}(f)$$

for λ, μ, ν with $|\lambda| + |\mu| + |\nu| = k$ and $2|\mu| \le k$, is proved by induction on $|\mu|$. We omit the details.

6. Miscellaneous Remarks

In [6, Remark 2], we showed that

Remark 1. If n is odd $(n \ge 3)$, then there exists such a map $f : M^n \to N^{2n-1}$ that is not homotopic to an embedding but cobordant to an embedding.

In general, if $f : M^n \to N_1$ is homotopic to an embedding and $i : N_1 \to N_2$ is a natural inclusion, then the composite $if : M^n \to N_2$ is also homotopic to an embedding. However, it is impossible to replace "homotopic" with "cobordant". We will show this by giving an example.

We denote by P^k the real projective k-space and by $g : P^2 \to P^2/P^1 = S^2$ the natural projection. We define a 9-manifold M^9, a 14- and 15- manifold N_1^{14} and N_2^{15} by

$$M^9 = P^2 \times P^7, \quad N_1^{14} = S^3 \times P^{11}, \quad N_2^{15} = S^3 \times P^{12}.$$

Let

$$i_1 : S^2 \subset S^3, \quad i_2 : P^7 \subset P^{11}, \quad i : S^3 \times P^{11} \subset S^3 \times P^{12},$$

be the natural inclusions, and let

$$f_1 = i_1 g \times i_2 : M^9 (= P^2 \times P^7) \to N_1^{14} (= S^3 \times P^{11}),$$

$$f_2 = if_1 : M^9 \to N_1^{14} \subset N_2^{15} (= S^3 \times P^{12}).$$

Then we have

Remark 2. (1) $f_1 : M^9 \to N_1^{14}$ is cobordant to an embedding, while

(2) $f_2 = if_1 : M^9 \to N_1^{14} \subset N_2^{15}$ is not cobordant to an embedding.

Proof. Let

$$H^1(P^2) = Z_2\langle x \rangle, \qquad H^1(P^7) = Z_2\langle y \rangle,$$

$$H^1(P^{11}) = Z_2\langle z_1 \rangle, \qquad H^1(P^{12}) = Z_2\langle z_2 \rangle.$$

Then

$$w(M^9) = 1 + x + x^2, \qquad \overline{w}(M^9) = 1 + x, \tag{6.1}$$

$$f_i^*(z_i) = y, \ (i = 1, 2), \tag{6.2}$$

$$w(N_1^{14}) = (1 + z_1)^{12}, \qquad w(N_2^{15}) = (1 + z_2)^{13}. \tag{6.3}$$

Hence

$$w(f_1) = (1+x)(1+y^4), \qquad w(f_2) = (1+x)(1+y+y^4+y^5). \tag{6.4}$$

By (6.1)-(6.4), we have

$$w_{9-i}(f_1) = 0,\ (i = 1,\ 2,\ 3),$$

$$w_4(M^9) = w_2^2(M^9) = w_2(M^9)w_1^2(M^9) = w_1^4(M^9) = 0, \quad w_3(M^9) = 0,$$

$$f_1^*(w_1(N_1)) = 0, \qquad f_1^*(w_2(N_1)) = 0.$$

Thus f_1 is cobordant to an embedding by Proposition 4.1.

On the other hand, because $(f_2 \times f_2)^* U_{N_2} = 0$, we have $f_2^* f_{2!}(1) = 0$ by [6, Lemma 2], and so

$$\theta(f_2) = w_6(f_2) = xy^5$$

by (6.4). Hence

$$w_1(M^9)\, f_2^*(w_1^2(N_2))\, \theta(f_2) \neq 0,$$

by (6.1) and (6.2). Therefore, by Proposition 3.1, $f_2 (= if_1)$ is not cobordant to an embedding.

On the other hand, even if $f : M^n \to N_1^{n+k}$ is not cobordant to an embedding, it sometimes happens that the composite of f and a map $p : N_1^{n+k} \to N_2^{n+k-i}$, $(i > 0)$ is cobordant to an embedding. Let K be the Klein bottle and $h : K \to P^2$ be the blowing-up at a point in P^2 (see, e.g. [4]). Then

$$H^1(K) = Z_2\langle x\rangle + Z_2\langle w_1\rangle,\ (w_1 = w_1(K)),$$

$$w_1^2 = 0, \quad w_1 x = x^2 \neq 0, \quad w(K) = \overline{w}(K) = 1 + w_1. \tag{6.5}$$

Let $i_1 : P^2 \subset P^3$ and $i_2 : P^{15} \subset P^{28}$ be the natural inclusions, and $p : P^3 \times P^{28} \to P^{28}$ the natural projection.

Remark 3. Let $f = i_1 h \times i_2 : K \times P^{15} \to P^3 \times P^{28}$.

(1) f is not cobordant to an embedding.

(2) $pf : K \times P^{15} \to P^{28}$ is cobordant to an embedding.

Sketch of the proof. Let $f_1 = f$, $f_2 = pf$ and let

$$H^1(P^{15}) = Z_2\langle y\rangle, \qquad H^1(P^{28}) = Z_2\langle x\rangle.$$

Then

$$w(K \times P^{15}) = \overline{w}(K \times P^{15}) = 1 + w_1, \quad w(P^{28}) = (1+z)^{29}, \quad f^*(z) = y.$$

Hence we have

$$w_{17-i}(f_j) = 0 \ (i \leq 2), \qquad w_{11}(f_j) = 0,$$

$$w_{14}(f_j) = w_1 y^{13}, \qquad w_{12}(f_j) = y^{12}, \ (j = 1, 2),$$

$$f_1^* f_{1!}(1) = xy^{13}, \qquad w_1 f_1^*(w_1^2(P^3 \times P^{28}))(f_1^* f_{1!}(1) - w_{14}(f_1)) \neq 0,$$

and so $f_1 = f$ is not cobordant to an embedding.

Using (6.5) and the fact $w_{11}(f_2) = 0$, we see easily that the condition (1) of Brown's theorem is satisfied. To prove (2), it is sufficient to show that $f_2^*(x_4) w_1 f_2^* f_{2!}(w_1) = 0$ for $x_4 \in H^4(P^{28})$, because $w_1^2 = 0$, and this equality follows from Lemma 2.2(2).

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