Bull. of Yamagata Uuiv., Nat. Sci., Vol. 13, No. 2, Jan. 1993

Enumerating embeddings of m-manifolds into $(2m+1)$-manifolds

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(August 7, 1992)

Abstract

For a closed connected differentiable manifold M, a differentiable manifold N and a differentiable map $f: M \to N$, we denote the set of isotopy classes of embeddings homotopic to f by $[M \subset N]_{[f]}$ and the set $\pi_1(N^M, Emb(M, N), f)$ by $[M \subset N]_f$, where $Emb(M, N)$ stands for the space of embeddings of M to N. In this paper, we will determine the set $[M \subset N]_f$ on the assumption that $\dim N = 2\dim M + 1 \geq 7$, along the lines of Larmore [4], [5]. If $N = L^m(p)$, the lens space mod p, we will determine the isotopy set $[M \subset N]_{[f]}$ for $\dim M = m \geq 3$.

§1. Introduction

Let M be a compact connected differentiable m-manifold without boundary and let $f: M \to N$ be a differeniable map of M to a differentiable manifold N without boundary. Denote the set of isotopy classes of embeddings homotopic to f by $[M \subset N]_{[f]}$, and let $\pi_1(N^M, Emb(M, N), f) = [M \subset N]_f$, where $Emb(M, N)$ stands for the space of embeddings of M to N. Under these circumstances, it is known that there is a $\pi_1(N^M, f)$-action on $[M \subset N]_f$ such that

$$[M \subset N]_f / \pi_1(N^M, f) = [M \subset N]_{[f]}$$

(see §2 or e.g. [4], [5], [9]).

If $\dim N > 2\dim M + 1$, then $[M \subset N]_{[f]}$ is a singleton [12]. If $\dim N = 2\dim M + 1$, then there exists an embedding $M \to N$ homotopic to a given map f [12], and if moreover $f_\# : \pi_1(M) \to \pi_1(N)$ is surjective, then $[M \subset N]_{[f]}$ is also a singleton [3]. However, in general, the set $[M \subset N]_{[f]}$ is not necessarily a singleton even if $\dim N = 2\dim M + 1$ (cf. [1], [2], [4], [7]).

The set $[M \subset N]_f$ has an affine abelian group structure with unit $[f]$ in the metastable range (cf. [4], [5], [9]). In this paper, we shall study the affine group $[M \subset N]_f$ on the assumption that $\dim N = 2\dim M + 1$.

For a manifold V, let $w_1(V): \pi_1(V) \to \mathrm{Aut}(Z)$ be the orientation homomorphism of V and define a number $(-1)^a$ for $a \in \pi_1(V)$ by the equation

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$$w_1(V)(a) = (-1)^{a} 1_Z.$$

For a map $f: M \to N$, the fundamental group $\pi_1(N)$ of N is expressed in the form

$$\pi_1(N) = \operatorname{Im} f_\# + \sum_{i \in I} ((\operatorname{Im} f_\#) a_i (\operatorname{Im} f_\#) \cup (\operatorname{Im} f_\#) a_i^{-1} (\operatorname{Im} f_\#)).$$

For $i \in I$, let

$$A_i = \{(b, c) \in \pi_1(M) \times \pi_1(M) \mid f_\#(b) a_i f_\#(c)^{-1} = a_i\},$$
$$B_i = \{(b, c) \in \pi_1(M) \times \pi_1(M) \mid f_\#(b) a_i^{-1} f_\#(c)^{-1} = a_i\},$$

and let $h_i: A_i \to \{\pm 1\}$ and $k_i: B_i \to \{\pm 1\}$ be the maps defined by

$$h_i(b, c) = (-1)^b (-1)^c (-1)^{f_\#(c)},$$
$$k_i(b, c) = (-1)^{m+1} (-1)^{a_i} (-1)^b (-1)^c (-1)^{f_\#(c)}, \quad (m = \dim M).$$

Then, using classical algebraic topology along the lines of Larmore [4], [5], we will prove the following theorem of Li [6]. The proof is different from that of Li, who used normal bordism theory due to Dax [1] and Salomonsen [9].

Theorem 1.1 (Li). *Assume that M is a compact connected differentiable m-manifold without boundary ($m \geq 3$) and N is a differentiable $(2m+1)$-manifold without boundary. Then for any embedding $f: M \to N$,*

$$[M \subset N]_f = Z_2^{\alpha} + Z^{\beta},$$

where α and β are the cardinalities of the set $\{i \in I \mid h_i \cup k_i : A_i \cup B_i \to \{\pm 1\}$ is surjective$\}$ and the set $\{i \in I \mid h_i \cup k_i$ is not surjective$\}$, respectivvely.

Corollary 1.2. *In addition to the assumption above, we assume that $f_\#: \pi_1(M) \to \pi_1(N)$ is trivial. Then*

$$[M \subset N]_f = \begin{cases} \Sigma_A Z_2 + \Sigma_{B \cup C} Z & \text{if } w_1(M) = 0,\ m \equiv 0(2), \\ \Sigma_B Z_2 + \Sigma_{A \cup C} Z & \text{if } w_1(M) = 0,\ m \equiv 1(2), \\ \Sigma_{A \cup B \cup C} Z_2 & \text{if } w_1(M) \neq 0, \end{cases}$$

where

$$A = \{a \in \pi_1(N) \mid a \neq 1,\ a^2 = 1,\ (-1)^a = 1\},$$
$$B = \{a \in \pi_1(N) \mid a^2 = 1,\ (-1)^a = -1\},$$
$$C = \{\{a, a^{-1}\} \mid a \in \pi_1(N),\ a^2 \neq 1\}.$$

If $N = L^m(p) = S^{2m+1}/Z_p$ ($p \geq 2$), the lens space mod p ($L^m(2) = P^{2m+1}$, the odd dimensional real projective space), then $[M \subset N]_f = [M \subset N]_{[f]}$ (see (2.3)) and hence we have the following

Theorem 1.3. *Assume that M is a compact connected differentiable m-manifold without boundary ($m \geq 3$). If $f: M \to L^m(p)$ ($p \geq 2$) is nullhomotopic,* *then*

$$[M \subset L^m(p)]_{[f]} = \begin{cases} Z^{(p-2)/2} & p \equiv 1(2),\ w_1(M)=0, \\ Z^{(p-2)/2} + Z & p \equiv 0(2),\ w_1(M)=0,\ m \equiv 1(2), \\ Z^{(p-2)/2} + Z_2 & p \equiv 0(2),\ w_1(M)=0,\ m \equiv 0(2), \\ Z_2^{[p/2]} & w_1(M) \neq 0, \end{cases}$$

where $[q]$ *denotes the integer part of* q.

Remark. If p is a prime and f is not nullhomotopic, then $f_{\#} : \pi_1(M) \to \pi_1(L^m(p))$ is surjective and hence $[M \subset L^m(p)]_{[f]}$ is a singleton [3].

Remark. If M is simply connected, then the results above are coincident with those of Li, Liu and Zhang [7].

The remainder of this paper is organized as follows: In §2, we recall the definition of $\pi_1(N^M, f)$-action on the set $[M \subset N]_f$ and prove that this action is trivial if $N = L^n(p)$ $(p \geq 2)$. In §3, we introduce Larmore's method of computing $[M \subset N]_f$. The proofs of the results in the introduction are given in §4. A key lemma used in proving Theorem 1.1 is proved in §5.

The author is grateful to Dr. Zhang for his valuable suggestions.

§2. The $\pi_1(N^M, f)$-action on $[M \subset N]_f$

In what follows, M is a compact connected differentiable m-manifold without boundary, N is a connected differentiable manifold without boundary and $f : M \to N$ is an embedding. The space N^M and $Emb(M, N)$ stand for the space of differentiable maps of M to N and its subspace consisting of embeddings, respectively. According to the notations used in [4], [5], we set $\pi_1(N^M, Emb(M, N), f) = [M \subset N]_f$. The group $\pi_1(N^M, f)$ acts on the left of $[M \subset N]_f$ as follows: Given a self-homotopy $\{g_t\}$ of f and a homotopy $\{f_t\}$ from f to an embedding, we define $[g_t][f_t] = [\{g_t\}\{f_t\}]$, where $\{g_t\}\{f_t\}$ denotes the join of the two homotopies $\{g_t\}$ and $\{f_t\}$. The natural map $\Delta : [M \subset N]_f \to [M \subset N]_{[f]}$, defined by $\Delta[f_t] = [f_1]$, the isotopy class of the embedding f_1, leads to the following

Theorem 2.1 (Larmore et al). *There is a bijection*

$$[M \subset N]_f / \pi_1(N^M, f) = [M \subset N]_{[f]}.$$

To determine the isotopy set $[M \subset N]_{[f]}$ along these lines, we have to study the set $[M \subset N]_f$ and the $\pi_1(N^M, f)$-action on it. In some cases, $\pi_1(N^M, f)$-actions are found to be trivial, even if $\pi_1(N^M, f)$ is not a trivial group.

Lemma 2.2. *If any generator of* $\pi_1(N^M, f)$ *is represented by the composi-*

tion $\{\Phi_t f\}$ *of* f *and a periodic flow* $\{\Phi_t\}$ *of* N, *then* $\pi_1(N^M, f)$ *acts trivially on* $[M \subset N]_f$, *and hence* $[M \subset N]_f = [M \subset N]_{[f]}$.

Here the periodic flow means a flow $\{\Phi_t\}$ such that $\Phi_{t+1} = \Phi_t$ for $t \in R$.

Proof. Given a homotopy $\{f_t\}$ of f to an embedding, let $F: M \times I \times I \to N \times I \times I$ be a homotopy defined by

$$F(x, t, u) = \begin{cases} (f_{(2t+u-1)/(u+1)}(x), t, u), & (1-u)/2 \le t \le 1, \\ (f(x), t, u), & 0 \le t \le (1-u)/2. \end{cases}$$

Let $G: N \times I \times I \to N$ be a homotopy defined by

$$G(y, t, u) = \begin{cases} y, & 1/2 + u \le t \le 1 \text{ or } 0 \le t \le u, \\ \Phi_{2(t-u)}(y), & u \le t \le \min\{1, 1/2 + u\}. \end{cases}$$

Then the composition $GF: M \times I \times I \to N$ is a homotopy of $\{\Phi_t f\}\{f_t\}$ to $\{f_t\}$, and hence $[\Phi_t f][f_t] = [f_t]$.

Example. For $p \ge 2$, let $L^n(p) = S^{2n+1}/Z_p$ be the lens space mod p ($L^n(2) = P^{2n+1}$, the odd dimensional real projective space). Then $\Phi_t: L^n(p) \to L^n(p)$, defined by

$$\Phi_t[x_0, \cdots, x_n] = [x_0 \exp(2\pi i t/p), \cdots, x_n \exp(2\pi i t/p)],$$

is a periodic flow, where x_k $(0 \le k \le n)$ are complex numbers with $\sum_{k=0}^{n} |x_k|^2 = 1$. If $2n > \dim M$, by the Eilenberg classification theorem (e.g., [11, (6.17)]), we get $\pi_1(L^n(p)^M, f) = \pi_1(L^n(p)) = Z_p$ generated by $[\Phi_t f]$. Therefore

(2.3) $$[M \subset L^n(p)]_{[f]} = [M \subset N]_f \quad \text{if } \dim M < 2n.$$

§3. Larmore's method

In this section, we shall recall Larmore's method [4], [5] of computing $[M \subset N]_f$.

For an n-manifold V, Let $RV = (V \times V - \Delta_V) \cup_\phi SV \times [0, \varepsilon)$, where $\phi: SV \times (0, \varepsilon) \to V \times V - \Delta_V$ is a map given by $\phi(v, t) = (\exp(tv), \exp(-tv))$. Here SV denotes the total space of the tangent sphere bundle of V and Δ_V means the diagonal of V. A free Z_2-action on RV is induced from the antipodal map on SV and the interchaging of elements of $V \times V$. We denote the quotient spaces RV/Z_2 and $(V \times V - \Delta_V)/Z_2$ by R^*V and V^*, respectivey. Then the space R^*V is a $2n$-manifold with boundary PV $(= SV/Z_2)$, and $R^*V - PV = V^*$, the reduced symmetric product of V. If V is compact, so is R^*V.

The pair of spaces $(R^*(V \times R^\infty), P(V \times R^\infty))$ denotes the inductive limit of $R^*(V \times R^k), P(V \times R^k))$. We convert the natural inclusion $R^* i_V: (R^*V, PV) \subset (R^*(V \times R^\infty), P(V \times R^\infty))$ into a pair fibration $\zeta_V: (Y_V, Z_V) \to (R^*(V \times R^\infty), P(V \times R^\infty))$ in a standard manner. For an embedding $f: M \to N$, let $R^*i_N R^* f = F$ for brevity's sake, and let $\Gamma(F^{-1}\zeta_N)$ be the set of homotopy classes of cross sections of the pull-back of ζ_N along F. Then Larmore [4], [5] has proved the following

Theorem 3.1 (Larmore). *If* $2\dim N > 3(\dim M + 1)$, *then*

$$[M \subset N]_f = \Gamma(F^{-1}\zeta_N).$$

Let $\theta_V: Y_V \to R^*(V \times R^\infty)$ and $\rho_V: Z_V \to P(V \times R^\infty)$ be the restrictions of ζ_V to Y_V and Z_V, respectively. Then θ_V and ρ_V are both ordinary fibrations. Let $\pi_q\theta_V$ and $\pi_q\rho_V$ be the local systems of q-th homotopy groups of θ_V and ρ_V, respectively, and let $\pi_q\zeta_V$ be a subsheaf of $\pi_q\theta_V$ such that

$$\pi_q\zeta_V = \begin{cases} \pi_q\rho_V & \text{over } P(V \times R^\infty), \\ \pi_q\theta_V & \text{over } R^*(V \times R^\infty) - P(V \times R^\infty). \end{cases}$$

Theorem 3.2 (Larmore). *Let* $\dim M = m \geq 2$ *and* $\dim N = 2m+1$. *Then for any embedding* $f: M \to N$

$$[M \subset N]_f = \Gamma(F^{-1}\zeta_N) = H^{2m}(R^*M\,;\, F^{-1}\pi_{2m}\zeta_N).$$

Now, we shall explain the sheaf $\pi_{n-1}\zeta_V$ $(n = \dim V)$. We set $(V \times V) \times_{Z_2} S^\infty = \Gamma V$. Then the natural projection $p: \Gamma V \to P^\infty$ is a fibration with fiber $V \times V$ and with cross section s. We denote the generator of $\pi_1(P^\infty) = Z_2$ by t and set $s_\#(t) = t$, and $T_2 = Z_2$ generated by t. Then $\pi_1(\Gamma V) = (\pi_1(V) \times \pi_1(V)) \times_\varphi T_2$, the semidirect product, where $\varphi: T_2 \to \mathrm{Aut}(\pi_1(V) \times \pi_1(V))$ is given by $\varphi(t)(b, c) = (c, b)$ and there is a homotopy equivalence $\Psi: R^*(V \times R^\infty) \to \Gamma V$ [5, p. 84]. We regard $\Psi_\#: \pi_1(R^*(V \times R^\infty)) \to (\pi_1(V) \times \pi_1(V)) \times_\varphi T_2$ as the identily.

Lemma 3.3 (Larmore). *Assume that* $\dim V \geq 3$.

(1) $(R^*i_V)_\#: \pi_1(R^*V) \to (\pi_1(V) \times \pi_1(V)) \times_\varphi T_2$ *is an isomorphism*,

(2) $(R^*i_V)_\#: \pi_1(PV) \to \pi_1(P(V \times R^\infty)) = \Delta_{\pi_1(V)} \times T_2$ *is an isomorphism*,

(3) *the natural inclusion* $P(V \times R^\infty) \subset R^*(V \times R^\infty)$ *induces a natural inclusion* $\Delta_{\pi_1(V)} \times T_2 \subset (\pi_1(V) \times \pi_1(V)) \times_\varphi T_2$.

For a manifold V, we denote by $w_1(V)$ both the first Stiefel-Whitney class of V and its orientation homomorphism $w_1(V): \pi_1(V) \to \mathrm{Aut}(Z)$. For an element $a \in \pi_1(V)$, we define a number $(-1)^a$ by the equation

$$w_1(V)(a) = (-1)^a 1_Z.$$

Given an abedian group G and a homomorphism $\mu: \pi_1(X) \to \mathrm{Aut}(G)$, we denote the local system over X associated with μ by $S(G, \mu)$.

Lemma 3.4 (Larmore). *Assume that* $\dim V = n \geq 5$.

(1) $\pi_{n-1}\theta_V = S(Z\pi_1(V), \mu)$ *over* $R^*(V \times R^\infty)$, *where* $\mu : (\pi_1(V) \times \pi_1(V)) \times_\varphi T_2 \to \mathrm{Aut}(Z\pi_1(V))$ *is given by*

$$\mu(b, c, 1)(a) = (-1)^c bac^{-1} \qquad \text{for } a, b, c \in \pi_1(V),$$
$$\mu(b, c, t)(a) = (-1)^n(-1)^a(-1)^c ba^{-1}c^{-1} \qquad \text{for } a, b, c \in \pi_1(V).$$

(2) $\pi_{n-1}\rho_V = S(Z, \mu')$ *over* $P(V \times R^\infty)$, *where Z is an infinite cyclic group generated by* $1 \in \pi_1(V)$ *and* $\mu' : \Delta_{\pi_1(V)} \times T_2 \to \mathrm{Aut}(Z)$ *is the restriction of* μ.

§4. Proofs of the results in the introduction

If $\dim M \geq 3$, then it is easily proved that for $f : M \to N$, its induced homomorphism $F_\# = (R^*i_N)_\#(R^*f)_\# : \pi_1(R^*M)$ $(= (\pi_1(M) \times \pi_1(M)) \times_\varphi T_2) \to \pi_1(R^*(N \times R^\infty))$ $(= (\pi_1(N) \times \pi_1(N)) \times_\varphi T_2)$ is given by

$$F_\#(b, c, t') = (f_\#(b), f_\#(c), t') \quad \text{for } b, c \in \pi_1(M).$$

Hence, by Lemma 3.4, we have the following

Lemme 4.1. *Assume that* $\dim N = n \geq 5$ *and* $\dim M = m \geq 3$. *If* $f : M \to N$ *is an embedding, then* $F^{-1}\pi_{n-1}\theta_N = S(\pi_1(N), \mu_M)$, *where* $\mu_M : (\pi_1(M) \times \pi_1(M)) \times_\varphi T_2 \to \mathrm{Aut}(Z\pi_1(N))$ *is given by*

$$\mu_M(b, c, 1)(a) = (-1)^{f_\#(c)} f_\#(b) a f_\#(c)^{-1}$$
$$\mu_M(b, c, t)(a) = (-1)^n(-1)^a(-1)^{f_\#(c)} f_\#(b) a^{-1} f_\#(c)^{-1},$$

for $a \in \pi_1(N)$, *and* $b, c \in \pi_1(M)$.

We study $F^{-1}\pi_{n-1}\theta_N$ more exactly. The group $\pi_1(N)$ can be described in the form

$$\pi_1(N) = \mathrm{Im}\, f_\# + \textstyle\sum_{i \in I}((\mathrm{Im}\, f_\#)a_i(\mathrm{Im}\, f_\#) \cup (\mathrm{Im} f_\#)a_i^{-1}(\mathrm{Im}\, f_\#)).$$

We set $(\mathrm{Im}\, f_\#)a(\mathrm{Im}\, f_\#) \cup (\mathrm{Im}\, f_\#)a^{-1}(\mathrm{Im}\, f_\#) = [a]$. From Lemma 4.1, it follows that

$$F^{-1}\pi_{n-1}\theta_N = A_\theta \oplus \textstyle\sum_{i \in I} A_{\theta i} \tag{4.2}$$

where

$$A_\theta = S(Z(\mathrm{Im}\, f_\#), \mu_M)$$
$$A_{\theta i} = S(\textstyle\sum_{a \in [a_i]} Z\langle a \rangle, \mu_M). \tag{4.3}$$

Here $Z\langle a\rangle$ denotes the infinite cyclic group generated by a. By the definition of the sheaf $\pi_{n-1}\zeta_N$, the induced sheaf $F^{-1}\pi_{n-1}\zeta_N$ is desciried in the form

$$F^{-1}\pi_{n-1}\zeta_N = A_\zeta \oplus \sum_{i\in I} A_{\zeta i},$$

where A_ζ and $A_{\zeta i}$ are subsheaves of A_θ and $A_{\theta i}$, respectively, satisfying the conditions

$$(4.4)\qquad \begin{array}{ll} A_\zeta|PM = S(Z, \mu_M) = F^{-1}\pi_{n-1}\rho_N, & A_{\zeta i}|PM = 0, \\ A_\zeta = A_\theta \text{ over } M^* = R^*M - PM, & A_{\zeta i} = A_{\theta i} \text{ over } M^*. \end{array}$$

From now on, we assume that $n = \dim N = 2\dim M + 1 = 2m + 1$. By Theorem 3.2, we have

$$[M \subset N]_f = H^{2m}(R^*M\,;A_\zeta) \oplus \sum_{i\in I} H^{2m}(R^*M\,;A_{\zeta i}).$$

The proof of Theorem 1.1 follows from Assertions 1 and 2 below.

Assertion 1. $H^{2m}(R^*M\,;A_\zeta) = 0$.

Let A_i and B_i be the sets defined by

$$A_i = \{(b, c) \in \pi_1(M)\times\pi_1(M) \mid f_\#(b)a_i f_\#(c)^{-1} = a_i\},$$
$$B_i = \{(b, c) \in \pi_1(M)\times\pi_1(M) \mid f_\#(b)a_i^{-1} f_\#(c)^{-1} = a_i\},$$

and let $h_i : A_i \to \{\pm 1\}$ and $k_i : B_i \to \{\pm 1\}$ be maps given by

$$h_i(b, c) = (-1)^b(-1)^c(-1)^{f\#(c)},$$
$$k_i(b, c) = (-1)^{m+1}(-1)^b(-1)^c(-1)^{f\#(c)}(-1)^{a_i}.$$

Assertion 2. *$H^{2m}(R^*M\,;A_{\zeta i}) = Z_2$ or Z according as $h_i \cup k_i : A_i \cup B_i \to \{\pm 1\}$ is surjective or not.*

In proving the assertions mentioned above, we use the following lemmas. The first one is shown in §5, while the other one is well-known.

Lemma 4.5. *Let $w_1(R^*M)$ and $w_1(PM)$ be the orientation homomorphisms of R^*M and PM, respectively. Then*

(1) $w_1(R^*M) : \pi_1(R^*M) = (\pi_1(M)\times\pi_1(M))\times_\phi T_2 \to \mathrm{Aut}(Z)$ *is given by*
$$w_1(R^*M)(b, c, t^\epsilon) = (-1)^{\epsilon(\dim M)}(-1)^b(-1)^c 1_Z,$$
(2) $w_1(PM) : \pi_1(PM) = \Delta_{\pi_1(M)}\times T_2 \to \mathrm{Aut}(Z)$ *is given by*
$$w_1(PM)(b, b, t^\epsilon) = (-1)^{\epsilon(\dim M)} 1_Z.$$

Lemma 4.6. *Let G be an abelian group and $S(G, \mu)$ a local system over a pathconnected space X. Let $\bar{G}$ be the subgroup of G generated by $\{g - \mu(a)g \mid a \in \pi_1(X),\ g \in G\}$. Then*

$$H_0(X\,;S(G, \mu)) = G/\bar{G}.$$

Proof of assertion 1. Let $i: A_\zeta \to A_\theta$ be the natural inclusion. Because of the fact that R^*M is a $2m$-manifold with boundary PM and the properties of A_ζ and A_θ in (4.4), we have a commutative diagram of exact sequences

$$\begin{array}{ccccc} H^{2m-1}(PM; F^{-1}\pi_{2m}\rho_N) & \xrightarrow{\delta} & H^{2m}(R^*M, PM; A_\zeta) & \to & H^{2m}(R^*M; A_\zeta) \to 0 \\ \downarrow i_* & & \downarrow \cong i_* & & \\ H^{2m-1}(PM; A_\theta|PM) & \xrightarrow{\delta'} & H^{2m}(R^*M, PM; A_\theta) & \to & 0. \end{array}$$

Here i_* in the right hand side is an isomorphism because $A_\zeta = A_\theta$ on $R^*M - PM$. Therefore it is enough to show that $i_*\delta$, or equivalently $\delta' i_*$, is surjective. By the Poincaré duality, it is sufficient to prove that $j_*(i\otimes 1)_*: H_0(PM; (F^{-1}\pi_{2m}\rho_N)\otimes S(Z, w_1(PM))) \to H_0(PM; (A_\theta|PM)\otimes S(Z, w_1(PM))) \to H_0(R^*M; A_\theta\otimes S(Z, w_1(R^*M)))$ is surjective, where $j: PM \subset R^*M$ is the natural inclusion. From (4.3), (4.4) and Lemmas 4.5-6, it follows that

$$H_0(PM; (F^{-1}\pi_{2m}\rho_N)\otimes S(Z, w_1(PM))) = \begin{cases} Z & \text{if } m \equiv 1(2),\ \mathrm{Im} f_\# \subset \mathrm{Ker}\ w_1(N), \\ Z_2 & \text{otherwise,} \end{cases}$$

$$H_0(R^*M; A_\theta\otimes S(Z, w_1(R^*M))) = \begin{cases} Z & \text{if } m \equiv 1(2),\ \mathrm{Im} f_\# \subset \mathrm{Ker}\, w_1(N),\ \mathrm{Ker} f_\# \subset \mathrm{Ker}\, w_1(M), \\ Z_2 & \text{otherwise,} \end{cases}$$

and that j_* is surjective on $\mathrm{Im}\,(i\otimes 1)_*$.

Proof of Assertion 2. By using (4.3)-(4.4) and the Poincaré duality, we have

$$H^{2m}(R^*M; A_{\zeta i}) = H_0(R^*M; S(\textstyle\sum_{a\in[a_i]} Z\langle a\rangle, \mu_M\otimes w_1(R^*M))).$$

It is easily proved that the right hand side is isomorphic to Z_2 or Z according as $h_i \cup k_i: A_i \cup B_i \to \{\pm 1\}$ is surjective or not.

Proof of Corollary 1.2. Assume that $f_\#: \pi_1(M) \to \pi_1(N)$ is trivial. Then for $a \in \pi_1(N)$, the coset $[a] = \{a\}$ or $\{a, a^{-1}\}$ according as $a^2 = 1$ or not.

Case $w_1(M) = 0$. In this case, $(-1)^b = (-1)^{f_\#(b)} = 1$ for $b \in \pi_1(M)$. If $a^2 = 1$ then $A_a = B_a = \pi_1(M)\times\pi_1(M)$ and $h_a(b, c) = 1$, $k_a(b, c) = (-1)^{m+1}(-1)^a$. Hence $h_a \cup k_a$ is surjective if and only if either $(-1)^a = 1$, $m \equiv 0(2)$ or $(-1)^a = -1$, $m \equiv 1(2)$. If $a^2 \neq 1$, then $A_a = \pi_1(M)\times\pi_1(M)$, $B_a = \phi$ and $h_a(b, c) = 1$ for any $b, c \in \pi_1(M)$. Hence $h_a \cup k_a$ is not surjective. Therefore we get

$$[M \subset N]_f = \begin{cases} \sum_{a\in A} Z_2 \oplus \sum_{a\in B} Z \oplus \sum_{\{a,a^{-1}\}\in C} Z & \text{if } m \equiv 0(2), \\ \sum_{a\in A} Z \oplus \sum_{a\in B} Z_2 \oplus \sum_{\{a,a^{-1}\}\in C} Z & \text{if } m \equiv 1(2). \end{cases}$$

Case $w_1(M) \neq 0$. For any $a\,(\neq 1) \in \pi_1(M)$, we have $A_a = \pi_1(M)\times\pi_1(M)$ and $h_a(b, c) = (-1)^b(-1)^c$ because $(-1)^{f_\#(c)} = 1$. From the assumption $w_1(M) \neq 0$, it follows that h_a is surjective and so is $h_a \cup k_a$. Then we get

$$[M \subset N]_f = \textstyle\sum_{a\in A\cup B} Z_2 \oplus \sum_{\{a,a^{-1}\}\in C} Z_2.$$

Proof of Corollary 1.3 follows this corollary and (2.3).

§ 5. The orientation homomorphism of R^*M

In this section, we shall prove the following

Lemma 4.5. *Assume that* $\dim M \geq 3$.

(1) *The orientation homomorphism* $w_1(R^*M) : (\pi_1(M) \times \pi_1(M)) \times_\phi T_2 \to \mathrm{Aut}(Z)$ *of* R^*M *is given by*

$$w_1(R^*M)(b, c, t') = (-1)^{\varepsilon(\dim M)}(-1)^b(-1)^c 1_Z.$$

(2) $w_1(PM) : \Delta\pi_1(M) \times T_2 \to \mathrm{Aut}(Z)$ *is given by*

$$w_1(PM)(b, b, t') = (-1)^{\varepsilon(\dim M)} 1_Z.$$

Frist of all, we recall some well-known results.

Lemma 5.1. *For a pathconnected space* X, *there is an isomorphism* ψ_X : $H^1(X ; Z_2) \to \mathrm{Hom}(\pi_1(X), \mathrm{Aut}(Z))$, *which is natural for maps, that is, for a map* $g : X \to Y$, *there is an equation*

$$\phi_X(g^*(y))(x) = \psi_Y(y)(g_\#(x)) \quad x \in \pi_1(X), y \in H^1(Y ; Z_2).$$

In this sense, our purpose is to determine the homomorphisms $\psi_{R^*M}(w_1(R^*M))$ and $\psi_{PM}(w_1(PM))$. By Lemma 5.1, we easily obtain the following

Lemma 5.2. *For spaces* X, Y, *the isomorphism* $\psi_{X\times Y}$ *is given by*

$$\psi_{X\times Y}(a\otimes 1 + 1\otimes a')(b, c) = \psi_X(a)(b)\, \psi_Y(a')(c).$$

Now, we return to the proof of Lemma 4.5. Since the natural inclusion $M^* = R^*M - PM \subset R^*M$ is a homotopy equivalence, we identify cohomology groups and homotopy groups of M^* and those of R^*M. The Z_2-cohomology of M^* (and hence R^*M) is calculated by Thomas [10]. The results stated in [10] are freely quoted hereafter. There is a commutative diagram

$$\begin{array}{ccc}
H^1(M\times M ; Z_2) & \xrightarrow{\rho_1} & H^1(M\times M - \Delta M ; Z_2) \\
\uparrow q^* & & \uparrow p^* \\
H^1(\Gamma M ; Z_2) & \xrightarrow{\rho} & H^1(M^* ; Z_2) = H^1(R^*M ; Z_2) \\
\uparrow k^* & & \uparrow j^* \\
H^1(M\times P^\infty ; Z_2) & \xrightarrow{\rho_2} & H^1(PM ; Z_2).
\end{array}$$

Here, the maps ρ, ρ_1, ρ_2 are all isomorphisms induecd by maps. In particular

$$\rho = (R^* i_M)^* \Psi^*$$

where the maps $R^* i_M$ and Ψ are given in §3. Let $u \in H^1(P^\infty; Z_2)$ be the generator. Then

$$H^1(\Gamma M; Z_2) = Z_2 \langle u \rangle \oplus \{x \otimes 1 + 1 \otimes x \mid x \in H^1(M; Z_2)\}.$$

Lemma 5.3. *The first Stiefel-Whitney class $w_1(R^*M)$ of R^*M is given by*

$$w_1(R^*M) = \rho(w_1(R^*M) \otimes 1 + 1 \otimes w_1(M)) + (\dim M)\rho(u).$$

Proof. $w_1(R^*M)$ is expressed in the form $w_1(R^*M) = \lambda\rho(u) + \rho(x \otimes 1 + 1 \otimes x)$ for some $x \in H^1(M; Z_2)$ and $\lambda \in Z_2$. Because $\rho_1(x \otimes 1 + 1 \otimes x) = p^* w_1(R^*M) = w_1(M \times M - \Delta_M) = \rho_1 w_1(M \times M) = \rho_1(w_1(M) \otimes 1 + 1 \otimes w_1(M))$, we have $x = w_1(M)$. Let $v \in H^1(PM; Z_2)$ be the first Stiefel-Whitney class of the double covering $SM \to PM$. Then $\lambda v = \rho_2 k^*(\lambda u + x \otimes 1 + 1 \otimes x) = j^* w_1(R^*M) = w_1(PM) = (\dim M)v$. Hence $\lambda \equiv \dim M(2)$.

Lemma 5.4. *The isomorphism $\psi_{R^*M} : H^1(R^*M; Z_2) \to \mathrm{Hom}(\pi_1(R^*M), \mathrm{Aut}(Z))$ is given by*

$$\psi_{R^*M}(\rho(a \otimes 1 + 1 \otimes a))(b, c, t^\varepsilon) = \psi_M(a)(bc),$$
$$\psi_{R^*M}(\rho(u)(b, c, t^\varepsilon) = (-1)^\varepsilon 1_Z,$$

for $a \in H^1(M; Z_2)$, $b, c \in \pi_1(M)$ and $\varepsilon = 0$ or 1.

Proof. Using Lemma 5.2 and the fact that $\rho = (R^* i_M)^* \Psi^*$, we have

$$\begin{aligned}\psi_{R^*M}(\rho(a \otimes 1 + 1 \otimes a))(b, c, 1) &= \psi_{\Gamma M}(a \otimes 1 + 1 \otimes a)(b, c, 1) \\ &= \psi_{M \times M}(a \otimes 1 + 1 \otimes a)(b, c) \\ &= \psi_M(a)(bc),\end{aligned}$$

$$\begin{aligned}\psi_{R^*M}(\rho(a \otimes 1 + 1 \otimes a))(1, 1, t) &= \psi_{\Gamma M}(a \otimes 1 + 1 \otimes a)(1, 1, t) \\ &= \psi_{\Gamma M}(a \otimes 1 + 1 \otimes a)(k_\#(1, t)) \\ &= \psi_{M \times P^\infty}(0)(1, t) = 1_Z.\end{aligned}$$

Thus we have the first half of the lemma. As for the second half, we have the following equations:

$$\psi_{R^*M}(\rho(u))(b, c, 1) = \psi_{\Gamma M}(u)(b, c, 1) = \psi_{P^\infty}(u)(p_\#(b, c, 1)) = 1_Z,$$

where $p : \Gamma M \to P^\infty$ is a projection (see §3), and

$$\psi_{R^*M}(\rho(u))(1, 1, t) = \psi_{P^\infty}(u)(t) = -1_Z.$$

Hence, the second half of the lemma is proved.

The proof of Lemma 4.5(1) follows immediately from Lemmas 5.1, and 5.3—4, while Lemma 4.5 (2) is easily obtained by using the fact that $w_1(PM)=(\dim M)v$.

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