

On the map defined by regarding embeddings as immersions

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Introduction

Let M be a closed connected smooth manifold of dimension n and R^m the m -dimensional Euclidean space. Denote by $[M \subseteq R^m]$ the set of regular homotopy classes of immersions of M in R^m and by $[M \subset R^m]$ the set of isotopy classes of embeddings of M in R^m , and consider the commutative diagram

$$\begin{array}{ccc} [M \subset R^{m+1}] & \xrightarrow{J_{m+1}} & [M \subseteq R^{m+1}] \\ E_m \uparrow & & I_m \uparrow \\ [M \subset R^m] & \xrightarrow{J_m} & [M \subseteq R^m], \end{array}$$

where E_m and I_m are the maps induced from the natural inclusion $R^m \subset R^{m+1}$ and J_k is the one defined by regarding embeddings as immersions.

The set $[M \subseteq R^m]$ for $2m > 3n + 1$ is an abelian group by taking 0 arbitrarily if it is not empty, and the map I_m is a homomorphism by taking $I_m(0) = 0$; while so are the set $[M \subset R^m]$ and the maps E_m and J_m for $2m > 3(n + 1)$ (see J. C. Becker [2]).

The purpose of this paper is to study the above commutative diagram when $m = 2n - 1$:

$$(*) \quad \begin{array}{ccc} [M \subset R^{2n}] & \xrightarrow{J_{2n}} & [M \subseteq R^{2n}] \\ E \uparrow & & I \uparrow \\ [M \subset R^{2n-1}] & \xrightarrow{J_{2n-1}} & [M \subseteq R^{2n-1}] \end{array} \quad (E = E_{2n-1}, I = I_{2n-1}),$$

(here we assume that the sets in consideration are not empty).

When $n \geq 4$, the upper groups are determined by A. Haefliger and M. W. Hirsch [3], [5], [6] and so is the group $[M \subseteq R^{2n-1}]$ by D. R. Bausum [1, Th. 37 and Prop. 41], L. L. Larmore and E. Thomas [10, Th. 5.1] and R. D. Rigdon [11, Th. 10.4], and moreover it is proved by R. D. Rigdon [11, Th. 10.4] that I is trivial for even n and is surjective for odd n , respectively. When $n \geq 6$, $[M \subset R^{2n-1}]$ is an abelian group and $\text{Im } E$ is determined by R. D. Rigdon [11, Th. 11.11 and Th. 11.26]. Together with these results, we have the following

MAIN THEOREM. *Let M be a closed connected smooth manifold of dimension*

n with the i -th Stiefel-Whitney class $w_i \in H^i(M; Z_2)$, and let

$$\begin{aligned} Sq^1: H^{n-1}(M; Z_2) &\longrightarrow H^n(M; Z_2), \\ \beta_2: H^{n-2}(M; Z_2) &\longrightarrow H^{n-1}(M; Z) \end{aligned}$$

be the squaring operation and the Bockstein operator, respectively, and $H^i(M; Z[w_1])$ be the integral cohomology twisted by w_1 . Then in the diagram (*) there hold the following properties (i)'s, ..., (iv)'s, respectively, when

- (i) n is even and $w_1 = 0$, (ii) n is even and $w_1 \neq 0$,
 (iii) n is odd and $w_1 = 0$, (iv) n is odd and $w_1 \neq 0$.

(1) Assume that $n \geq 4$. Then

- (i) $[M \subset R^{2n}] = H^{n-1}(M; Z_2)$, $[M \subseteq R^{2n}] = Z$, $J_{2n} = 0$,
 $[M \subseteq R^{2n-1}] = \begin{cases} H^{n-1}(M; Z_2) & \text{if } n \equiv 0(4), \\ H^{n-1}(M; Z_2) + Z_2 & \text{if } n \equiv 2(4), \end{cases} \quad I = 0;$
 (ii) $[M \subset R^{2n}] = Z + \text{Ker } Sq^1$, $[M \subseteq R^{2n}] = Z$, $J_{2n}(a, b) = 2a$,
 $[M \subseteq R^{2n-1}] = \begin{cases} H^{n-1}(M; Z_2) & \text{if } n \equiv 0(4), \\ \text{Ker } Sq^1 + Z_4 & \text{if } n \equiv 2(4), \end{cases} \quad I = 0;$
 (iii) $[M \subset R^{2n}] = H^{n-1}(M; Z)$, $[M \subseteq R^{2n}] = Z_2$, $J_{2n} = 0$,
 $[M \subseteq R^{2n-1}] = \begin{cases} H^{n-1}(M; Z) + Z_2 + Z_2, & I(a, b, c) = b \text{ if } n \equiv 1(4), \\ H^{n-1}(M; Z) + Z_4, & I(a, b) \equiv b(2) \text{ if } n \equiv 3(4); \end{cases}$
 (iv) $[M \subset R^{2n}] = H^{n-1}(M; Z_2)$, $[M \subseteq R^{2n}] = Z_2$, $J_{2n} = 0$,
 $[M \subseteq R^{2n-1}] = H^{n-1}(M; Z[w_1]) + Z_2$, $I(a, b) = b$.

(2) Assume that $n \geq 6$. Then

- (i) $\text{Im } E = [M \subset R^{2n}]$,
 $\text{Im } J_{2n-1} = \begin{cases} H^{n-1}(M; Z_2) & \text{if } n \equiv 2(4) \text{ and } w_2(\text{Ker } \beta_2) = 0, \\ [M \subseteq R^{2n-1}] & \text{otherwise;} \end{cases}$
 (ii) $\text{Im } E = \text{Ker } Sq^1$,
 $\text{Im } J_{2n-1} = \begin{cases} \text{Ker } Sq^1 + Z_2 & \text{if } n \equiv 2(4) \text{ and } w_1^2 + w_2 \neq 0, \\ \text{Ker } Sq^1 & \text{otherwise;} \end{cases}$
 (iii) $\text{Im } E = \text{Im } \beta_2$,

$$\text{Im } J_{2n-1} = \begin{cases} \text{Im } \beta_2 + 0 + Z_2 & \text{if } n \equiv 1(4) \text{ and } w_2(\text{Ker } \beta_2) \neq 0, \\ \text{Im } \beta_2 + Z_2 & \text{if } n \equiv 3(4) \text{ and } w_2(\text{Ker } \beta_2) \neq 0, \\ \text{Im } \beta_2 & \text{otherwise;} \end{cases}$$

(iv) $\text{Im } E = [M \subset R^{2n}]$, $\text{Im } J_{2n-1} = H^{n-1}(M; Z[w_1])$.

The group $[M \subset R^{2n-1}]$ will be studied in the forthcoming paper [14].

In §1, the group structures and the filtrations on $[M \subseteq R^m]$ and $[M \subset R^m]$ are recalled according to [1], [2], [8], [11] and [13], and the methods for computing I_m , E_m and J_m are stated. The groups $[M \subseteq R^{2n}]$, $[M \subset R^{2n}]$ and $[M \subseteq R^{2n-1}]$ are restated in §2 and the results on J_{2n} and I are proved. The map J_{2n-1} is investigated in §§3-4, by using the results on the cohomology of $(\Lambda^2 M, \Delta M)$ due to L. L. Larmore [7] together with the remarks given in §5. In §5, the twisted integral cohomology groups $H^i(\Lambda^2 M, \Delta M; Z[v])$ for $i \geq 2n-3$ ($v \in H^1(\Lambda M^2 - \Delta M; Z_2)$) are treated.

§1. Preliminaries

Let M be a closed connected smooth manifold of dimension n . Then there is a fixed point free involution on the tangent sphere bundle SM over M , which is the antipodal map on each fibre S^{n-1} . Thus, for an immersion $f: M \subseteq R^m$, we have the Z_2 -equivariant map

$$\pi S(f): SM \xrightarrow{S(f)} R^m \times S^{m-1} \xrightarrow{\pi} S^{m-1},$$

where $S(f)$ is the Z_2 -equivariant map induced from the derivation of f and π is the projection.

THEOREM (Haefliger-Hirsch [4]). *If $2m > 3n + 1$, then the correspondence which associates the Z_2 -equivariant homotopy class of $\pi S(f)$ with a regular homotopy class of an immersion f is a bijection between $[M \subseteq R^m]$ and the set of Z_2 -equivariant homotopy classes of Z_2 -equivariant maps of SM to S^{m-1} .*

On the other hand, let ΔM be the diagonal of $M \times M$. Then there is a fixed point free involution on $M \times M - \Delta M$ defined by the interchange of factors. Thus, for an embedding $f: M \subset R^m$, we have the Z_2 -equivariant map

$$f': M \times M - \Delta M \longrightarrow S^{m-1},$$

$$f'(x, y) = (f(x) - f(y)) / \|f(x) - f(y)\| \quad (x, y \in M, x \neq y).$$

THEOREM (Haefliger [3]). *If $2m > 3(n + 1)$, then the correspondence which associates the Z_2 -equivariant homotopy class of f' with an isotopy class of an*

embedding f is a bijection between $[M \subset R^m]$ and the set of Z_2 -equivariant homotopy classes of Z_2 -equivariant maps of $M \times M - \Delta M$ to S^{m-1} .

Let $PM = SM/Z_2$ and $M^* = (M \times M - \Delta M)/Z_2$ be the tangent projective bundle over M and the reduced symmetric product of M , respectively. Moreover, let

$$\eta: PM \longrightarrow P^\infty \quad \text{and} \quad \xi: M^* \longrightarrow P^\infty$$

be the classifying maps of the double coverings $SM \rightarrow PM$ and $M \times M - \Delta M \rightarrow M^*$, respectively. Now, $S^\infty \rightarrow P^\infty$ is the universal double covering and $S^\infty \times_{Z_2} S^{m-1} \rightarrow P^\infty$ is homotopically equivalent to the natural inclusion $P^{m-1} \subset P^\infty$. Therefore the above theorems are restated as follows, where

$$[X, P^{m-1}; \alpha] = [X, S^\infty \times_{Z_2} S^{m-1}; \alpha] \quad \text{for} \quad \alpha: X \longrightarrow P^\infty$$

denotes the homotopy sets of liftings of α to $S^\infty \times_{Z_2} S^{m-1}$:

THEOREM 1.1. *There exist bijections*

$$\begin{aligned} A: [M \subseteq R^m] &\cong [PM, P^{m-1}; \eta] && \text{if } 2m > 3n + 1, \\ B: [M \subset R^m] &\cong [M^*, P^{m-1}; \xi] && \text{if } 2m > 3(n+1). \end{aligned}$$

Each set of the right hand sides has the structure of an abelian group by [2] if it is not empty, which induces those of $[M \subseteq R^m]$ and $[M \subset R^m]$.

Now PM is a manifold of dimension $2n - 1$ and M^* has the homotopy type of a CW-complex of dimension less than $2n$.

PROPOSITION 1.2 (Bausum [1, Prop. 5 and Prop. 6], Larmore-Rigdon [8, Prop. 4.1], Yasui [13, Prop. 1.1]). *Assume that X has the homotopy type of a CW-complex of dimension less than $2n$ ($n \geq 4$). Then for a map $\alpha: X \rightarrow P^\infty$, there exist decreasing filtrations*

$$\begin{aligned} [X, P^{2n-1}; \alpha] &= G_0(\alpha) \supset G_1(\alpha) = 0, \quad G_0(\alpha) = H^{2n-1}(X; Z); \\ [X, P^{2n-2}; \alpha] &= F_0(\alpha) \supset F_1(\alpha) \supset F_2(\alpha) = 0, \\ F_0(\alpha)/F_1(\alpha) &= H^{2n-2}(X; Z[v]), \\ F_1(\alpha) &= \text{Coker}(\Theta: H^{2n-3}(X; Z[v]) \longrightarrow H^{2n-1}(X; Z_2)), \end{aligned}$$

where $H^i(X; Z[v])$ is the integral cohomology of X twisted by $v = \alpha^*u$ ($u \in H^1(P^\infty; Z_2)$ is the generator) and

$$\Theta = Sq^2 \tilde{\rho}_2 + \binom{2n-1}{2} v^2 \tilde{\rho}_2$$

($\tilde{\rho}_2: H^i(X; Z[v]) \rightarrow H^i(X; Z_2)$ is the reduction mod 2).

By the definitions of the maps I_m, E_m and J_m in the introduction and the bijections A and B in Theorem 1.1, we have the commutative diagram

$$\begin{array}{ccccccc}
 [M \subset R^{m+1}] & \xleftarrow{E_m} & [M \subset R^m] & \xrightarrow{J_m} & [M \subseteq R^m] & \xrightarrow{I_m} & [M \subseteq R^{m+1}] \\
 B \downarrow \cong & & B \downarrow \cong & & A \downarrow \cong & & A \downarrow \cong \\
 [M^*, P^m; \xi] & \xleftarrow{i_*} & [M^*, P^{m-1}; \xi] & \xrightarrow{j^*} & [PM, P^{m-1}; \eta] & \xrightarrow{i_*} & [PM, P^m; \eta]
 \end{array}$$

for $2m > 3(n+1)$ (cf. [8], [11]), where $i: P^{m-1} \subset P^m$ is the natural inclusion and

$$j: PM \longrightarrow M^* \text{ is the embedding with } \xi j = \eta$$

induced from the Z_2 -equivariant map $j: SM \rightarrow M \times M - \Delta M$ defined by $j(u) = (\exp(u), \exp(-u))$.

PROPOSITION 1.3 (Larmore-Rigdon [8, Prop. 5.1 and Prop. 6.1]). *Let (X, α) represent (PM, η) or (M^*, ξ) , and consider the filtrations of $[X, P^{m-1}; \alpha]$ for $m = 2n - 1, 2n$ given in Proposition 1.2. Then*

(1) $i_*: [X, P^{2n-2}; \alpha] \rightarrow [X, P^{2n-1}; \alpha]$ preserves the filtrations and the induced homomorphism

$$i_*: F_0(\alpha)/F_1(\alpha) = H^{2n-2}(X; Z[v]) \longrightarrow G_0(\alpha) = H^{2n-1}(X; Z)$$

is just the multiplication by $V = \tilde{\beta}_2(1) \in H^1(X; Z[v])$ ($\tilde{\beta}_2: H^i(X; Z_2) \rightarrow H^{i+1}(X; Z[v])$ is the twisted Bockstein operator);

(2) $j^*: [M^*, P^{m-1}; \xi] \rightarrow [PM, P^{m-1}; \eta]$ preserves the filtrations and $j^*: G_0(\xi) \rightarrow G_0(\eta)$ and $j^*: F_i(\xi)/F_{i+1}(\xi) \rightarrow F_i(\eta)/F_{i+1}(\eta)$ are j^* on the cohomology groups and moreover j^* for $m = 2n - 1$ induces the map

$$j_0^*: \text{Ker}(j^*: F_0(\xi)/F_1(\xi) \longrightarrow F_0(\eta)/F_1(\eta)) \longrightarrow \text{Coker}(j^*: F_1(\xi) \longrightarrow F_1(\eta)),$$

which is equal to the functional operation

$$\Theta_j: \text{Ker } j^*(\subset H^{2n-2}(M^*; Z[v])) \longrightarrow H^{2n-1}(PM; Z_2)/(\text{Im } \Theta + \text{Im } j^*)$$

given by $\delta^{-1} \Theta i^{*-1}$ in the commutative diagram

$$\begin{array}{ccccc}
 \dots \xrightarrow{j^*} H^{2n-3}(PM; Z[j^*v]) \xrightarrow{\delta} H^{2n-2}(M^*, PM; Z[v]) \xrightarrow{i^*} H^{2n-2}(M^*; Z[v]) \xrightarrow{j^*} \dots \\
 \theta \downarrow & & \theta \downarrow & & \theta \downarrow \\
 \dots \xrightarrow{j^*} H^{2n-1}(PM; Z_2) \xrightarrow{\delta} H^{2n}(M^*, PM; Z_2) \xrightarrow{i^*} H^{2n}(M^*; Z_2) (=0)
 \end{array}$$

of the exact sequences of the pair (M^*, PM) , where $v = \xi^*u$ and $i: M^* \subset (M^*, PM)$.

Furthermore, let $\Lambda^2 M = (M \times M)/Z_2$ be the 2-fold symmetric product of M , the set of unordered pairs of M . Then $\Lambda^2 M - \Delta M = M^*$ and $PM = j(PM)$ bounds a tubular neighborhood N of ΔM in $\Lambda^2 M$, and the natural inclusions

$$(M^*, PM) \subset (\Lambda^2 M, N) \supset (\Lambda^2 M, \Delta M)$$

induce isomorphisms of cohomology groups (cf. [8, §5]). Thus we have the following

LEMMA 1.4. *The cohomology exact sequence of (M^*, PM) with any coefficients (e.g., the one in the diagram in Proposition 1.3) can be replaced by the exact sequence*

$$\begin{aligned} \dots \longrightarrow H^{i-1}(M^*) \xrightarrow{j^*} H^{i-1}(PM) \xrightarrow{\delta} H^i(\Lambda^2 M, \Delta M) \\ \xrightarrow{i^*} H^i(M^*) \xrightarrow{j^*} H^i(PM) \longrightarrow \dots \end{aligned}$$

Our study is based on these results. Moreover the cohomology of $(\Lambda^2 M, \Delta M)$ is investigated by L. L. Larmore[7]. The notations Λx and $\Delta(x, y)$ and the results stated in [7, pp. 908–915] are freely quoted hereafter. We also use the following lemma and the results remarked in §5.

LEMMA 1.5. (1) $\tilde{\rho}_r(\Lambda x) = \Lambda(\rho_r x)$ and $\tilde{\rho}_r(\Delta(x, y)) = \Delta(\rho_r x, \rho_r y)$ for $x, y \in H^*(M; Z_s)$, where $r | s, s \leq \infty$ and $\rho_r, \tilde{\rho}_r$ are the reductions mod r .

(2) $\Delta(x, y) = \Lambda x \Lambda y + \Lambda(xy)$ for $x, y \in H^*(M; Z_2)$.

(3) $\delta(v^i x) = v^{i+1} \Lambda x$ for $x \in H^*(M; Z_2)$, where $v^i x = j^* v^i \cdot \pi^* x$ ($\pi: PM \rightarrow M$ is the projection).

PROOF. The relations (1) and (2) are easily obtained by chasing the constructions of Λx and $\Delta(x, y)$ given in [7]. The relation (3) follows from the equality $\delta x = v \Lambda x$ ($\delta: H^{i-1}(M) = H^{i-1}(\Delta M) \rightarrow H^i(\Lambda^2 M, \Delta M)$) in [7, Lemma 6], by noticing that the restriction of the projection $N \rightarrow M$ on PM is equal to π and the one on ΔM is the identity $\Delta M \rightarrow M$. q. e. d.

§2. J_{2^n}, I, E and $[M \subseteq R^{2^n-1}]$

The following results are well-known:

(2.1) *Let $v \in H^1(PM; Z_2)$ be the first Stiefel-Whitney class of the double covering $SM \rightarrow PM$. Then $1, v, \dots, v^{n-1}$ form a base of the $H^*(M; Z_2)$ -module $H^*(PM; Z_2)$ with the relation*

$$v^n = \sum_{i=1}^n v^{n-i} w_i \quad (w_i = w_i(M)).$$

(2.2) $[M \subseteq R^{2^n}] = H^{2^n-1}(PM; Z)$ in Theorem 1.1 and Proposition 1.2 is isomorphic to Z if n is even and Z_2 if n is odd.

(2.3) ([3], [5] and [11]) $[M \subseteq R^{2^n}] = H^{2^n-1}(M^*; Z)$ in Theorem 1.1 and Proposition 1.2 is isomorphic to

$$\begin{array}{ll}
 H^{n-1}(M; Z) & \text{if } n \text{ is odd and } w_1 = 0, \\
 Z + K \ (K = \text{Ker}(Sq^1: H^{n-1}(M; Z_2) \rightarrow H^n(M; Z_2))) & \text{if } n \text{ is even and } w_1 \neq 0, \\
 H^{n-1}(M; Z_2) & \text{otherwise.}
 \end{array}$$

PROOF OF MAIN THEOREM ON J_{2n} . By the results stated in §1, we have a commutative diagram

$$\begin{array}{ccccccc}
 [M \subset R^{2n}] & \xrightarrow{J_{2n}} & [M \subseteq R^{2n}] & & & & \\
 \parallel & & \parallel & & & & \\
 H^{2n-1}(M^*; Z) & \xrightarrow{j^*} & H^{2n-1}(PM; Z) & \xrightarrow{\partial} & H^{2n}(\Lambda^2 M, \Delta M; Z) & \longrightarrow & 0,
 \end{array}$$

where the lower sequence is exact by Lemma 1.4, while by Proposition 5.2(2),

$$H^{2n}(\Lambda^2 M, \Delta M; Z) = \begin{cases} Z & \text{if } n \text{ is even and } w_1 = 0, \\ Z_2 & \text{otherwise.} \end{cases}$$

Thus if n is even and $w_1 \neq 0$ then $\text{Im}(j^*: Z + K \rightarrow Z) = 2Z$, and if it is not then $j^* = 0$. q. e. d.

We now recall that the filtration

$$[M \subseteq R^{2n-1}] = [PM, P^{2n-2}; \eta] = F_0 \supset F_1 \supset 0 \quad (F_i = F_i(\eta))$$

satisfies

$$\begin{aligned}
 F_0/F_1 &= H^{2n-2}(PM; Z[v]), \\
 F_1 &= \text{Coker}(\Theta: H^{2n-3}(PM; Z[v]) \longrightarrow H^{2n-1}(PM; Z_2))
 \end{aligned}$$

where $\Theta = Sq^2 \tilde{\rho}_2 + (n-1)v^2 \tilde{\rho}_2$.

The twisted integral cohomology of PM is investigated by R. D. Rigdon and is given as follows:

PROPOSITION 2.4 (Rigdon [11, Prop. 9.2 and 9.13]). *Let $M \in H^n(M; Z_2)$ be the generator. Then*

- (1) *if n is even, there exist isomorphisms*

$$H^{2n-1}(PM; Z[v]) = Z_2,$$

$$\theta: H^{n-1}(M; Z_2) \cong H^{2n-2}(PM; Z[v]), \quad \theta(x) = \tilde{\beta}_2(v^{n-2}x) \ (x \in H^{n-1}(M; Z_2));$$

- (2) *if n is odd, there exist isomorphisms*

$$H^{2n-1}(PM; Z[v]) = Z,$$

$$\theta: H^{n-1}(M; Z[w_1]) + H^n(M; Z_2) \cong H^{2n-2}(PM; Z[v]),$$

$$\theta(M) = \tilde{\beta}_2(v^{n-3}M), \tilde{\rho}_2\theta(y) = (v^{n-1} + v^{n-2}w_1)\tilde{\rho}_2y \quad (y \in H^{n-1}(M; Z[v])).^*)$$

Let $M' \in H^{n-1}(M; Z_2)$ be the element with $Sq^1M' = M$ when $w_1 \neq 0$ and let $K = \text{Ker}(Sq^1: H^{n-1}(M; Z_2) \rightarrow H^n(M; Z_2))$. Then $H^{2n-2}(PM; Z[v])$ is the following form by Proposition 2.4, $(Z_r\langle a \rangle)$ denotes the cyclic group of order r generated by a):

$$(2.5) \quad \begin{aligned} F_0/F_1 &= \theta H^{n-1}(M; Z_2) && \text{if } n \text{ is even and } w_1 = 0, \\ &= \theta K + Z_2\langle \theta M' \rangle && \text{if } n \text{ is even and } w_1 \neq 0, \\ &= \theta H^{n-1}(M; Z[w_1]) + Z_2\langle \theta M \rangle && \text{if } n \text{ is odd.} \end{aligned}$$

Further, by studying Θ , we have

$$(2.6) \quad F_1 = \text{Coker } \Theta = \begin{cases} H^{2n-1}(PM; Z_2) = Z_2 & \text{if } n \text{ is odd and } w_1 = 0, \\ & \text{or } n \equiv 2(4), \\ 0 & \text{otherwise.} \end{cases}$$

In case of $F_1 = Z_2$, the group extension ϕ_2 of $0 \rightarrow F_1 \rightarrow F_0 \rightarrow F_0/F_1 \rightarrow 0$ is given by

$$\begin{aligned} \phi_2 &= Sq^2\tilde{\beta}_2^{-1} + (n-1)v^2\tilde{\beta}_2^{-1} + Sq^1\tilde{\rho}_2: \{z \in F_0/F_1 \mid 2z = 0\} = \tilde{\beta}_2 H^{2n-3}(PM; Z_2) \\ &\longrightarrow F_1 = H^{2n-1}(PM; Z_2), \end{aligned}$$

which is proved by using [10, Th. 4.1] (cf. [9, Cor. 3.7]), and so we have the following:

(2.7) *The group extension ϕ_2 is trivial except for*

$$\begin{aligned} \phi_2(\theta M') &= v^{n-1}M && \text{if } n \equiv 2(4) \text{ and } w_1 \neq 0, \\ \phi_2(\theta M) &= v^{n-1}M && \text{if } n \equiv 3(4) \text{ and } w_1 = 0. \end{aligned}$$

THEOREM 2.8 (Bausum [1, Th. 37 and Prop. 41], Larmore-Thomas [10, Th. 5.1], Rigdon [11, Th. 10.4]). *Let $n \geq 4$. Then the group $[M \subseteq R^{2n-1}] = [PM, P^{2n-2}; \eta]$ is as follows:*

$$\begin{aligned} [M \subseteq R^{2n-1}] &= \theta H^{n-1}(M; Z_2) && \text{if } n \equiv 0(4), \\ &= \theta H^{n-1}(M; Z_2) + Z_2 && \text{if } n \equiv 2(4) \text{ and } w_1 = 0, \\ &= \theta K + Z_4 && \text{if } n \equiv 2(4) \text{ and } w_1 \neq 0, \\ &= \theta H^{n-1}(M; Z) + Z_2 + Z_2 && \text{if } n \equiv 1(4) \text{ and } w_1 = 0, \\ &= \theta H^{n-1}(M; Z) + Z_4 && \text{if } n \equiv 3(4) \text{ and } w_1 = 0, \\ &= \theta H^{n-1}(M; Z[w_1]) + Z_2 && \text{if } n \equiv 1(2) \text{ and } w_1 \neq 0. \end{aligned}$$

*) This relation is different from that of Rigdon [11], but his relation can be modified as stated in the proposition by chasing his construction of θ .

PROOF OF MAIN THEOREM ON I AND E . By (2.2), (2.4) and Proposition 1.3(1), we see that

(2.9) ([11, Th. 10.4]) I is trivial if n is even.

Assume that n is odd and consider the homomorphism

$$\begin{aligned} \rho_2 i_* \theta: H^{n-1}(M; Z[w_1]) + H^n(M; Z_2) &\cong H^{2n-2}(PM; Z[v]) (= F_0/F_1) \\ &\xrightarrow{i_*} H^{2n-1}(PM; Z) (= Z_2) \xrightarrow{\rho_2} H^{2n-1}(PM; Z_2). \end{aligned}$$

Then the relation $\rho_2 i_* \theta(x, y) = v^{n-1}y$ follows from Propositions 2.4, 1.3(1) and (2.1). Therefore, by (2.6–8), we have the equalities

$$\begin{aligned} I(a, b, c) = b &\quad \text{if } n \equiv 1(4) \text{ and } w_1 = 0, \\ I(a, b) \equiv b(2) &\quad \text{if } n \equiv 3(4) \text{ and } w_1 = 0, \\ I(a, b) = b &\quad \text{if } n \equiv 1(2) \text{ and } w_1 \neq 0. \end{aligned}$$

These and (2.9) show the desired results on I . The results on E is proved by R. D. Ridgon [11, Th. 11.11 and Th. 11.26]. q. e. d.

§ 3. $j^*: F_i(\xi)/F_{i+1}(\xi) \rightarrow F_i(\eta)/F_{i+1}(\eta)$ in Proposition 1.3

In this and next sections, we investigate the homomorphism

$$J_{2n-1} = j^*: [M \subset R^{2n-1}] = [M^*, P^{2n-2}; \xi] \longrightarrow [M \subseteq R^{2n-1}] = [PM, P^{2n-2}; \eta].$$

in Proposition 1.3(2), which preserves the filtrations

$$[M^*, P^{2n-2}; \xi] = F_0(\xi) \supset F_1(\xi) \supset 0, \quad [PM, P^{2n-2}; \eta] = F_0(\eta) \supset F_1(\eta) \supset 0$$

given in Proposition 1.2.

LEMMA 3.1. $j^* = j^*: F_1(\xi) = H^{2n-1}(M^*; Z_2) \rightarrow F_1(\eta) = H^{2n-1}(PM; Z_2)$ is trivial.

PROOF. This is an immediate consequence of E. Thomas [12, Prop. 2.9(c)]. q. e. d.

Next, we study the homomorphism

$$\begin{aligned} j^* = j^*: F_0(\xi)/F_1(\xi) &= H^{2n-2}(M^*; Z[v]) \\ &\longrightarrow F_0(\eta)/F_1(\eta) = H^{2n-2}(PM; Z[v]), \end{aligned}$$

where the range $H^{2n-2}(PM; Z[v])$ is given in Proposition 2.4. Hereafter, we use essentially Propositions 5.2–3 given in §5 below.

- LEMMA 3.2. (1) *If n is even and $w_1=0$, then j^* is surjective.*
 (2) *If n is even and $w_1 \neq 0$, then $\text{Im } j^* = \theta \rho_2 H^{n-1}(M; Z) = \theta K$.*
 (3) *If n is odd and $w_1=0$, then $\text{Im } j^* = \theta \beta_2 H^{n-2}(M; Z_2)$.*
 (4) *If n is odd and $w_1 \neq 0$, then $\text{Im } j^* = \theta H^{n-1}(M; Z[w_1])$.*

PROOF. We prove the lemma by using the exact sequence

$$(3.3) \quad \cdots \longrightarrow H^{2n-2}(M^*; Z[v]) \xrightarrow{j^*} H^{2n-2}(PM; Z[v]) \\ \xrightarrow{\hat{\delta}} H^{2n-1}(\Lambda^2 M, \Delta M; Z[v]) \xrightarrow{i^*} H^{2n-1}(M^*; Z[v]) \\ \xrightarrow{j^*} H^{2n-1}(PM; Z[v]) \xrightarrow{\hat{\delta}} H^{2n}(\Lambda^2 M, \Delta M; Z[v]) \longrightarrow 0$$

in Lemma 1.4. In this sequence, the following is given by R. D. Rigdon [11, Prop. 11.9 and Prop. 11.19]:

$$(3.4) \quad H^{2n-1}(M^*; Z[v]) \cong H^{n-1}(M; Z) \quad \text{if } n \text{ is even and } w_1=0, \\ \cong Z + K \quad \text{if } n \text{ is odd and } w_1 \neq 0, \\ \cong H^{n-1}(M; Z_2) \quad \text{otherwise.}$$

(1) Assume that n is even and $w_1=0$. Then for any $z \in H^{n-1}(M; Z)$, we have $\delta \tilde{\beta}_2(v^{n-2}z') = \tilde{\beta}_2 \delta(v^{n-2}z') = \tilde{\beta}_2(v^{n-1}\Lambda z') = \tilde{\beta}_2(\Lambda z' \Lambda z' + v^{n-2}\Lambda(Sq^1 z')) = \tilde{\beta}_2 \tilde{\rho}_2 \Delta(z, z) = 0$ ($\rho_2 z = z'$) by Lemma 1.5 and [7, Lemma 10]. Therefore the first δ in (3.3) is trivial by Proposition 2.4 (1) and so (1) is shown.

(2) Assume that n is even and $w_1 \neq 0$. Then the exact sequence (3.3) is equal to

$$H^{2n-2}(M^*; Z[v]) \xrightarrow{j^*} \theta K + Z_2 \xrightarrow{\hat{\delta}} K + Z_4 \longrightarrow K + Z_2 \longrightarrow Z_2 \longrightarrow Z_2 \longrightarrow 0$$

by Proposition 2.4(1), (3.4) and Propositions 5.2–3, and so $\text{Im } \delta = Z_2$. Now $\delta \theta K = 0$ is proved in the above case. Thus $\text{Im } j^* = \text{Ker } \delta = \theta K$.

(3) Assume that n is odd and $w_1=0$. Then (3.3) induces an exact sequence

$$(3.5) \quad H^{2n-2}(M^*; Z[v]) \xrightarrow{j^*} \theta G + Z_2 \langle \tilde{\beta}_2(v^{n-3}M) \rangle \\ \xrightarrow{\hat{\delta}} G + Z_2 \langle \tilde{\beta}_2(v^{n-2}\Lambda M) \rangle \xrightarrow{i^*} K = \rho_2 G, (G \cong H^{n-1}(M; Z)),$$

by Proposition 2.4(2), (3.4) and Propositions 5.2–3. Here the relation

$$\delta \tilde{\beta}_2(v^{n-3}M) = \tilde{\beta}_2(v^{n-2}\Lambda M)$$

holds by Lemma 1.5(3), and the relation

$$(3.6) \quad \delta(\theta G) \subset G$$

holds, because $\tilde{\rho}_2 \tilde{\beta}_2(v^{n-2}\Lambda M) = v^{n-1}\Lambda M$ in $H^{2n-1}(\Lambda^2 M, \Delta M; Z_2)$ by [7, Lemma

10] and $\tilde{\rho}_2\delta\theta G = \delta\tilde{\rho}_2\theta G = 0$ by Lemma 1.5 and Proposition 2.4(2). Therefore the sequence (3.5) induces an exact sequence

$$(3.5)' \quad H^{2n-2}(M^*; Z[v]) \xrightarrow{\theta^{-1}j^*} G \xrightarrow{f} G \xrightarrow{g} K \longrightarrow 0, \quad (f = \delta\theta, g = i^*).$$

Here $K = \rho_2 G = G/2G$. Hence $g(2G) = 0$ and g induces an epimorphism $g': G/2G \rightarrow K$, which is isomorphism because $G/2G$ is finite. Therefore $2G = \text{Ker } g = \text{Im } f$. Since $\text{rank } G = \text{rank } 2G$ and $2G = \text{Im } f$, we see that

$$\text{Ker } f \subset T \text{ and } f(T) = 2T \text{ (} T \text{ is the torsion subgroup of } G\text{)}$$

by noticing that the torsion subgroup of $2G$ is equal to $2T$. Thus f determines an epimorphism

$$f|T: T \longrightarrow 2T.$$

If we can prove

$$(3.7) \quad {}_2G (= \{x \in G \mid 2x = 0\}) = \beta_2 H^{n-2}(M; Z_2) \subset \text{Ker } f \text{ in (3.5)'}$$

then ${}_2T (= \{x \in T \mid 2x = 0\}) = {}_2G \subset \text{Ker } (f|T)$ and $f|T$ induces an epimorphism $T/{}_2T \rightarrow 2T$, which is isomorphic because the orders of the two groups are finite and coincident with each other. Hence $\text{Ker } f = {}_2G$ and Lemma 3.2(3) is proved.

To show (3.7), we notice that $\theta(\beta_2 H^{n-2}(M; Z_2)) \subset \tilde{\beta}_2 H^{2n-3}(PM; Z_2)$. For any element $X \in \theta(\text{Im } \beta_2)$, there is an element $Y \in H^{2n-3}(PM; Z_2)$ such that $\tilde{\beta}_2 Y = X$ and

$$Y = \lambda v^{n-3}M + v^{n-2}x + (v^{n-1}y + v^{n-3}Sq^2y)$$

for some $\lambda \in Z_2, x \in H^{n-1}(M; Z_2)$ and $y \in H^{n-2}(M; Z_2)$ by (2.1). For $x \in H^{n-1}(M; Z_2)$, there is a relation $\tilde{\rho}_2 \tilde{\beta}_2 (v^{n-3}x) = v^{n-2}x$ and so $\tilde{\beta}_2 (v^{n-2}x) = 0$. Further the relation $\delta \tilde{\beta}_2 (v^{n-1}y + v^{n-3}Sq^2y) = 0$ for $y \in H^{n-2}(M; Z_2)$ follows from Lemma 1.5 and [7, Th. 11]. Thus $\delta X = \delta \tilde{\beta}_2 Y = \lambda \tilde{\beta}_2 (v^{n-2}AM)$ and so $\tilde{\rho}_2 \delta X = \lambda v^{n-1}AM$. This and (3.6) imply $\lambda = 0$ and so $\delta X = 0$. This completes the proof of (3.7).

(4) Assume that n is odd and $w_1 \neq 0$. Then $\tilde{\rho}_2: H^{2n-1}(\Lambda^2 M, \Delta M; Z[v]) \rightarrow H^{2n-1}(\Lambda^2 M, \Delta M; Z_2)$ is monomorphic by Proposition 5.3(iv). Further, by Lemma 1.5 and Proposition 2.4, we see that

$$\tilde{\rho}_2 \delta \tilde{\beta}_2 (v^{n-3}M) = v^{n-1}AM, \quad \tilde{\rho}_2 \delta \theta(x) = 0 \quad \text{for } x \in H^{n-1}(M; Z[w_1]).$$

Therefore $\text{Im } j^* = \text{Ker } \delta = \theta H^{n-1}(M; Z[w_1])$. q. e. d.

§4. $J_{2n-1}: [M \subset R^{2n-1}] \rightarrow [M \subseteq R^{2n-1}]$

This section is a continuation of §3 and we will determine $\text{Im } J_{2n-1}$ by using Proposition 1.3(2).

If $F_1(\eta)=0$, then $\text{Im } J_{2n-1} = \text{Im } (j^\sharp: F_0(\xi)/F_1(\xi) \rightarrow F_0(\eta)/F_1(\eta))$ and so by Proposition 1.3(2), (2.6) and Lemma 3.2, we have the following

- PROPOSITION 4.1. (1) If $n \equiv 0(4)$ and $w_1 = 0$, then $\text{Im } J_{2n-1} = [M \subseteq R^{2n-1}]$.
 (2) If $n \equiv 0(4)$ and $w_1 \neq 0$, then $\text{Im } J_{2n-1} = \theta \rho_2 H^{n-1}(M; Z) = \theta K$.
 (3) If $n \equiv 1(2)$ and $w_1 \neq 0$, then $\text{Im } J_{2n-1} = \theta H^{n-1}(M; Z[w_1])$.

In the rest of this section, we study J_{2n-1} in case when $n \equiv 1(2)$ and $w_1 = 0$, or $n \equiv 2(4)$. In these cases, $F_1(\eta) = H^{2n-1}(PM; Z_2)$ and we have to study the homomorphism

$$j_0^\sharp: \text{Ker } (j^\sharp: F_0(\xi)/F_1(\xi) \rightarrow F_0(\eta)/F_1(\eta)) \rightarrow \text{Coker } (j^\sharp: F_1(\xi) \rightarrow F_1(\eta))$$

induced from $j^\sharp: (F_0(\xi), F_1(\xi)) \rightarrow (F_0(\eta), F_1(\eta))$. By Lemma 3.1,

$$\text{Coker } (j^\sharp: F_1(\xi) \rightarrow F_1(\eta)) = F_1(\eta) = H^{2n-1}(PM; Z_2) = Z_2.$$

Further by the second half of Proposition 1.3(2),

$$(4.2) \quad \text{Im } j_0^\sharp = \text{Im } \delta^{-1} \Theta$$

where

$$\Theta = Sq^2 \tilde{\rho}_2 + (n-1)v^2 \tilde{\rho}_2: H^{2n-2}(\Lambda^2 M, \Delta M; Z[v]) \rightarrow H^{2n}(\Lambda^2 M, \Delta M; Z_2).$$

Because $H^{2n}(\Lambda^2 M, \Delta M; Z_2) = Z_2$,

(4.3) the homomorphism $\delta: H^{2n-1}(PM; Z_2) \rightarrow H^{2n}(\Lambda^2 M, \Delta M; Z_2)$ in (4.2) is an isomorphism.

We now assume that the integral cohomology groups $H^i(M; Z)$ for $i = n, n-1$ are given as in (5.1). Let K_i ($i = 1, \dots, 4$) be the subgroups of $H^{2n-2}(\Lambda^2 M, \Delta M; Z_2)$ defined as follows:

- $K_1 = \{ \Lambda \rho_2 x \Lambda \rho_2 y \mid x, y \in H^{n-1}(M; Z) \},$
- $K_2 = \{ \Lambda \rho_2 x \Lambda M \mid x \in H^{n-2}(M; Z) \}, (M = \rho_2 M \text{ if } w_1 = 0),$
- $K_3 = Z_2 \langle v^{n-2} \Lambda M \rangle,$
- $K_4 = \sum_{i=a+1}^{\beta} Z_2 \langle \Lambda M \Lambda \rho_2 y_i + (r(i)/2) \Lambda M' \Lambda \rho_2 x_i \rangle \text{ if } w_1 \neq 0.$

LEMMA 4.4. With the above notation, $\tilde{\rho}_2 H^{2n-2}(\Lambda^2 M, \Delta M; Z[v])$ is

- (1) $\sum_{i=1}^3 K_i$ if n is even and $w_1 = 0$,
- (2) $\sum_{i=1}^4 K_i$ if n is even and $w_1 \neq 0$,
- (3) $K_1 + K_2$ if n is odd and $w_1 = 0$.

PROOF. (1) Assume that n is even and $w_1 = 0$. Then $H^{2n-2}(\Lambda^2 M, \Delta M; Z_2)$ is given by [7, Th. 11] as follows:

$$H^{2n-2}(\Lambda^2 M, \Delta M; Z_2) = K_1 + K_2 + K_3 + K_5,$$

where

$$K_5 = \sum_{i=\alpha+1}^{\beta} Z_2 \langle \Lambda \rho_2 y_i \Lambda M \rangle.$$

By Lemma 1.5 and the relation $\tilde{\rho}_2 \tilde{\beta}_2 = Sq^1 + v$, we have the relations

$$\tilde{\rho}_2 \Delta(x, y) = \Lambda \rho_2 x \Lambda \rho_2 y \quad \text{for } x, y \in H^{n-1}(M; Z),$$

$$\tilde{\rho}_2 \Delta(x, M) = \Lambda \rho_2 x \Lambda \rho_2 M = \Lambda \rho_2 x \Lambda M \quad \text{for } x \in H^{n-2}(M; Z),$$

$$\tilde{\rho}_2 \tilde{\beta}_2(v^{n-3} \Lambda M) = v^{n-2} \Lambda M,$$

$$\tilde{\beta}_2(\Lambda \rho_2 y_i \Lambda M) = \tilde{\beta}_2 \tilde{\rho}_2 \Delta(y_i, \rho_{r(i)} M) = (r(i)/2) \tilde{\beta}_{r(i)} \Delta(y_i, \rho_{r(i)} M),$$

and so

$$K_1 + K_2 + K_3 \subset \tilde{\rho}_2 H^{2n-2}(\Lambda^2 M, \Delta M; Z[v]).$$

On the other hand, $(r(i)/2) \tilde{\beta}_{r(i)} \Delta(y_i, \rho_{r(i)} M)$ for $\alpha < i \leq \beta$ form a base of $\tilde{\beta}_2 H^{2n-2}(\Lambda^2 M, \Delta M; Z_2)$ by Proposition 5.3(i). This completes the proof of (1).

(2) Assume that n is even and $w_1 \neq 0$. Then we have, in the same way as the above proof,

$$H^{2n-2}(\Lambda^2 M, \Delta M; Z_2) = \sum_{i=1}^4 K_i + K_6, \quad K_6 = \{ \Lambda M' \Lambda x \mid x \in H^{n-1}(M; Z_2) \},$$

and

$$K_1 + K_3 \subset \tilde{\rho}_2 H^{2n-2}(\Lambda^2 M, \Delta M; Z[v]).$$

Moreover, we have the relations

$$\tilde{\rho}_2 \tilde{\beta}_2(\Lambda \rho_2 x \Lambda M') = \Lambda \rho_2 x \Lambda M,$$

$$\tilde{\rho}_2 \tilde{\beta}_2(\Lambda M' \Lambda \rho_2 y_i) = \Lambda M \Lambda \rho_2 y_i + (r(i)/2) \Lambda M' \Lambda \rho_2 x_i \quad \text{for } \alpha < i \leq \beta,$$

and so $K_2 + K_4 \subset \tilde{\rho}_2 H^{2n-2}(\Lambda^2 M, \Delta M; Z[v])$. On the other hand, we see that $\dim_{Z_2} \tilde{\beta}_2 H^{2n-2}(\Lambda^2 M, \Delta M; Z_2) = \beta + 1$ by Proposition 5.3(ii) and $\dim_{Z_2} K_6 = \beta + 1$. This implies (2).

(3) is obtained by the method similar to those of the above cases. q. e. d.

LEMMA 4.5. $\text{Im } j_0^{\xi}(\subset H^{2n-1}(PM; Z_2) = Z_2)$ is given as follows:

- (1) When $n \equiv 2(4)$ and $w_1 = 0$, $\text{Im } j_0^{\xi} = 0$ if and only if $w_2 \rho_2 H^{n-2}(M; Z) = 0$.
- (2) When $n \equiv 2(4)$ and $w_1 \neq 0$, $\text{Im } j_0^{\xi} = 0$ if and only if $w_2 + w_1^2 = 0$.
- (3) When $n \equiv 1(2)$ and $w_1 = 0$, $\text{Im } j_0^{\xi} = 0$ if and only if $w_2 \rho_2 H^{n-2}(M; Z) = 0$.

PROOF. The $(Sq^2 + (n-1)v^2)$ -image of K_i ($i=1, \dots, 4$) are easily obtained by using [7, Lemmas 7 and 10] as follows:

$$\begin{aligned} (Sq^2 + (n-1)v^2)(K_1 + K_3) &= 0, \\ (Sq^2 + (n-1)v^2)K_2 &= \{ASq^2\rho_2xAM \mid x \in H^{n-2}(M; Z)\}, \\ (Sq^2 + v^2)(AM\Lambda\rho_2y_i + (r(i)/2)AM'\Lambda\rho_2x_i) &= AM\Lambda Sq^2\rho_2y_i \quad (\alpha < i \leq \beta). \end{aligned}$$

Using these relations and the well-known fact that $Sq^2x = (w_2 + w_1^2)x$ for $x \in H^{n-2}(M; Z_2)$, we have

$$\begin{aligned} \Theta H^{2n-2}(\Lambda^2M, \Delta M; Z[v]) &= \{\Lambda w_2\rho_2xAM \mid x \in H^{n-2}(M; Z)\} \\ &\quad \text{in cases (1) and (3),} \\ &= \{\Lambda(w_2 + w_1^2)xAM \mid x \in H^{n-2}(M; Z_2)\} \text{ in case (2).} \end{aligned}$$

This and (4.3) show the lemma. q. e. d.

We are now ready to determine $\text{Im } J_{2n-1}$ for $n \equiv 1(2)$ and $w_1 = 0$, or $n \equiv 2(4)$.

PROPOSITION 4.6. (1) *Assume that $n \equiv 2(4)$ and $w_1 = 0$. Then*

$$\text{Im } J_{2n-1} = \begin{cases} [M \subseteq R^{2n-1}] & \text{if } w_2\rho_2H^{n-2}(M; Z) \neq 0, \\ \theta H^{n-1}(M; Z_2) & \text{otherwise.} \end{cases}$$

(2) *Assume that $n \equiv 2(4)$ and $w_1 \neq 0$. Then*

$$\text{Im } J_{2n-1} = \begin{cases} \theta K + Z_2 & \text{if } w_2 + w_1^2 \neq 0, \\ \theta K & \text{otherwise.} \end{cases}$$

(3) *Assume that $n \equiv 1(2)$ and $w_1 = 0$. Then*

$$\begin{aligned} \text{Im } J_{2n-1} &= \theta\beta_2H^{n-2}(M; Z_2) && \text{if } w_2\rho_2H^{n-2}(M; Z) = 0, \\ &= \theta\beta_2H^{n-2}(M; Z_2) + 0 + Z_2 && \text{if } n \equiv 1(4) \text{ and } w_2\rho_2H^{n-2}(M; Z) \neq 0, \\ &= \theta\beta_2H^{n-2}(M; Z_2) + Z_2 && \text{otherwise.} \end{aligned}$$

PROOF. This is an immediate consequence of Lemmas 3.1, 3.2, 4.5 and (2.7). q. e. d.

Propositions 4.1 and 4.6 give the results on J_{2n-1} in Main Theorem. Thus Main Theorem in the introduction is proved.

§5. Appendix on $H^i(\Lambda^2M, \Delta M; Z[v])$ for $i \geq 2n-3$

In the previous sections, the cohomology of $(\Lambda^2M, \Delta M)$ plays an important part. L. L. Larmore [7] investigated it but the author can not understand the proof of [7, Th. 20]. Therefore we should like to try to describe the cohomology

groups $H^{2n}(\Lambda^2 M, \Delta M; \mathbb{Z})$ and $H^i(\Lambda^2 M, \Delta M; \mathbb{Z}[v])$ for $i \geq 2n - 3$ by using the notations and the results stated in [7, pp. 908–915]. We note that Propositions 5.4–5 for $i = 2n - 2, 2n - 3$ are not used in this paper and are prepared for the forthcoming paper [14].

Let M be a closed connected n -manifold and assume that

$$(5.1) \quad \begin{aligned} H^n(M; \mathbb{Z}) &= \mathbb{Z}\langle M \rangle \text{ if } w_1 = 0, \quad = \mathbb{Z}_2\langle \beta_2 M' \rangle \text{ (Sq}^1 M' = M) \text{ if } w_1 \neq 0, \\ H^m(M; \mathbb{Z}) &= \sum_{i=1}^{\gamma(m)} \mathbb{Z}_{r(m,i)}\langle x_{m,i} \rangle \text{ (direct sum) for } m \leq n - 1, \\ x_{m,i} &= \beta_{r(m,i)} y_{m,i} \text{ (} y_{m,i} \in H^{m-1}(M; \mathbb{Z}_{r(m,i)}) \text{ for } \alpha(m) < i \leq \gamma(m), \end{aligned}$$

where the order $r(m, i)$ is infinite for $1 \leq i \leq \alpha(m)$, a power of 2 for $\alpha(m) < i \leq \beta(m)$ and a power of an odd prime for $\beta(m) < i \leq \gamma(m)$, and if $\alpha(m) < i < j$ then either $(r(m, i), r(m, j)) = 1$ or $r(m, i) \mid r(m, j)$ holds.

Furthermore, for the simplicity,

(5.1)' denote $\alpha(m), \beta(m), \gamma(m), r(m, i), x_{m,i}$ and $y_{m,i}$ in (5.1) respectively by

$$\begin{aligned} \alpha, \beta, \gamma, r(i), x_i \text{ and } y_i & \quad \text{when } m = n - 1, \\ \alpha', \beta', \gamma', r'(i), x'_i \text{ and } y'_i & \quad \text{when } m = n - 2. \end{aligned}$$

Then we have the following propositions, where (i)'s, ..., (iv)'s hold respectively when

- (i) n is even and $w_1 = 0$, (ii) n is even and $w_1 \neq 0$,
- (iii) n is odd and $w_1 = 0$, (iv) n is odd and $w_1 \neq 0$.

PROPOSITION 5.2. (1) $H^{2n}(\Lambda^2 M, \Delta M; \mathbb{Z}[v])$ is

- (i) $\mathbb{Z}_2\langle \tilde{\beta}_2(v^{n-1} \Lambda \rho_2 M) \rangle$, (ii) $\mathbb{Z}_2\langle \tilde{\beta}_2(v^{n-1} \Lambda M) \rangle$,
- (iii) $\mathbb{Z}\langle \Delta(M, M) \rangle$, (iv) $\mathbb{Z}_2\langle \tilde{\beta}_2(\Lambda M' \Lambda M) \rangle$.

(2) $H^{2n}(\Lambda^2 M, \Delta M; \mathbb{Z})$ is

- (i) $\mathbb{Z}\langle \Lambda M \Lambda M \rangle$, (ii) $\mathbb{Z}_2\langle \beta_2(\Lambda M' \Lambda M) \rangle$,
- (iii) $\mathbb{Z}_2\langle \beta_2(v^{n-1} \Lambda \rho_2 M) \rangle$, (iv) $\mathbb{Z}_2\langle \beta_2(v^{n-1} \Lambda M) \rangle$.

PROPOSITION 5.3. $H^{2n-1}(\Lambda^2 M, \Delta M; \mathbb{Z}[v])$ is

- (i) G , (ii) $\mathbb{Z}_4\langle (1/2)\tilde{\beta}_2(v^{n-1} \Lambda M') \rangle + K$,
- (iii) $\mathbb{Z}_2\langle \tilde{\beta}_2(v^{n-2} \Lambda \rho_2 M) \rangle + G$,
- (iv) $\mathbb{Z}_2\langle \tilde{\beta}_2(v^{n-2} \Lambda M) \rangle + K$, and $\tilde{\rho}_2: H^{2n-1}(\Lambda^2 M, \Delta M; \mathbb{Z}[v]) \rightarrow H^{2n-1}(\Lambda^2 M, \Delta M; \mathbb{Z}_2)$ is monomorphic,

where

$$G = \sum_{i=1}^{\alpha} Z \langle \Delta(x_i, M) \rangle + \sum_{i=\alpha+1}^{\gamma} Z_{r(i)} \langle \tilde{\beta}_{r(i)} \Delta(y_i, \rho_{r(i)} M) \rangle (\cong H^{n-1}(M; Z)),$$

$$K = \sum_{i=1}^{\beta} Z_2 \langle \tilde{\beta}_2 \Delta(M', \rho_2 x_i) \rangle (\cong \text{Im}(\rho_2: H^{n-1}(M; Z) \longrightarrow H^{n-1}(M; Z_2))).$$

PROPOSITION 5.4. $H^{2n-2}(\Lambda^2 M, \Delta M; Z[v])$ is

- (i) $Z_2 \langle \tilde{\beta}_2(v^{n-3} \Lambda \rho_2 M) \rangle + G_1 + G_2 + G_3 + G_4 + G_6,$
- (ii) $Z_2 \langle \tilde{\beta}_2(v^{n-3} \Lambda M) \rangle + G_1 + G_2 + G_3 + G_4 + G_7,$
- (iii) $G_1 + G_3 + G_5 + G_6,$
- (iv) $Z_2 \langle \tilde{\beta}_2(v^{n-2} \Lambda M') \rangle + G_1 + G_3 + G_5 + G_7,$

where

$$G_1 = \sum_{1 \leq i < j \leq \alpha} Z \langle \Delta(x_i, x_j) \rangle, \quad G_2 = \sum_{i=1}^{\alpha} Z \langle \Delta(x_i, x_i) \rangle,$$

$$G_3 = (\sum_{1 \leq i \leq \alpha < j \leq \gamma} + \sum_{\alpha < j < i \leq \gamma}) Z_{r(j)} \langle \tilde{\beta}_{r(j)} \Delta(y_j, \rho_{r(j)} x_i) \rangle,$$

$$G_4 = \sum_{i=\alpha+1}^{\gamma} Z_{r(i)} \langle \tilde{\beta}_{r(i)} \Delta(y_i, \rho_{r(i)} x_i) \rangle, \quad G_5 = \sum_{i=1}^{\beta} Z_2 \langle \tilde{\beta}_2(v^{n-2} \Lambda \rho_2 x_i) \rangle,$$

$$G_6 = \sum_{k=1}^{\alpha'} Z \langle \Delta(x'_k, M) \rangle + \sum_{k=\alpha'+1}^{\gamma'} Z_{r'(k)} \langle \tilde{\beta}_{r'(k)} \Delta(y'_k, \rho_{r'(k)} M) \rangle (\cong H^{n-2}(M; Z)),$$

$$G_7 = \sum_{k=1}^{\beta'} Z_2 \langle \tilde{\beta}_2 \Delta(M', \rho_2 x'_k) \rangle + \sum_{i=\alpha+1}^{\beta} Z_2 \langle \tilde{\beta}_2 \Delta(M', \rho_2 y_i) \rangle (\cong H^{n-2}(M; Z_2)).$$

PROPOSITION 5.5. $\tilde{\rho}_2 H^{2n-3}(\Lambda^2 M, \Delta M; Z[v]) / \delta H^{2n-4}(PM; Z_2)$ is isomorphic to

- (i) $H,$ (ii) $H + H_4,$ (iii) $H + H_5,$ (iv) $H + H_4 + H_5,$

where

$$H = H_1 + H_2 + H_3,$$

$$H_1 = \begin{cases} \{\Lambda \rho_2 x \Lambda \rho_2 M \mid x \in H^{n-3}(M; Z)\} & \text{if } w_1 = 0, \\ \{\Lambda \rho_2 x \Lambda M \mid x \in H^{n-3}(M; Z)\} & \text{if } w_1 \neq 0, \end{cases}$$

$$H_2 = \{\Lambda \rho_2 x \Lambda \rho_2 y \mid x \in H^{n-2}(M; Z), y \in H^{n-1}(M; Z)\},$$

$$H_3 = \sum_{\alpha < i < j \leq \beta} Z_2 \langle \Lambda \rho_2 x_i \Lambda \rho_2 y_j + (r(j)/r(i)) \Lambda \rho_2 y_i \Lambda \rho_2 x_j \rangle,$$

$$H_4 = \sum_{k=\alpha'+1}^{\beta'} Z_2 \langle \Lambda \rho_2 y'_k \Lambda M + (r'(k)/2) \Lambda \rho_2 x'_k \Lambda M' \rangle,$$

$$H_5 = \sum_{i=\alpha+1}^{\beta} Z_2 \langle \Lambda \rho_2 y_i \Lambda \rho_2 x_i \rangle.$$

To prove these propositions, we use the following results frequently:

(5.6) ([7, p. 914]) For any cyclic group G , there is an exact sequence

$$\dots \longrightarrow H^{i-1}(\Lambda^2 M, \Delta M; G) \xrightarrow{V_*} H^i(\Lambda^2 M, \Delta M; G[v])$$

$$\xrightarrow{\pi^*} H^i(M^2, \Delta M; G) \longrightarrow H^i(\Lambda^2 M, \Delta M; G) \longrightarrow \dots,$$

where $\pi: (M^2, \Delta M) \rightarrow (\Lambda^2 M, \Delta M)$ is the natural projection, v is the first Stiefel-Whitney class of the double covering $M^2 - \Delta M \rightarrow \Lambda^2 M - \Delta M = M^*$ and $V = \tilde{\beta}_2(1) \in H^1(M^*; Z[v])$.

(5.7) For any positive integer p , there is the Bockstein exact sequence

$$\begin{aligned} \dots \longrightarrow H^{i-1}(\Lambda^2 M, \Delta M; Z_p[v]) \xrightarrow{\tilde{\beta}_p} H^i(\Lambda^2 M, \Delta M; Z[v]) \\ \xrightarrow{\times p} H^i(\Lambda^2 M, \Delta M; Z[v]) \xrightarrow{\tilde{\beta}_p} H^i(\Lambda^2 M, \Delta M; Z_p[v]) \xrightarrow{\tilde{\beta}_p} \dots \end{aligned}$$

(5.8) ([7, Remark 13]) For any odd prime p , $\pi^*: H^*(\Lambda^2 M, \Delta M; Z_p[v]) \rightarrow H^*(M^2, \Delta M; Z_p)$ is monomorphic and

$$\text{Im } \pi^* = \{x \mid x \in H^*(M^2, \Delta M; Z_p), t^*x = -x\},$$

where $t: (M^2, \Delta M) \rightarrow (M^2, \Delta M)$ is a map defined by $t(x, y) = (y, x)$.

(5.9) (cf. [7, p. 914]) For $x \in H^r(M; Z_t)$ and $y \in H^s(M; Z_t)$ ($t \leq \infty$),

$$\pi^* \Delta(x, y) = x \otimes y - (-1)^{rs} y \otimes x \quad \text{and} \quad \pi^* \Delta x = x \otimes 1 - 1 \otimes x,$$

and moreover the order of $\Delta(x, y)$ for $x \neq y$ is the greatest common factor of those of x and y , and the order of Δx is equal to that of x .

We now sketch the proofs of Propositions 5.2-5.

By (5.6), the following relation holds:

$$\text{rank } H^i(\Lambda^2 M, \Delta M; Z[v]) + \text{rank } H^i(\Lambda^2 M, \Delta M; Z) = \text{rank } H^i(M^2, \Delta M; Z).$$

By using (5.9) and Lemma 1.5(1), we can choose generators (mod torsions) of $H^i(\Lambda^2 M, \Delta M; Z[v])$ and $H^i(\Lambda^2 M, \Delta M; Z)$. In particular we have

LEMMA 5.10. There hold the following congruences mod torsions:

$$(1) \quad H^{2n}(\Lambda^2 M, \Delta M; Z) \equiv \begin{cases} Z\langle \Lambda M \Lambda M \rangle & \text{if } n \text{ is even and } w_1 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$(2) \quad H^{2n}(\Lambda^2 M, \Delta M; Z[v]) \equiv \begin{cases} Z\langle \Delta(M, M) \rangle & \text{if } n \text{ is odd and } w_1 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$(3) \quad H^{2n-1}(\Lambda^2 M, \Delta M; Z[v]) \equiv \begin{cases} \sum_{i=1}^n Z\langle \Delta(x_i, M) \rangle & \text{if } w_1 = 0, \\ 0 & \text{if } w_1 \neq 0. \end{cases}$$

(4) $H^{2n-2}(\Lambda^2 M, \Delta M; Z[v])$ is congruent mod torsion to the direct sum of G_1 and

$$\begin{aligned} G_2 & \quad \text{if } n \text{ is even,} \\ \sum_{k=1}^{n-1} Z\langle \Delta(x'_k, M) \rangle & \quad \text{if } w_1 = 0. \end{aligned}$$

To determine the odd torsion subgroup of $H^i(\Lambda^2 M, \Delta M; Z[v])$, let p be an odd prime. Then the Z_p -base of $H^i(\Lambda^2 M, \Delta M; Z_p[v])$ can be determined by (5.8–9). Thus the p -primary component and its generators of $H^i(\Lambda^2 M, \Delta M; Z[v])$ are determined by the exact sequence (5.7) for odd prime p and [7, Remark 16]. In particular we have

LEMMA 5.11. Denote by T_o^i the odd torsion subgroup of $H^i(\Lambda^2 M, \Delta M; Z[v])$. Then

- (1) $T_o^{2n} = 0$;
- (2) $T_o^{2n-1} = \begin{cases} (G)_o = \sum_{i=\beta+1}^{\gamma} Z_{r(i)} \langle \tilde{\beta}_{r(i)} \Delta(y_i, \rho_{r(i)} M) \rangle & \text{if } w_1 = 0, \\ 0 & \text{if } w_1 \neq 0; \end{cases}$
- (3) T_o^{2n-2} is the direct sum of

$$(G_3)_o = (\sum_{1 \leq i \leq \alpha, \beta < j \leq \gamma} + \sum_{\beta < j < i \leq \gamma}) Z_{r(j)} \langle \tilde{\beta}_{r(j)} \Delta(y_j, \rho_{r(j)} x_i) \rangle$$

and

$$\begin{aligned} (G_4)_o &= \sum_{i=\beta+1}^{\gamma} Z_{r(i)} \langle \tilde{\beta}_{r(i)} \Delta(y_i, \rho_{r(i)} x_i) \rangle && \text{if } n \text{ is even,} \\ (G_6)_o &= \sum_{k=\beta'+1}^{\gamma'} Z_{r'(k)} \langle \tilde{\beta}_{r'(k)} \Delta(y'_k, \rho_{r'(k)} M) \rangle && \text{if } w_1 = 0. \end{aligned}$$

The proof of (1) of Proposition 5.2 is given by using Lemmas 5.10–11, (5.7) for $p=2$ and [7, Th. 11], and that of (2) is given by using the ordinary Bockstein exact sequence instead of (5.7).

In the rest of this section, we study the 2-primary components of $H^i(\Lambda^2 M, \Delta M; Z[v])$ for $2n-3 \leq i \leq 2n-1$. First we consider the case (ii) n is even and $w_1 \neq 0$. By (5.7) for $p=2$, Lemmas 5.10–11 and [7, Th. 11], we have

$$\begin{aligned} H^{2n-1}(\Lambda^2 M, \Delta M; Z[v]) &= K + Z_s (K = \sum_{i=1}^{\beta} Z_2 \langle \tilde{\beta}_2 \Delta(M', \rho_2 x_i) \rangle), \\ \tilde{\rho}_2 Z_s &= Z_2 \langle v^{n-1} \Lambda M + \Lambda M' \Lambda M \rangle, \text{ for some integer } s \geq 2. \end{aligned}$$

In the exact sequence (3.3), both groups $H^{2n-2}(PM; Z[v])$ and $H^{2n-1}(M^*; Z[v])$ are isomorphic to $H^{n-1}(M; Z_2)$ by Proposition 2.4 and (3.4) and so $s \leq 4$. On the other hand,

$$\tilde{\rho}_2 \tilde{\beta}_2 H^{2n-2}(\Lambda^2 M, \Delta M; Z_2) \ni v^{n-1} \Lambda M + \Lambda M' \Lambda M$$

follows briefly. Thus $s \geq 4$ and so $s=4$. Moreover by (5.7) for $p=2$, (5.9) and Lemmas 1.5 and 5.10, we see that

$$Z_s = Z_4 \langle (1/2) \tilde{\beta}_2 (v^{n-1} \Lambda M') \rangle,$$

and

$$H^{2n-2}(\Lambda^2 M, \Delta M; Z[v]) \equiv Z_2 \langle \tilde{\beta}_2(v^{n-3} \Lambda M) \rangle + G_1 + G_2 + G_7 \\ + (\sum_{1 \leq i \leq \alpha < j \leq \beta} + \sum_{\alpha < j < i \leq \beta}) Z_{r(j)} \langle \Delta(x_j, x_i) \rangle + \sum_{j=\alpha+1}^{\beta} Z_{s(j)} \langle \Delta(x_j, x_j) \rangle \\ \text{mod odd torsion,}$$

where $s(j)$ is the order of $\Delta(x_j, x_j)$. As for the element $\Delta(x_j, x_i)$, if $i \neq j$ then $\pi^* \tilde{\beta}_{r(j)} \Delta(y_j, \rho_{r(j)} x_i) = \pi^* \Delta(x_j, x_i)$ by (5.9) and hence

$$\tilde{\beta}_{r(j)} \Delta(y_j, \rho_{r(j)} x_i) = \Delta(x_j, x_i) + VX_{j,i} \text{ for some } X_{j,i} \in H^{2n-3}(\Lambda^2 M, \Delta M; Z)$$

by (5.6). Further we see easily that

$$\tilde{\rho}_2 \tilde{\beta}_{r(j)} \Delta(y_j, \rho_{r(j)} x_i) = \Lambda \rho_2 x_j \Lambda \rho_2 x_i + v \rho_2 X_{j,i} \neq 0$$

by Lemma 1.5. Therefore we can replace $\Delta(x_j, x_i)$ by $\tilde{\beta}_{r(j)} \Delta(y_j, \rho_{r(j)} x_i)$. If $i=j$ and $r(j)=2$, then

$$\tilde{\rho}_2 \tilde{\beta}_2 \Delta(y_j, \rho_2 x_j) = \Lambda \rho_2 x_j \Lambda \rho_2 x_j = \tilde{\rho}_2 \Delta(x_j, x_j)$$

by Lemma 1.5, and so $s(j)=2$ and $\Delta(x_j, x_j)$ can be replaced by $\tilde{\beta}_2 \Delta(y_j, \rho_2 x_j)$. If $i=j$ and $r(j) \geq 4$, then we see easily that

$$\tilde{\beta}_{r(j)} \Delta(y_j, \rho_{r(j)} x_j) = \Delta(x_j, x_j) + VY_j \quad (Y_j \in H^{2n-3}(\Lambda^2 M, \Delta M; Z))$$

by (5.6) and (5.9), and that

$$\tilde{\beta}_2(\Lambda \rho_2 y_j \Lambda \rho_2 x_j) \neq 0$$

by (5.7) for $p=2$. Using Lemma 1.5 and the relation $\tilde{\beta}_2 \tilde{\rho}_2 = (r/2) \tilde{\beta}_r: H^{i-1}(\Lambda^2 M, \Delta M; Z_r[v]) \rightarrow H^i(\Lambda^2 M, \Delta M; Z[v])$, we see that

$$(r(j)/2) \tilde{\beta}_{r(j)} \Delta(y_j, \rho_{r(j)} x_j) = \tilde{\beta}_2 \tilde{\rho}_2 \Delta(y_j, \rho_{r(j)} x_j).$$

The above three relations imply that $s(j)=r(j)$ and $\Delta(x_j, x_j)$ can be replaced by $\tilde{\beta}_{r(j)} \Delta(y_j, \rho_{r(j)} x_j)$. This completes the proofs of (ii)'s of Propositions 5.3–4. The proof of Proposition 5.5(ii) is given by Lemma 1.5(3) and (5.7) for $p=2$ immediately.

The proofs of (i)'s, (iii)'s and (iv)'s of Propositions 5.3–5 are similar to, but simpler than, those of (ii)'s except the results concerning H_5 of Proposition 5.5 for odd n .

Let n be odd. By simple calculations, using Lemma 1.5, Proposition 5.4 and (5.9), we see that

$$\tilde{\beta}_2(\Lambda \rho_2 y_j \Lambda \rho_2 x_j) \in \text{Ker } \pi^* \quad (\subset H^{2n-2}(\Lambda^2 M, \Delta M; Z[v]))$$

and

$$\text{Ker } \pi^* = \{ \tilde{\beta}_2(v^{n-2} \Lambda x) \mid x \in H^{n-1}(M; Z_2) \}.$$

This implies that there is an element $X_j \in H^{n-1}(M; \mathbb{Z}_2)$ such that

$$\Lambda \rho_2 y_j \Lambda \rho_2 x_j + v^{n-2} \Lambda X_j \in \text{Im } \tilde{\rho}_2.$$

Using this result, Lemma 1.5(3) and (5.7) for $p=2$, we have Proposition 5.5(iii)–(iv) completely.

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