

Analysis of Potential Problems by the Use of the Cylindrical Harmonics Involving the Modified Bessel Functions of Purely Imaginary Order $K_{is}(x)$ and $I_{is}(x)$

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It is well known that there are three kinds of cylindrical harmonics which are used to express the potential functions in the cylindrical coordinates. The three kinds of cylindrical harmonics have their own regions of rapid convergence and are called the z -, r - and φ -form. The cylindrical harmonics of the z - and r -form are very popular and are oftenly used in many field of physics and engineering. However, the cylindrical harmonics of the φ -form which involve the modified Bessel functions of purely imaginary order $K_{is}(x)$ and $I_{is}(x)$ have not been used for the analysis of some practical problems.

In this paper, we discussed the possibility of analysis of potential problems based on the cylindrical harmonics of the φ -form. Usually the expression of the φ -form is given in a series of some kinds of integral with respect to the order s . It is shown that this type of integration is carried out efficiently by a method which is a variation of the Double Exponential quadrature formula recently presented by Takahashi and Mori. By comparing the computing times which are required for the analysis of two analogous problems with non-axisymmetry, it is concluded that the complexity of this new method of analysis is the same in order as that of ordinary method using the expressions of the r -form. As an applications of this method, we analyzed some potential problems in connection with the four point probe technique.

§ 1 Introduction

It is known¹⁾ that the potential function can be expressed by the three different cylindrical harmonics, what are called, the z -, r - and φ -forms. It is advantageous to make these cylindrical harmonics usable, as each has its own region of rapid convergence. The potential functions of the z - and r -forms are expressed in terms of well known Bessel functions or Modified Bessel functions and they have been used for the analysis of many potential problems. However in the cylindrical harmonics of the φ -form the special form of Modified Bessel functions $K_\nu(x)$ and $I_\nu(x)$ when ν is purely imaginary is are used. Since the property of these functions and the method of evaluation of these function were not known at all, the cylindrical harmonics of the φ -form had not been used for the analysis of practical potential problem.

In the preceeding papers²⁾, the authors had given the detailed investigation on the Modified Bessel functions of purely imaginary order $K_{is}(x)$ and $I_{is}(x)$.

As a second step of study, we here discuss the possibility of analysis of potential problems based on the cylindrical harmonics of the φ -form.

The expression of the φ -form usually has an integral form with respect to the order s . In order to integrate the integral effectively, the bahaviors of the integrand must be known.

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At first, in this section, the behaviors of $K_{is}(x)$ and its related functions are shown as functions of the order s and it is shown that a modified method of the Double Exponential quadrature formula recently presented by Takahashi and Mori³⁾ is suitable for such integrals.

Some examples of numerical integrations with respect to the order s are given and computing times of the new method of analysis based on the expression of the φ -form are compared with that of ordinary one based on the r -form. Last of all, it is concluded that the new method of analysis using the φ -form is available for some kinds of potential problems as well as other methods based on the expression of the r - and z -form.

§ 2 Double Exponential Quadrature Formula

Recently Takahashi and Mori³⁾ have presented Double Exponential Quadrature Formula which is to apply the trapezoidal rule after the integral variable is transformed such that the integrand decays double exponentially.

We treat here a following integral:

$$I = \int_0^{\infty} f(x) e^{-x} dx, \quad (1)$$

where the function $f(x)$ is assumed to vary quietly.

Applying a following transformation to (1)

$$x = \exp \{u - \exp(-u)\}, \quad (2)$$

we obtain

$$I = \int_{-\infty}^{+\infty} f[\exp \{u - \exp(-u)\}] \times \{1 + \exp(-u)\} \times \exp \{u - \exp(-u)\} \\ \times \exp[-\exp \{u - \exp(-u)\}] du. \quad (3)$$

It should be noticed that the integral over semi-infinite region from 0 to $+\infty$ is transformed into an integral over infinite region from $-\infty$ to $+\infty$ and the integrand of (3) decays double exponentially as the variable tends to $\pm\infty$.

Integrating (3) by trapezoidal rule with step width h , we obtain

$$I_h = h \sum_{n=-\infty}^{n=+\infty} f[\exp \{nh - \exp(-nh)\}] \times \{1 + \exp(-nh)\} \times \exp \{nh - \exp(-nh)\} \\ \times \exp[-\exp \{nh - \exp(-nh)\}] \quad (4)$$

According to Takahashi and Mori's theory, if $f(x)$ varies quietly the error of above integral ΔI_h is roughly estimated by

$$|\Delta I_h| \simeq \exp(-\pi^2/h) \quad (5)$$

The pivots and weights of Double exponential quadrature formula applied to semi-infinite region 0 to infinity are shown in Table 1.

The Laguerre-Gauss quadrature formula is considered to be suitable for an integral such as is given in (1). However when $f(x)$ does not vary quietly, it becomes unsuitable because

that the computations corresponding to pivots far from zero become meaningless.

For such case, the Double exponential rule becomes suitable. In a practical application of this formula the error estimations must be carried out at every computation and the integration must be stopped when the contribution of a pivot becomes negligible. When the combination of the step width h and the error bound used for the error estimation is pertinent, this Double exponential quadrature formula works very efficiently. One remarkable merit of this quadrature formula is that we need not to provide data for pivots and weights. This merit becomes prominent when some quadrature formulas of different number of pivots must

Table 1. Coefficients of Double exponential quadrature formula for semi-infinite region from 0 to ∞ .

$h=0.5$			
u	pivots X_i	weights W_i	$W_i \exp(X_i)$
-2.50	0.42021727E-06	0.55395092E-05	0.55395116E-05
-2.00	0.83634362E-04	0.70155467E-03	0.70161335E-03
-1.50	0.25245585E-02	0.13803952E-01	0.13838845E-01
-1.00	0.24275642E-01	0.88098851E-01	0.90263678E-01
-0.50	0.11663320E 00	0.27491935E 00	0.30892885E 00
0.0	0.36787944E 00	0.50929276E 00	0.73575888E 00
0.50	0.89894749E 00	0.58778082E 00	0.14441867E 01
1.00	0.18815964E 01	0.39210950E 00	0.25737970E 01
1.50	0.35853993E 01	0.12158810E 00	0.43854100E 01
2.00	0.64537717E 01	0.11537197E-01	0.73271947E 01
2.50	0.11222436E 02	0.16237024E-03	0.12143630E 02
3.00	0.19110022E 02	0.10068972E-06	0.20061454E 02
$h=0.3$			
u	pivots X_i	weights W_i	$W_i \exp(X_i)$
-3.30	0.61938279E-13	0.17412484E-11	0.17412484E-11
-3.00	0.94206089E-10	0.19863859E-08	0.19863859E-08
-2.70	0.23185673E-07	0.36818225E-06	0.36818226E-06
-2.40	0.14804328E-05	0.17799478E-04	0.17799504E-04
-2.10	0.34790377E-04	0.31888341E-03	0.31889450E-03
-1.50	0.25245587E-02	0.13803953E-01	0.13838846E-01
-1.20	0.10887751E-01	0.46527015E-01	0.47036357E-01
-0.90	0.34749049E-01	0.11611221E 00	0.12021792E 00
-0.60	0.88733411E-01	0.22915326E 00	0.25041623E 00
-0.30	0.19207703E 00	0.37247678E 00	0.45135389E 00
0.0	0.36787944E 00	0.50929276E 00	0.73575888E 00
0.30	0.64350967E 00	0.58862107E 00	0.11202334E 01
0.60	0.10525211E 01	0.56901709E 00	0.16301570E 01
0.90	0.16379252E 01	0.44783026E 00	0.23038559E 01
1.20	0.24566675E 01	0.27401312E 00	0.31966016E 01
1.50	0.35853992E 01	0.12158811E 00	0.43854099E 01
1.80	0.51279250E 01	0.35428231E-01	0.59755653E 01
2.10	0.72249734E 01	0.59052689E-02	0.81097178E 01
2.40	0.10067194E 02	0.46611608E-03	0.10980469E 02
2.70	0.13912594E 02	0.13473904E-04	0.14847597E 02
3.00	0.19110022E 02	0.10068978E-06	0.20061454E 02
3.30	0.26130855E 02	0.12144995E-09	0.27094644E 02
3.60	0.35611771E 02	0.12511376E-13	0.36584818E 02

Table 2. Coefficients of modified Double exponential quadrature formula based on the transformation (6) for $h=0.3$. and $\alpha=2.25$.

u	pivots X_i	weights W_i	$W_i \exp(X_i)$
-1.50	0.45353742E-13	0.30275727E-11	0.30275727E-11
-1.20	0.10391094E-06	0.35827860E-05	0.35827864E-05
-0.90	0.20838756E-03	0.37598304E-02	0.37606140E-02
-0.60	0.11592169E-01	0.11090983E 00	0.11220301E 00
-0.30	0.10393049E 00	0.50761171E 00	0.56320703E 00
0.0	0.36787944E 00	0.82760074E 00	0.11956082E 01
0.30	0.81126832E 00	0.77336465E 00	0.17406589E 01
0.60	0.14060154E 01	0.54566485E 00	0.22261309E 01
0.90	0.21554643E 01	0.32386940E 00	0.27956073E 01
1.20	0.31043194E 01	0.16029984E 00	0.35737309E 01
1.50	0.43309281E 01	0.61361331E-01	0.46643696E 01
1.80	0.59451610E 01	0.16177533E-01	0.61782133E 01
2.10	0.80940503E 01	0.25208515E-02	0.82556003E 01
2.40	0.10973501E 02	0.19011015E-03	0.11085017E 02
2.70	0.14845553E 02	0.53271651E-05	0.14922366E 02
3.00	0.20062032E 02	0.38966161E-07	0.20114885E 02
3.30	0.27096479E 02	0.46306671E-10	0.27132826E 02

be provided. At the neighbor of the zero, the distances between the adjacent pivots in the Table 1 are very close to each other, in other words, this quadrature formula involves a fair over-estimation. Such an over-estimation near zero will be removed, keeping its character of double exponential rule, by the method based on the following transformation:

$$x = \exp(u - \exp(-\alpha u)), \quad (6)$$

where $\alpha > 1$.

In Table 2, the coefficients of modified Double exponential quadrature formula based on the transformation (6) are given. As will be shown later this rule works more efficiently in some cases than the original Double exponential quadrature formula presented by Takahashi and Mori.

§3 Behaviors of the Expression of the φ -form as functions of the order

If we want to analyze the problem, we must often carry out some kinds of integral with respect to the order s such as:

$$I_0(a) = \int_0^\infty \{\cosh(s\pi/2) - 1\} K_{is}^2(a) ds, \quad (7)$$

$$I_1(a, \rho) = \int_0^\infty \{\cosh(s\pi/2) - 1\} \frac{\{I_{is}(\rho a) K_{is}(a) - I_{is}(a) K_{is}(\rho a)\}^2}{I_{is}(\rho a) I_{-is}(\rho a)} ds,$$

where $\rho > 1$.

These two integrals occur in the analyses of some kinds of potential problems in the four-point probe technique.

As shown already, $K_{is}(x)$ has the following asymptotic property²⁾:

$$K_{is}(x) \simeq \sqrt{\frac{2\pi}{s}} e^{-s\pi/2} \sin(\pi/4 + s \log(2s/ex)), \quad \text{for large } s. \quad (9)$$

In the Fig. 1, the absolute values of $K_{is}(x)$ are plotted in semi-logarithmic scale. From the figure we can see that for small x , $K_{is}(x)$ oscillates rapidly and its envelope is nearly equal to $\sqrt{2\pi/s} e^{-s\pi/2}$, while for large x , the oscillation becomes slow and the values of $K_{is}(x)$ are far from the envelope, i.e., actual values of $K_{is}(x)$ decay more slowly than $\sqrt{2\pi/s} e^{-s\pi/2}$.

By the use of (9), the integrand of (7) is given by

$$\sqrt{\frac{2\pi}{s}} e^{-s\pi/2} \sin^2(\pi/4 + s \log(2s/ex)), \quad \text{for large } s. \quad (10)$$

The actual behaviors of the integrand of (7) are conjectured fairly from Fig. 1.

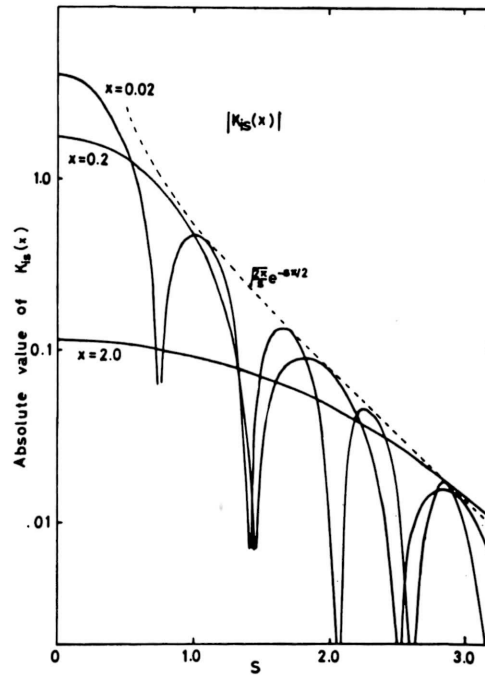


Fig. 1. Behaviors of $K_{is}(x)$ as a function of the order s .

When x is large, the integrand of (6) can not be considered to decay as $(2\pi/s)^{1/2} e^{-\pi s/2}$, it decays more slowly.

While the other integral (8), which coincides with (7) for large ρ , is unwieldy, because the functions $I_{is}(x)$ and $I_{-is}(x)$ are complex. As shown in the previous paper,

$$I_{is}(x) = \frac{\cosh(s\pi)}{\pi} M_{is}(x) - i \frac{\sinh(s\pi)}{\pi} K_{is}(x), \quad (11)$$

$$I_{-is}(x) = \frac{\cosh(s\pi)}{\pi} M_{is}(x) + i \frac{\sinh(s\pi)}{\pi} K_{is}(x), \quad (12)$$

Hence, we obtain

$$\frac{\{I_{is}(\rho a)K_{is}(a) - K_{is}(\rho a)M_{is}(a)\}^2}{I_{is}(\rho a)I_{-is}(\rho a)} = \frac{\{M_{is}(\rho a)K_{is}(a) - K_{is}(\rho a)M_{is}(a)\}^2}{M_{is}^2(\rho a) + \{\tanh(s\pi)K_{is}(\rho a)\}^2} \quad (13)$$

Consider the following function $f_1(x, s, \rho)$

$$f_1(x, s, \rho) = \frac{M_{is}(\rho x)K_{is}(x) - K_{is}(\rho x)M_{is}(x)}{[M_{is}^2(\rho x) + \{\tanh(s\pi)K_{is}(\rho x)\}^2]^{1/2}}. \quad (14)$$

If ρ tends to infinity, this function $f_1(x, s, \rho)$ approaches to $K_{is}(x)$. The asymptotic property of this function for large order is

$$f_1(x, s, \rho) \simeq \sqrt{\frac{2\pi}{s}} e^{-s\pi/2} \sin(s \log \rho), \quad (15)$$

where

$$M_{is}(x) \simeq \sqrt{\frac{2\pi}{s}} e^{-s\pi/2} \cos(\pi/4 + s \log(2s/ex)),$$

is used. Some numerical values of $f_1(x, s, \rho)$ are plotted in Fig. 2. When ρ approaches to unity, the oscillation of $f_1(x, s, \rho)$ becomes slow as is shown easily from (15).

For the problem with Neumann condition on the cylindrical surface, the following integral appears:

$$I_2(a, \rho) = \int_0^\infty \{\cosh(s\pi/2) - 1\} \frac{\{I'_{is}(\rho a)K_{is}(a) - K'_{is}(\rho a)I_{is}(a)\}^2}{I'_{is}(\rho a)I'_{-is}(\rho a)} ds. \quad (16)$$

For this case, the function

$$f_2(x, s, \rho) = \frac{M'_{is}(\rho x)K_{is}(x) - K'_{is}(\rho x)M_{is}(x)}{\sqrt{M_{is}^{\prime 2}(\rho x) + \{\tanh(s\pi)K'_{is}(\rho x)\}^2}} \quad (17)$$

plays the corresponding role to $f_1(x, s, \rho)$ or $K_{is}(x)$. This function $f_2(x, s, \rho)$ also approaches to $K_{is}(x)$ when ρ becomes large, and has the following asymptotic property for large s :

$$f_2(x, s, \rho) \simeq \sqrt{\frac{2\pi}{s}} e^{-s\pi/2} \cos(s \log \rho). \quad (18)$$

The behaviors of $f_2(x, s, \rho)$ are shown in Fig. 3.

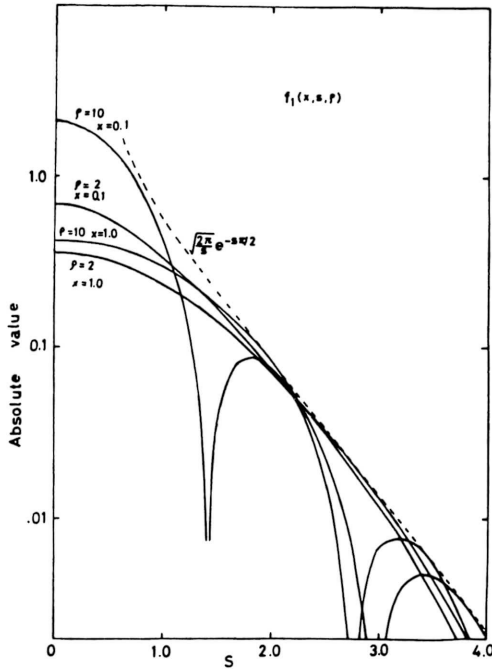


Fig. 2. Behaviors of $f_1(x, s, \rho)$ as a function of the order s .

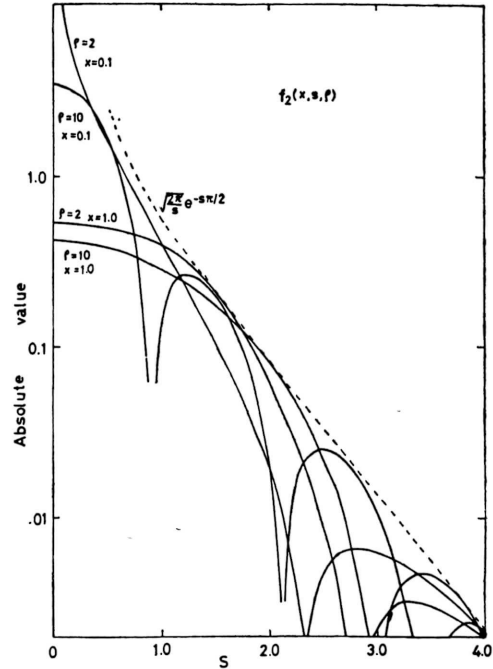


Fig. 3. Behaviors of $f_2(x, s, \rho)$ as a function of the order s .

§ 4 Numerical Integration with respect to the order s

In the book by Erdelyi⁽⁴⁾ et al, the following formulas are given:

$$\int_0^\infty K_{is}(z) \cos(sx) ds = \frac{\pi}{2} e^{-z} \cosh(y), \quad (19)$$

$$\int_0^\infty K_{is}(z) \cosh(\pi s/2) \cos(sy) ds = \frac{\pi}{2} \cos(z \sinh(y)), \quad (20)$$

$$\int_0^\infty K_{is}(z) \sinh(\pi s/2) \sin(sy) ds = \frac{\pi}{2} \sin(z \sinh(y)). \quad (21)$$

Example 1

Putting $y=0$ in (19), we obtain

$$\int_0^\infty K_{is}(z) ds = \frac{\pi}{2} e^{-z}. \quad (22)$$

The integrand of (22) decays exponentially. Hence it may be used as a test device of the expression of the φ -form. In Table 3, the relative errors of numerical integration of (22) are shown. For numerical integration, the procedure of evaluating the values of $K_{is}(x)$ stated in the previous paper²⁾ is used. This procedure has an accuracy of at least eight decimal places

for x and $s < 30$. In the table, a blank column means that the accuracy higher than 10^{-8} is obtained. The number of times of evaluating $K_{is}(x)$ is denoted by N . In the table, the errors of integration increase as z becomes small, because that the oscillation of $K_{is}(x)$ is rapid for small z . From these results we see that when z is large, the Laguerre-Gauss quadrature formula yields a higher accuracy than the Double exponential quadrature rule. While for small z , we cannot say necessarily that Double exponential rule is superior to the Laguerre-Gauss rule. This is because of a fair over-estimation near the zero mentioned previously.

In Table 4, the relative errors of numerical integration of (22) by the aid of the modified Double exponential rule for $\alpha=2.25$ are shown. From these results, we see that the over-estimation near zero is removed and the number of times of evaluation of $K_{is}(x)$ is reduced to a large extent.

Table 3. Relative errors of numerical integration of (22) using the original Double exponential quadrature formula.

Z	Double exponential quadrature formula based on (2).								Laguerre- Gauss 15-point formula.
	h=0.4		h=0.25		h=0.15		h=0.1		
	rel.error	N	rel.error	N	rel.error	N	rel.error	N	
0.01	3.9×10^{-2}	11	1.3×10^{-3}	18	1.2×10^{-4}	33	5.7×10^{-6}	60	5.1×10^{-2}
0.02	3.9×10^{-2}	11	3.4×10^{-3}	18	3.9×10^{-5}	34	1.4×10^{-6}	60	1.2×10^{-3}
0.05	2.3×10^{-2}	11	8.0×10^{-4}	19	1.7×10^{-5}	35	3.3×10^{-7}	60	4.5×10^{-3}
0.1	1.0×10^{-3}	12	5.2×10^{-4}	20	1.2×10^{-5}	35	$7. \times 10^{-8}$	62	3.2×10^{-3}
0.2	6.6×10^{-3}	12	1.2×10^{-4}	20	6.1×10^{-7}	36			9.2×10^{-4}
0.5	1.9×10^{-3}	12	8.6×10^{-5}	20	2.8×10^{-7}	40			1.8×10^{-5}
1.0	7.7×10^{-4}	13	1.3×10^{-6}	21	1.0×10^{-8}	42			1.1×10^{-5}
2.0	1.4×10^{-4}	13	3.5×10^{-6}	20					6.6×10^{-7}
5.0	9.4×10^{-5}	14	4.6×10^{-7}	23					1.5×10^{-8}
10.0	2.3×10^{-5}	14	9.0×10^{-8}	24					

Table 4. Relative errors of numerical integration of (22) using the modified Double exponential quadrature formula ($\alpha=2.25$).

Z	Modified Double exponential quadrature formula based on (6).								Laguerre- Gauss 15- point formula.
	h=0.4		h=0.25		h=0.15		h=0.1		
	rel.error	N	rel.error	N	rel.error	N	rel.error	N	
0.01	3.7×10^{-2}	8	7.9×10^{-4}	15	1.5×10^{-4}	24	2.8×10^{-6}	40	5.1×10^{-2}
0.02	1.7×10^{-2}	8	2.0×10^{-4}	15	2.0×10^{-5}	26	9.8×10^{-7}	42	1.2×10^{-3}
0.05	1.5×10^{-2}	8	9.0×10^{-5}	15	1.1×10^{-5}	26	9.9×10^{-8}	43	4.5×10^{-3}
0.1	6.5×10^{-3}	9	1.5×10^{-5}	15	6.1×10^{-6}	28	3.4×10^{-8}	43	3.2×10^{-3}
0.2	6.1×10^{-4}	9	1.0×10^{-5}	15	9.0×10^{-7}	28			9.2×10^{-4}
0.5	9.4×10^{-4}	9	9.5×10^{-6}	15	3.3×10^{-8}	30			1.8×10^{-5}
1.0	5.1×10^{-4}	9	7.0×10^{-6}	16	1.0×10^{-8}	30			1.1×10^{-5}
2.0	1.8×10^{-4}	10	1.9×10^{-6}	17					6.6×10^{-7}
5.0	4.0×10^{-5}	11	2.0×10^{-8}	20					1.5×10^{-8}
10.0	1.5×10^{-5}	11							

This modified Double exponential rule works more efficiently than the original Double exponential rule or Laguerre-Gauss rule. For large z , the Laguerre Gauss rule is still superior

to all. However, since the integration for small values of z is essential in most cases in the potential theory as shown in a forthcoming example, this modified Double exponential quadrature formula will become more available in practical cases.

Example 2

As an instructive example of the integration with respect to the order s , we take the problem of the flow of electricity in a parallel plate conductor with thickness c and of infinite extent. Consider four points A, B, C , and D which are placed squarely on the surface of the conductor. How is the potential difference between A and B , if the current source and sink are placed at C and D ?

Using the following condition $A(1, 0, 0)$, $B(1, \pi/2, 0)$, $C(1, \pi, 0)$ and $D(1, 3\pi/2, 0)$ the potential difference is determined as follows

$$\Delta V = \frac{I\rho}{4\pi c} \left[\log 2 + \frac{8}{\pi} \sum_{p=1}^{\infty} \int_0^{\infty} \{ \cosh(s\pi/2) - 1 \} K_{is}^2(p\pi/c) ds \right], \quad (23)$$

from the expression of the φ -form (11), where I is the current and ρ is the resistivity of the conductor.

This problem also can be treated by the method of images as follows:

$$\Delta V = \frac{I\rho}{4\pi c} \left[\frac{\sqrt{2}-1}{2\alpha} + \sum_{n=1}^{\infty} \left\{ \frac{1}{\sqrt{\alpha^2/2+n^2}} - \frac{1}{\sqrt{\alpha^2+n^2}} \right\} \right] \quad (24)$$

where $\alpha = l/c$.

In order to integrate (23) effectively, we must vary the step width h of trapezoidal rule depending on the variable x . In the numerical experiments, we adopt the following step width as an example:

$$h = \exp \{ -\log_{10}(20/x) \} \times k \quad (25)$$

In other words, when $k=1$, (25) gives the following values:

$$\begin{aligned} \text{for } x &= 20, 2, 0.2, 0.02, \\ h &= 1, 1/e, 1/e^2, 1/e^3. \end{aligned}$$

The numerical results for the case $k=1.0, 0.5$ and 0.3 are shown in Table 5. The data in the column of Laguerre-Gauss rule are the results of applying indiscriminately this 15-point rule to (23) whether x is large or not. In the table the net number of times of evaluating $K_{is}(x)$ is denoted by N .

From the table when $k=0.3$, the accuracy of seven decimal digits is obtained. Hence the step width of trapezoidal rule (25) used for the numerical integration of (23) seems to be reasonable.

Example 3

In the analysis of some kinds of potential problems in the four-point probe technique, the following integral appears⁵⁾.

$$\Delta V = \frac{I\rho}{4\pi c} \left[\log 2 - \log \frac{\{1 + (l/a)^2\}^2}{1 + (l/a)^4} \right]$$

$$+ \frac{8}{\pi} \sum_{p=1}^{\infty} \int_0^{\infty} \{ \cosh(s\pi/2) - 1 \} \times f_1(p\pi l/c, s, a/l)^2 ds \}, \quad (26)$$

and

$$\Delta V = \frac{I\rho}{4\pi c} \left[\log 2 + \log \frac{\{1 + (l/a)^2\}^2}{1 + (l/a)^4} \right. \\ \left. + \frac{8}{\pi} \sum_{p=1}^{\infty} \int_0^{\infty} \{ \cosh(s\pi/2) - 1 \} \times f_2(p\pi l/c, s, a/l)^2 ds \right]. \quad (27)$$

Table 5. Numerical integration of (23) using the modified Double exponential quadrature formula ($\alpha=2.25$).

l/c	15-point Laguerre -Gauss	Modified Double Exponential formula			Exact values
		$k=1.0$	$k=0.5$	$k=0.3$	
1.5	2.51846 $N=45$	2.519 $N=18$	2.5184 586 $N=30$	2.5184 5861 $N=43$	2.5184 5861
1.0	1.7309 5 $N=75$	1.7304 $N=27$	1.7309 38 $N=40$	1.7309 368 $N=63$	1.7309 3641
0.8	1.4561 9 $N=75$	1.455 $N=33$	1.4562 4 $N=44$	1.4562 578 $N=70$	1.4562 5763
0.6	1.2320 $N=105$	1.233 $N=47$	1.2318 0 $N=68$	1.2318 205 $N=108$	1.2318 2039
0.4	1.0802 $N=135$	1.078 $N=66$	1.0803 3 $N=98$	1.0803 431 $N=170$	1.0803 4343
0.3	1.038 $N=180$	1.038 $N=88$	1.0360 25 $N=135$	1.0360 2413 $N=215$	1.0360 2415
0.25	1.020 $N=210$	1.0209 $N=112$	1.0213 75 $N=158$	1.0213 737 $N=248$	1.0213 7371
0.20	1.007 $N=270$	1.01118 $N=131$	1.0111 6 $N=194$	1.0111 737 $N=321$	1.0111 7393
0.16	1.009 $N=315$	1.0060 $N=160$	1.0057 8 $N=245$	1.0057 989 $N=380$	1.0057 9920

In these expressions, the functions $f_1(x, s, \rho)$ and $f_2(x, s, \rho)$ play the same role as that of $K_{is}(x)$ in eq. (23). Since these functions oscillate more quietly than the function $K_{is}(x)$ for $\rho \gg 1$ as stated previously, the integrations of (26) and (27) are comparatively easy. The integration of (26) and (27) with respect to the order s were carried out utilizing the modified Double Exponential quadrature formula the step width of which is varied depending on the integrand behavior.

The numerical results are tabulated in Table 6 and 7 with the parameters of the ratio l to a and the ratio l to c . In these Tables, the values of $C.D. = \frac{\sqrt{2}-1}{2} \frac{4\pi l}{\rho} \left(\frac{\Delta V}{I} \right)$ are given⁵⁾.

If we use the other forms, i.e., the z - or r -form for the analysis of these problems, even the accuracy of only one or two decimal places will not be obtained with a considerable computational labor.

Table 6. The Results of Numerical Estimation of eq. (26).

a/l l/c	0.2	0.3	0.4	0.5	0.6	0.7	0.8	1.0	1.2	1.5
1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.9	0.0337614	0.0337615	0.0337615	0.0337631	0.0337753	0.0338281	0.0339819	0.0348710	0.0368287	0.0417641
0.8	0.131686	0.131866	0.131687	0.131710	0.131846	0.132364	0.133627	0.139539	0.150665	0.175561
0.7	0.281688	0.281688	0.281701	0.281869	0.282717	0.285174	0.290221	0.310254	0.343107	0.408649
0.6	0.460726	0.460728	0.460833	0.461748	0.465219	0.473368	0.487701	0.535959	0.605467	0.732603
0.5	0.640382	0.640477	0.641133	0.644923	0.655619	0.676232	0.707655	0.799126	0.917703	1.12187
0.4	0.794522	0.794806	0.797935	0.809929	0.835456	0.876171	0.931251	1.07549	1.24857	1.53493
0.3	0.906065	0.908093	0.918924	0.946775	0.994309	1.05869	1.13835	1.33423	1.55964	1.92364
0.2	0.970729	0.980722	1.00436	1.05454	1.12183	1.20569	1.30623	1.54505	1.81325	2.24072

Table 7. The Results of Numerical Estimation of eq.(27).

a/l l/c	0.2	0.3	0.4	0.5	0.6	0.7	0.8	1.0	1.2	1.5
1.0	2.95016	2.95073	2.95197	2.96928	3.01662	3.10014	3.22753	3.61732	4.14192	5.05732
0.9	2.65317	2.65387	2.65869	2.68968	2.75691	2.86904	3.03176	3.48799	4.05546	4.99871
0.8	2.35741	2.35857	2.37096	2.41553	2.50947	2.65725	2.85315	3.34982	3.92901	4.86299
0.7	2.04724	2.05030	2.07326	2.13987	2.26150	2.43474	2.64916	3.16965	3.72965	4.62877
0.6	1.74772	1.75494	1.79350	1.88378	2.02722	2.21391	2.43227	2.92805	3.46568	4.30437
0.5	1.48138	1.49672	1.55401	1.66269	1.81523	1.99981	2.20668	2.66286	3.15241	3.91509
0.4	1.26988	1.29756	1.37023	1.48450	1.62863	1.79552	1.98053	2.38687	2.82206	3.50175
0.3	1.12733	1.16737	1.24079	1.34512	1.47122	1.61406	1.77291	2.12748	2.51154	3.11352
0.2	1.05142	1.09392	1.15249	1.24015	1.34444	1.46541	1.60309	1.91608	2.25848	2.79727

§ 5 Computing time of this method of analysis

In order to check the complexity of this method of analysis based on the expression of the φ -form, the computing time of (26) is compared with that of an analogous and non-axisymmetric problem which must be analyzed by the use of the expression of the r -form.

In a certain potential problem for which the expression of the r -form is suitable, the following expression appears:

$$\Delta V = \frac{I\rho}{\pi c} \left[\log 2 + \log \frac{4a^2 + 3s^2}{4a^2 - 3s^2} + 8 \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \frac{I_{2m-1}(p\pi s/2c)}{I_{2m-1}(p\pi a/c)} \{ I_{2m-1}(p\pi a/c) K_{2m-1}(3p\pi s/2c) - I_{2m-1}(3p\pi s/2c) K_{2m-1}(p\pi a/c) \} \right]. \quad (28)$$

In Table 8, the number of times of computation required for the estimations of (26) or (28) accurately to six decimal digits are given with some parameters.

The total computing time of the expression of the φ -form (26) is given by $T = 14\text{ms} \times 2 \times N$, where 14ms is the mean computing time of the procedure developed in the previous paper and the factor 2 denotes that the twice use of the procedure is necessary for the estimation of $f_1(x, s, \rho)$. The corresponding total computing time of the expression of the r -form (28) is given by $T = 3\text{ms} \times 6 \times N$, where 3ms is the mean computing time of ordinary Bessel function and the factor 6 is used because that for estimation of the expression under

the double series sign of (28) we must compute six Bessel functions.

Table 8. Comparison of computing time of the methods of analysis.

	Number of times of computation					Computing time
The φ -form (26)	l/a	1.0	1.0	0.7	0.7	$T = 14 \text{ ms} \times 2 \times N$
	l/c	0.5	0.3	0.5	0.3	
	N	57	98	71	118	
The r -form (28)	s/a	2/3	2/3	0.5	0.5	$T = 3 \text{ ms} \times 6 \times N$
	s/c	0.5	0.25	0.5	0.25	
	N	42	78	42	78	

§ 6 Conclusion

The cylindrical harmonics of the φ -form which involve $K_{is}(x)$ and $I_{is}(x)$ are available for some kinds of potential problems which can not be analyzed by the ordinary cylindrical harmonics. In this paper, the numerical method of analysis of potential problems using the cylindrical harmonics of the φ -form is investigated. Usually the potential function of the φ -form has an integral form with respect to the order s . It was shown that the Double exponential type quadrature formulas are suitable for the numerical integration of the integral involved in the expressions of the φ -form. However, it was also shown that the original Double exponential quadrature formula contained a fair over-estimation. And a more efficient quadrature formula suitable for our purpose were derived by modifying the original formula.

The complexity of this new method is compared with that of ordinary one and concluded to be equal in its order to that of ordinary method based on the cylindrical harmonics of the r -form.

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