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Numerical Calculation of Tide in Kagoshima Bay

Part 1. Two Dimensional Explicit Method

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Abstract

The tide in Kagoshima bay is calculated by an explicit two dimensional numerical method. For the time differentiation, the two-step Lax-Wendroff difference method is adopted and for the space, the two dimensional finite element method is used. The estimated tidal residual flow shows the existence of three large vortices in Kagoshima bay, i.e., clockwise vortices at south and north of Nishi-Sakurajima channel and an anti-clockwise vortex at the center of the bay. These vortices qualitatively agree with the experimental ones.

1. Introduction

Water exchange in a semi-closed bay is essential for the environment of fish cultures and especially for comfortable human lives around the bay. In addition to experimental investigation, we believe, the numerical model calculation is respectable to make clear the primary factors of the water exchange in the bay. Kawaharada et al.[1] investigated the tidal current in Kagoshima bay by using one dimensional implicit model and showed that the tidal current induces only rather small movement of water mass. Although the one dimensional model is much advantageous economically, geographical features can not be accounted in this scheme. In this paper, we improve their analysis to the two dimensional model by applying an explicit numerical method. Recently, explicit numerical method is shown[2] to be more appropriate than the implicit one in order to solve the hyperbolic type differential equation. According to Kawahara et al.[3], we adopt the two-step Lax-Wendroff difference method[4] for the time differentiation and divide the surface of the bay by the two dimensional simplex elements.

The water exchange is caused by the tidal residual flow, the density current and the wind-driven current. The last two effects can be rightly estimated only by the three dimensional formulation[5]. The three dimensional calculation have to be finally performed. It requires, however, large capacity of a computer and long computational time. Even the two dimensional estimation could give the rough features of the water exchange in the bay, especially if the water in the bay is stratified and the wind blows gently.

In section 2, we give the formulation. Results are given in section 3. Section 4 is devoted to discussions.

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2. Formulation

For complete fluid, Euler's equations of motion and equation of continuity are written as:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+g\frac{\partial \eta}{\partial x}-fv=0, \tag{2.1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+g\frac{\partial \eta}{\partial y}+fu=0, \tag{2.2}$$

$$\frac{\partial \eta}{\partial t}+\frac{\partial (Hu)}{\partial x}+\frac{\partial (Hv)}{\partial y}=0, \tag{2.3}$$

where u and v are vertically averaged water velocities of x and y directions respectively, η is the water elevation, $H=h+\eta$ with h denoting the depth, g is the gravitational constant and f is the Coriolis parameter. In order to solve Eqs. (2.1)–(2.3) numerically in the bay, we first divide the surface of the bay by the two dimensional simplex element (see Fig. 1). Concentrating on one of the elements and following the usual formulations of the method of weighted residuals[6] we consider in that element, the variational functionals:

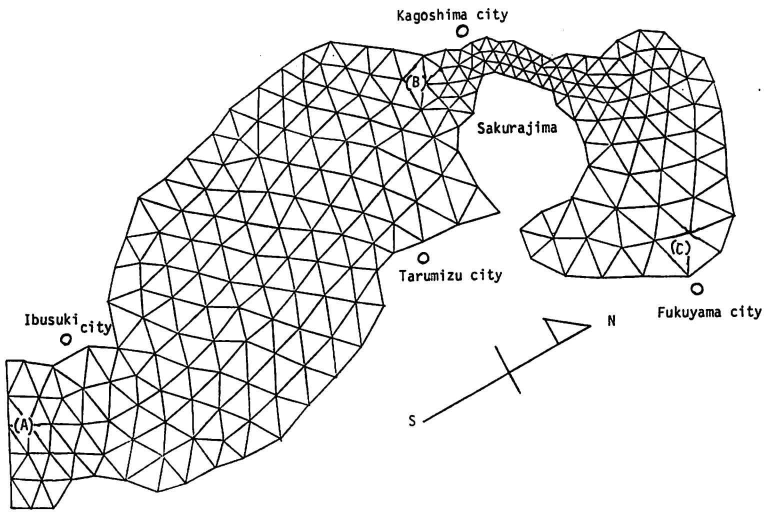

Fig. 1. Division of Kagoshima bay by the two dimensional simplex elements.

$$\delta\chi_u^e=\int_A u^*\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+g\frac{\partial \eta}{\partial x}-fv\right)dA, \tag{2.4}$$

$$\delta\chi_v^e=\int_A v^*\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+g\frac{\partial \eta}{\partial y}+fu\right)dA, \tag{2.5}$$

$$\delta\chi_\eta^e=\int_A \eta^*\left(\frac{\partial \eta}{\partial t}+\frac{\partial H}{\partial x}u+H\frac{\partial u}{\partial x}+\frac{\partial H}{\partial y}v+H\frac{\partial v}{\partial y}\right)dA, \tag{2.6}$$

where dA denotes the two dimensional integral and u^*, v^* and η^* are weighting functions.

Assuming linear dependences on coordinates for u, v, η and h in a simplex element, it follows:

$$q = q_\alpha L_\alpha \ (\alpha = 1-3), \tag{2.7}$$

where q represents u, v, η and h and summation convention is used. In Eq. (2.7) q_α is the value of q at the α-th vertex of the triangle and L_α ($\alpha=1$–3) are the barycentric coordinates:

$$L_\alpha = \frac{1}{2A}(a_\alpha + b_\alpha x + c_\alpha y), \tag{2.8}$$

where A is the area of the simplex element and

$$\begin{aligned} a_\alpha &= x_\beta y_\gamma - x_\gamma y_\beta, \\ b_\alpha &= y_\beta - y_\gamma, \\ c_\alpha &= x_\gamma - x_\beta. \ (\alpha,\ \beta,\ \gamma \text{ cyclic}) \end{aligned} \tag{2.9}$$

According to the Galerkin's method, weighting function q^* representing u^*, v^* and η^* is also written as:

$$q^* = q^*_\alpha L_\alpha \ (\alpha = 1-3), \tag{2.10}$$

where q^*_α is the arbitrary variation at the α-th vertex of the triangle element. Substituting Eqs. (2.7)–(2.10) into Eqs. (2.4)–(2.6) we obtain:

$$\begin{aligned} \delta\chi^e_u = u^*_\alpha \int_A (&L_\alpha L_\beta \dot{u}_\beta + L_\alpha L_\beta L_{\gamma,x} u_\beta u_\gamma + L_\alpha L_\beta L_{\gamma,y} v_\beta u_\gamma \\ &+ g L_\alpha L_{\beta,x} \eta_\beta - f L_\alpha L_\beta v_\beta)\, dA, \end{aligned} \tag{2.11}$$

$$\begin{aligned} \delta\chi^e_v = v^*_\alpha \int_A (&L_\alpha L_\beta \dot{v}_\beta + L_\alpha L_\beta L_{\gamma,x} u_\beta v_\gamma + L_\alpha L_\beta L_{\gamma,y} v_\beta v_\gamma \\ &+ g L_\alpha L_{\beta,y} \eta_\beta + f L_\alpha L_\beta u_\beta)\, dA, \end{aligned} \tag{2.12}$$

$$\begin{aligned} \delta\chi^e_\eta = \eta^*_\alpha \int_A (&L_\alpha L_\beta \dot{\eta}_\beta + L_\alpha L_{\beta,x} L_\gamma H_\beta u_\gamma + L_\alpha L_\beta L_{\gamma,x} H_\beta u_\gamma \\ &+ L_\alpha L_{\beta,y} L_\gamma H_\beta v_\gamma + L_\alpha L_\beta L_{\gamma,y} H_\beta v_\gamma)\, dA, \end{aligned} \tag{2.13}$$

where

$$H_\alpha = h_\alpha + \eta_\alpha.$$

We require the functionals $\delta\chi^e_u$, $\delta\chi^e_v$ and $\delta\chi^e_\eta$ vanish for arbitrary u^*_α, v^*_α and η^*_α.

The time developments of u, v and η are given by using the two-step Lax-Wendroff method as follows:

$$q^{n+\frac{1}{2}}=q^n+\frac{\Delta t}{2}\dot{q}^n, \tag{2.14}$$

$$q^{n+1}=q^n+\Delta t\dot{q}^{n+\frac{1}{2}}. \tag{2.15}$$

Our final equations for one element are given by:

$$M_{\alpha\beta}u_\beta^{n+\frac{1}{2}}=M_{\alpha\beta}u_\beta^n-\frac{\Delta t}{2}(K^x_{\alpha\beta\gamma}u^n_\beta u^n_\gamma+K^y_{\alpha\beta\gamma}v^n_\beta u^n_\gamma$$
$$+gN^x_{\alpha\beta}\eta^n_\beta-fM_{\alpha\beta}v^n_\beta), \tag{2.16}$$

$$M_{\alpha\beta}v_\beta^{n+\frac{1}{2}}=M_{\alpha\beta}v_\beta^n-\frac{\Delta t}{2}(K^x_{\alpha\beta\gamma}u^n_\beta v^n_\gamma+K^y_{\alpha\beta\gamma}v^n_\beta v^n_\gamma$$
$$+gN^y_{\alpha\beta}\eta^n_\beta+fM_{\alpha\beta}u^n_\beta) \tag{2.17}$$

$$M_{\alpha\beta}\eta_\beta^{n+\frac{1}{2}}=M_{\alpha\beta}\eta_\beta^n-\frac{\Delta t}{2}\{K^x_{\alpha\beta\gamma}(H^n_\beta u^n_\gamma+u^n_\beta H^n_\gamma)$$
$$+K^y_{\alpha\beta\gamma}(H_{\beta n}v^n_\gamma+v^n_\beta H^n_\gamma)\}, \tag{2.18}$$

$$M_{\alpha\beta}u_\beta^{n+1}=M_{\alpha\beta}u_\beta^n-\Delta t(K^x_{\alpha\beta\gamma}u_\beta^{n+\frac{1}{2}}u_\gamma^{n+\frac{1}{2}}+K^y_{\alpha\beta\gamma}v_\beta^{n+\frac{1}{2}}u_\gamma^{n+\frac{1}{2}}$$
$$+gN^x_{\alpha\beta}\eta_\beta^{n+\frac{1}{2}}-fM_{\alpha\beta}v_\beta^{n+\frac{1}{2}}), \tag{2.19}$$

$$M_{\alpha\beta}v_\beta^{n+1}=M_{\alpha\beta}v_\beta^n-\Delta t(K^x_{\alpha\beta\gamma}u_\beta^{n+\frac{1}{2}}v_\gamma^{n+\frac{1}{2}}+K^y_{\alpha\beta\gamma}v_\beta^{n+\frac{1}{2}}v_\gamma^{n+\frac{1}{2}}$$
$$+gN^y_{\alpha\beta}\eta_\beta^{n+\frac{1}{2}}+fM_{\alpha\beta}u_\beta^{n+\frac{1}{2}}), \tag{2.20}$$

$$M_{\alpha\beta}\eta_\beta^{n+1}=M_{\alpha\beta}\eta^n-\Delta t\{K^x_{\alpha\beta\gamma}(H_\beta^{n+\frac{1}{2}}u_\gamma^{n+\frac{1}{2}}+u_\beta^{n+\frac{1}{2}}H_\gamma^{n+\frac{1}{2}})$$
$$+K^y_{\alpha\beta\gamma}(H_\beta^{n+\frac{1}{2}}v_\gamma^{n+\frac{1}{2}}+v_\beta^{n+\frac{1}{2}}H_\gamma^{n+\frac{1}{2}})\}, \tag{2.21}$$

where

$$M_{\alpha\beta}=\int_A L_\alpha L_\beta dA,$$

$$K^x_{\alpha\beta\gamma}=\int_A L_\alpha L_\beta L_{\gamma,x}dA,\quad K^y_{\alpha\beta\gamma}=\int_A L_\alpha L_\beta L_{\gamma,y}dA,$$

$$N^x_{\alpha\beta}=\int_A L_\alpha L_{\beta,x}dA,\quad N^y_{\alpha\beta}=\int_A L_\alpha L_{\beta,y}dA. \tag{2.22}$$

The integrals in Eq. (2.22) are easily performed by using the formula[7]:

$$\int_A L_1^a L_2^b L_3^c dA=\frac{a!b!c!}{(a+b+c+2)!}2A. \tag{2.23}$$

As the matrix M is not a diagonal one, $q_\alpha^{n+1/2}$ and q_α^{n+1} can not be solved explicitly. We then adopt the lamped mass matrix technique[8], that is, instead of the original mass matrix M in the left-hand sides of Eqs. (2.16)–(2.21) we use the diagonal mass matrix in which all of the off-diagonal elements are summed to the diagonal ones. Owing to this technique our scheme becomes completely explicit.

By summing Eqs. (2.16)–(2.21) over all elements, the velocities and elevations of all nodal points at time t can be explicitly calculated if those at time $t-\Delta t$ are known.

3. Results

Our division of Kagoshima bay by the two dimensional simplex elements is given in Fig. 1. The numbers of element and nodal point are 417 and 255, respectively. We have carefully divided the bay to use only acute type triangles. In the explicit numerical method, the time step is limited[9] by the inequality:

$$\Delta t \leqq \frac{\Delta x}{\sqrt{gH}}. \tag{3.1}$$

Nishi-Sakurajima channel gives the upper limit of Δt to be about 50 seconds. For safety, we have chosen 30 seconds for the time step. As our purpose is the model calculation, we give the sine curve with amplitude 1.0 meter and period 12.5 hours for the elevation at the entrance of Kagoshima bay. We adopt the slip boundary condition, i.e., the velocity perpendicular to the shoreline vanishes. Initial condition is that the vertically averaged velocities at all nodal points are vanished and the surface of the bay is flat. The real depths at all nodal points are read out from the chart and utilized.

Calculation was performed over 30 hours. The vertically averaged velocity vectors after 20 and 26 hours are given in Figs. 2 and 3 respectively. The 12.5 hours averaged

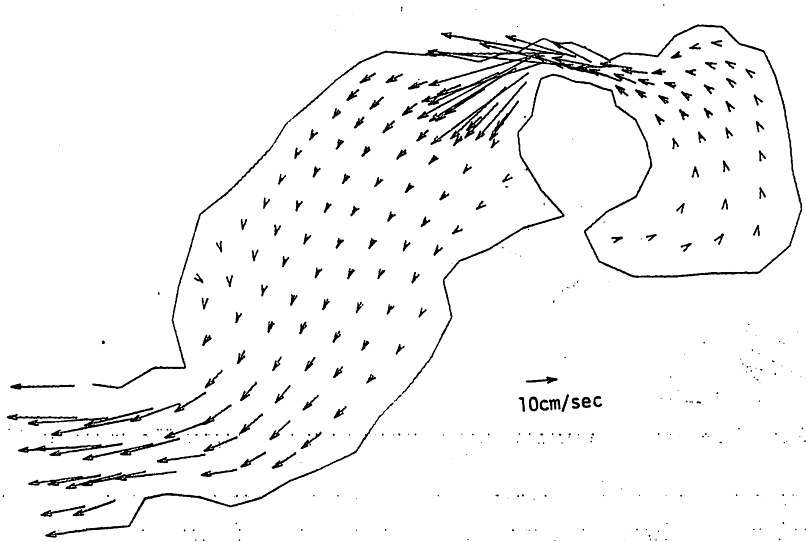

Fig. 2. Vertically averaged velocity vectors after 20 hours.

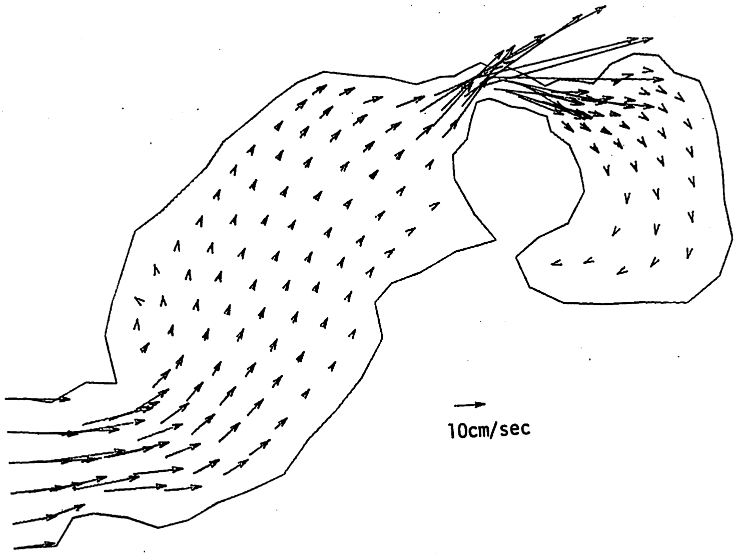

Fig. 3. Vertically averaged velocity vectors after 26 hours.

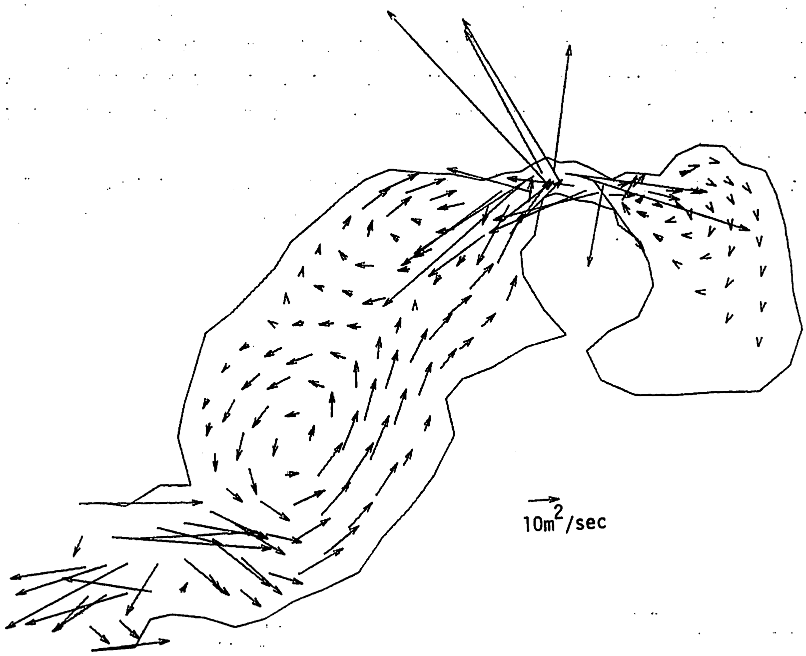

Fig. 4. Sum of the vertically averaged velocity vectors times their depths H over the last 12.5 hours.

mass transport (vertically averaged velocity vectors times their depth H $(=h+\eta)$) is shown in Fig. 4. Three large vortices can be seen in the figure, i.e., clockwise vortices at south and north of Nishi-Sakurajima channel and an anti-clockwise vortex at the

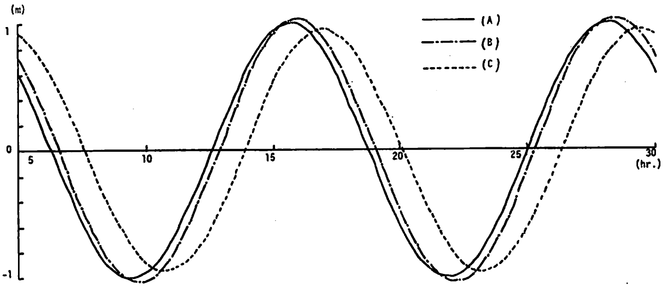

Fig. 5. Elevations at (A), (B) and (C) shown in Fig. 1.

center of the bay. The experiment[10] also seems to show such vortices. The elevations at the entrance (A) (close to our input), at the off Kagoshima city (B) and off Fukuyama city (C) are given in Fig. 5. The phase differences of (B) and (C) with respect to (A) can be seen to be about 20 and 90 minutes respectively. The preliminary experiment at the coast of Fukuyama city implies that these values are larger than the experimental ones. We have to perform more careful experiment to see whether the disagreement is fateful or not. The amplitude of (B) is 4% larger and that of (C) is 7.5% smaller than the amplitude of (A).

4. Discussion

Recently Morihira et al.[11] implied that in the ocean with violently varied depth, it is desirable for the two dimensional calculation of tidal current to ignore most

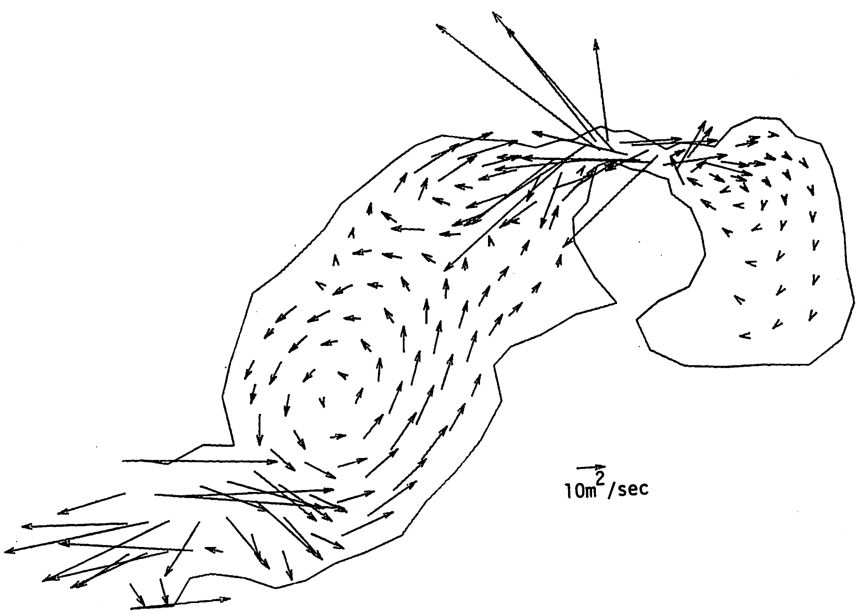

Fig. 6. Tidal residual flow in the case of averaged depth.

deep place. According to them, we tried to ignore the depth deeper than 80 (60) meters in the southern (northern) place of Nishi-Sakurajima channel. The resulted tidal residual current does not change so much compared with the result of calculation using the real depth (see Fig. 6). The amplitude at the off Kagoshima city is, however, not greater than that at the entrance in this case (see Fig. 7).

Geographically the depth increases rapidly at the most part of the coast of Kagoshima bay. As we have assumed the linear dependences of depth on coordinates, this geographical effect may not be estimated rightly. The effect will be important especially at Nishi-Sakurajima channel and be hoped to improve the agreement of phase difference.

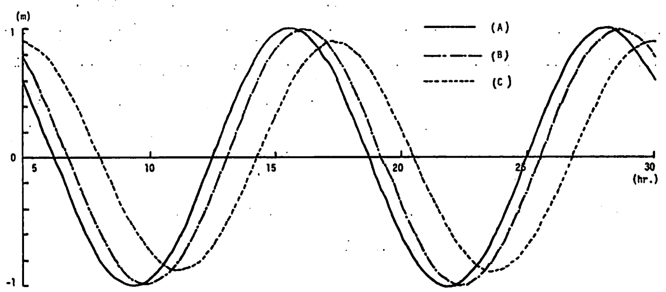

Fig. 7. Elevations at (A), (B) and (C) shown in Fig. 1 in the case of averaged depth.

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References

1) 河原田礼次郎，戸原義男・加藤　治（1976，1980）：内湾海水の動態についての水理学的研究 I, II, 鹿児島大学農学部学術報告，第26号 231-235, 第30号 233-238.

2) H. Fujii (1972): *Finite element schemes: stability and convergence,* Advances in Computational Method in Structual Mechanics and Design, HAH Press, 201-218.

3) 川原睦人，竹内則雄，首藤伸夫（1976）：2段階ラックス・ウェンドロフ有限要素法による潮汐流解析，第23回海岸工学講演会論文集，498-501.

4) R. D. Richtmyer (1963): *A survey of difference methods for non-steady fiuid dynamics,* N. C. A. R. Tech. Note, **63-2**.

5) 大西行雄（1979）：大阪湾の恒流に関する数値実験，第26回海岸工学講演会論文集，514-518.
J. Sudermann (1977): *Computation of barotropic tides by the finite element method,* Int. Conf. Finite Element Water Resour., 51-67.

6) L. J. Segerlind (1976): Applied Finite Element Analysis, John Wiley & Sons.
O. C. Zienkiewicz (1977): The Finite Element Method, 3-rd Eddition, McGraw-Hill.

7) M. A. E. EISENBERG and L. E. MALVERN (1973): *On finite element integration in natural coordinates,* Int. Jour. Numer. Meth. in Eng. **7** 574–575.

8) See O. C. ZIENKIEWICZ in ref. 5, 535–539.

9) A. F. BLUMBERG (1977): *Numerical model oт esturine circulation,* Jour. Hydraul. Div. ASCE **103** 295–310.

10) 第10管区海上保安本部 (1977)：鹿児島湾の潮流.

11) 森平倫生，笹嶋　博，久保省吾，西田寿夫，川原睦人 (1979)：有限要素法を用いた海域の三次元流れ・拡散解析，第26回海岸工学講演論文集，595–599.