Mem. Fac. Fish., Kagoshima Univ.
Vol. 36, No. 1, pp. 177～187 (1987)

Numerical Simulation of Tide in Kagoshima Bay by Two-Dimensional Subdomain Finite Element Method

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Abstract

The tidal flow in Kagoshima Bay is numerically simulated by the horizontally two-dimensional subdomain finite element method with the two-step Lax-Wendroff time stepping scheme. The bay is divided into quadrilateral elements. The real depth at every node is determined from the chart. At the open boundary, the water surface is forced to be oscillatory with the period of 12.5 hours.

Stable solutions were obtained after four tidal periods. In the calculated tidal residual flow pattern, an anti-clockwise and a clockwise vortices appear in the north of the open boundary and a large anti-clockwise vortex in the center of the bay. The numerical result that the water flows northward along the east coast and southward along the west coast in the center of the bay agrees with the oceanographical observations and the distribution of planktons.

Introduction

Kagoshima Bay is located in the most southern part of Kyushu, Japan. It has two deep basins connected by Sakurajima Channel (Fig. 1). The water renovations of the inner and the outer basins are the important matters for our concern. The water will be renewed due to the periodic tidal flow, the constant residual flow, the density flow and the wind driven flow. The main purpose of this article is to estimate the residual tidal flow in Kagoshima Bay by the numerical method in the horizontally two-dimensinal space.

The finite element method (FEM) is suitable for modelling complicated topography of natural bays and inland seas. It also has the advantage of investigating the interesting domain in detail by using smaller size elements. However, large computing region and long CPU time are generally needed in FEM calculation. Recently, particular attention has been devoted to the explicit FEM[1),2),3)], which could remarkably save the computer cost. In 1986, the conservative region method (CRM) was proposed[4)] as one of the candidates of the explicit FEM for tidal flow problems. CRM is based on the direct integral balance of the water mass and the momentum in some a priori defined subdomain, in contrast to the usual FEM, which is formulated by the weighted residual method.

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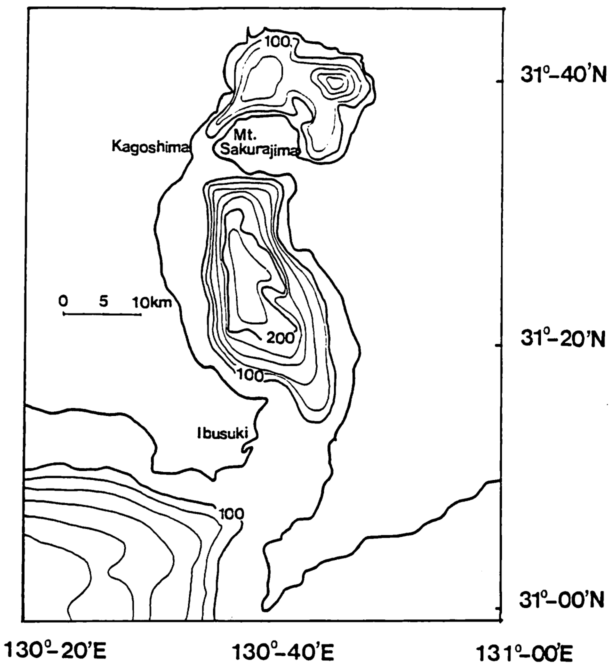


Fig. 1. Topography of Kagoshima Bay. The numbers denote the depth in meter.

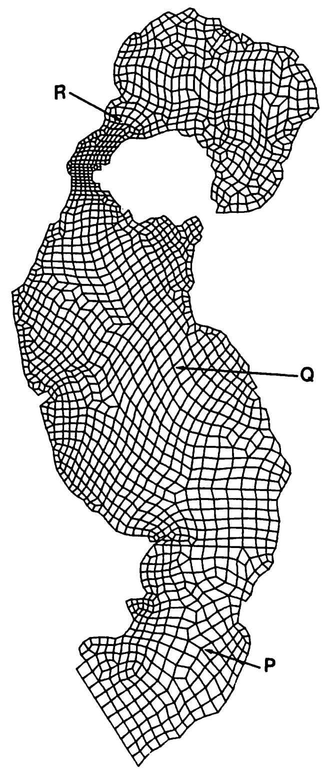


Fig. 2. Division of Kagoshima Bay into quadrilateral elements.

In this article, another explicit FEM will be proposed for the two-dimensional tidal flow problems. The philosophy of the method is the same as CRM. But its subdomain is now variable in contrast to the fixed one of CRM. The programming becomes simpler with less computer cost. To distinguish the method from CRM, it will be named as the subdomain finite element method (SDFEM). SDFEM will be applied to Kagoshima Bay and some interesting results will be presented.

Subdomain finite element method

In SDFEM, a bay or an inland sea is first divided into quadrilateral elements (see Fig. 2). As in CRM, the surface elevation and the velocity vector at some node α are determined by considering the conservations of the water mass and the momentum in an apriori defined subdomain. The subdomain in CRM is shown in Fig. 3 by the area surrounded by the broken lines. Although the calculated tidal residual flow in a rectangular model basin was similar to the Yasuda's analytic solution[5),6)], the programming of CRM was a little complicated due to the isoparametric parametrization. In SDFEM, the subdomain

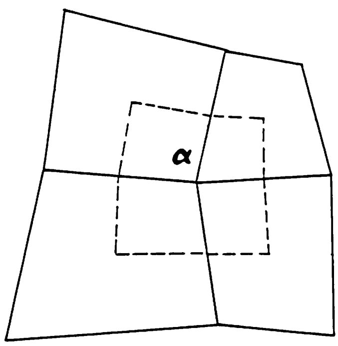


Fig. 3. Subdomain of the isoparametric parametrization (CRM).

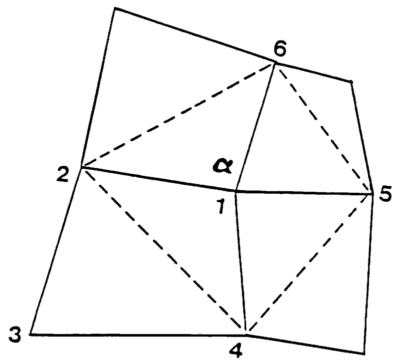


Fig. 4. Subdomain of the simplex parametrization (SDFEM).

adopted is the area surrounded by the broken lines in Fig. 4. The parametrization is the simplex one, so the programming is very concise. For a rectangular model basin, the calculated residual flow in SDFEM is almost qual to that in CRM.

As in the usual FEM, SDFEM equations are first written down for a quadrilateral element and then summed up over all elements. For an element, the conservative subdomain for node 1 ($=\alpha$) is the triangle constructed by nodes 1, 2 and 4 (hereafter referred to as the triangle 124) (see Fig. 4). Similarly, the conservative ones for nodes 2, 3 and 4 are the triangles 231, 342 and 413 respectively. The adoption of these variant triangles for each vertex of a quadrilateral element makes the method highly stable. In the following, the vertex 1 of a quadrilateral element is concentrated on.

Equation of continuity

The water mass entering the triangle 124 througth the side between the vertices 2 and 4 (hereafter referred to as the side $\overline{24}$) lifts the water surface η as ;

$$\rho A\Delta\eta = -\int_2^4 \rho(h+\eta)(\boldsymbol{v}\cdot\boldsymbol{n})d\Gamma\Delta t$$
$$= -\frac{1}{2}\rho(b_\alpha u_{1\alpha} + c_\alpha u_{2\alpha})\Delta t \qquad (1)$$

where ρ is the density, which is assumed to be constant, A the area of the triangle 124, h the depth, $\boldsymbol{v}$ the velocity vector, $\boldsymbol{n}$ the unit vector outward normal to the boundary and

$$\boldsymbol{u} \equiv (h+\eta)\boldsymbol{v} \qquad (2)$$

$$b_\alpha = x_{2\beta} - x_{2\gamma}, \qquad c_\alpha = x_{1\gamma} - x_{1\beta} \qquad (3)$$

In Eq. (3), α, β and γ are cyclic with respect to the vertices 1, 2 and 4. The integral in Eq. (1) was performed by assuming simplex parametrization of $\boldsymbol{u}$ and making use of Gauss' theorem. Equation (1) is the fundamental element equation of continuity.

Similar element equations can be also obtained for other element triangles 145, 156 and 162 (see Fig. 4). Since the conservation of water mass is required only in the subdomain surrounded by the broken lines in Fig. 4 and not in each element triangle, these element equations are summed up in one equation. The change of the water surface at node 1, i. e. $\Delta\eta_1$, will be calculated by replacing $\Delta\eta$ by $\Delta\eta_1$ in the equation.

Equation of momentum

The momentum vector will be changed due to four factors, i. e. the momentum accompanied with the water mass entering through the side $\overline{24}$ (advective term), the pressure by the water outside the element subdomain (pressure term), the Coriolis force (Corioris term) and the friction along the side $\overline{24}$ (viscous term). The gravity does not directly cause the change of momentum in our horizontally two-dimensional case. But indirectly it affects through the pressure term by adopting the static pressure approximation ;

$$p=\rho g(\eta-z) \tag{4}$$

where g denotes the gravitational acceleration.

The fundamental element equation of motion in the element triangle 124 is then written as

$$\begin{aligned}\rho A\Delta\boldsymbol{u}=&-\int_2^4\rho(h+\eta)(\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{v}d\Gamma\Delta t+\int_2^4(h+\eta)p(-\boldsymbol{n})d\Gamma\Delta t\\&-f\int_{\Delta 124}\rho(h+\eta)(\boldsymbol{k}\times\boldsymbol{v})ds\Delta t+\mu\int_2^4(h+\eta)(\boldsymbol{n}\cdot\nabla)\boldsymbol{v}d\Gamma\Delta t\end{aligned} \tag{5}$$

where f denotes Coriolis coefficient, $\boldsymbol{k}$ the unit vector directed vertically upward and μ is the viscous coefficient. In Eq. (5), the friction force is supposed to be proportional to $(\boldsymbol{n}\cdot\nabla)\boldsymbol{v}$. The integrations in Eq. (5) are carried out by assuming simplex parametrization for physical quantities and with the help of Gauss' theorem. The final equations are

$$\begin{aligned}\rho A\frac{\Delta u_1}{\Delta t}=&-\frac{\rho}{6}\left\{(b_\alpha u_{1\alpha})(\sum_\beta v_{1\beta})+(c_\alpha u_{1\alpha})(\sum_\beta v_{2\beta})+(\sum_\alpha u_{1\alpha})(b_\beta v_{1\beta}+c_\beta v_{2\beta})\right\}\\&-\frac{\rho}{6}g[\sum_\alpha(h+\eta)_\alpha](b_\beta\eta_\beta)\\&+\frac{\rho}{3}fA(\sum_\alpha u_{2\alpha})\\&-\frac{\mu}{2A}\{b_1(b_\alpha u_{1\alpha})+c_1(c_\alpha u_{1\alpha})\}\\\rho A\frac{\Delta u_2}{\Delta t}=&-\frac{\rho}{6}\{(b_\alpha u_{2\alpha})(\sum_\beta v_{1\beta})+(c_\alpha u_{2\alpha})(\sum_\beta v_{2\beta})+(\sum_\alpha u_{2\alpha})(b_\beta v_{1\beta}+c_\beta v_{2\beta})\}\\&-\frac{\rho}{6}g[\sum_\alpha(h+\eta)\alpha](c_\beta\eta_\beta)\\&-\frac{\rho}{3}fA(\sum_\alpha u_{1\alpha})\\&-\frac{\mu}{2A}\{b_1(b_\alpha u_{2\alpha})+c_1(c_\alpha u_{2\alpha})\}\end{aligned} \tag{6}$$

As in the case of the equation of continuity, similar element equations for the triangles 145, 156 and 162 are summed up with Eq. (6) and $\Delta u_i(=1,\ 2)$ in the left hand sides are replaced by the ones at node 1. The change of the momentum vector at node 1 will thus be calculated explicitly.

Numerical simulation in Kagoshima Bay

The division of Kagoshima Bay into quadrilateral elements is given in Fig. 2. The number of elements is 1633 and the number of nodes is 1812. The real depth at every node is determined from the chart. The physical parameters are chosen to be

$$\rho=10^3(kg\cdot m^{-3}),\quad g=9.8(m\cdot s^{-2}),$$
$$f=0.3745\times10^{-4}(s^{-1})\ \text{for Latitude } 31°\text{N},$$
$$\frac{\mu}{\rho}=\nu=\frac{\ell}{\ell_0}\nu_0,\ \ \nu_0=5\times10^2(m^2\cdot s^{-1}),\ \ \ell_0=10^3(m),\ \ \ell=\sqrt{b_1{}^2+c_1{}^2} \tag{7}$$

The kinetic eddy viscosity ν is chosen to be proportional to the length of the side $\overline{24}$ (ℓ). That is to say that the effect of an eddy is perceived only by the element with larger size than the eddy ; a small size element can only recognize the effects of smaller size eddys. The very definition of eddy viscosity[7] supports the above postulation ; as the most important eddy in the turbulent flow is the largest one, the size of which is the length of the flow region L, the eddy viscosity is proportional to L.

Initilly, the water is assumed to be at rest with flat water surface. The water surface at the open boundary is forced to be oscillatory as

$$\eta_{open\ boundary}=a\sin(-2\pi t/T)$$
$$a=1\,(m),\quad T=4.5\times10^4\,(s) \tag{8}$$

The non-slip boundary condition is adopted at the coast.

The fundamental equations are Eqs. (1) and (6). For the time stepping, the two-step Lax-Wendroff scheme will be employed ;

$$\rho A\eta^{n+\frac{1}{2}}=\rho A\eta^n+\frac{\Delta t}{2}\rho A\left(\frac{\Delta\eta}{\Delta t}\right)^n,\quad \rho Au_i^{n+\frac{1}{2}}=\rho Au_i^n+\frac{\Delta t}{2}\rho A\left(\frac{\Delta u_i}{\Delta t}\right)^n$$
$$\rho A\eta^{n+1}=\rho A\eta^n+\Delta t\rho A\left(\frac{\Delta\eta}{\Delta t}\right)^{n+\frac{1}{2}},\quad \rho Au_i^{n+1}=\rho Au_i^n+\Delta t\rho A\left(\frac{\Delta u_i}{\Delta t}\right)^{n+\frac{1}{2}} \tag{9}$$

where n denotes the time step. The time difference Δt is chosen to be 3 seconds, because $\Delta t=5$ seconds leads divergence. Stable solutions are obtained after four tidal periods.

In Fig. 5 is given the distribution of $\boldsymbol{u}$ at the maximum ebb current as an example of the calculated results. The hodographs at several points in the bay are shown in Fig. 6, where all hodographs rotate clockwise. The maximum value of velocity at the Sakurajima Channel is about 0. 5 m/s, which is similar to observations. The distributions of the tidal velocity at every half an hour over one tidal period in the Sakurajima Channel are depicted in Fig. 7. In Fig. 8 are presented the variations of η and $u_i (i=1,\ 2)$ at the points P, Q, R in Fig. 2. In Figs. 7 and 8, 0 hour is the time, when the sea level is zero at the open boundary. It is seen from Fig. 8 that the phases of P and Q are almost the same and the one of R delays about 30 minutes in η and u_2, while in u_1, the phases of Q and R are almost the same and the one of P advances about 30 minutes. The phase delay of η only occurs in the Sakurajima Channel. In Fig. 9 are given the distributions of the relative phase delays of η in the Sakurajima Channel at the highest (Fig. 9a) and the lowest (Fig. 9b) sea levels.

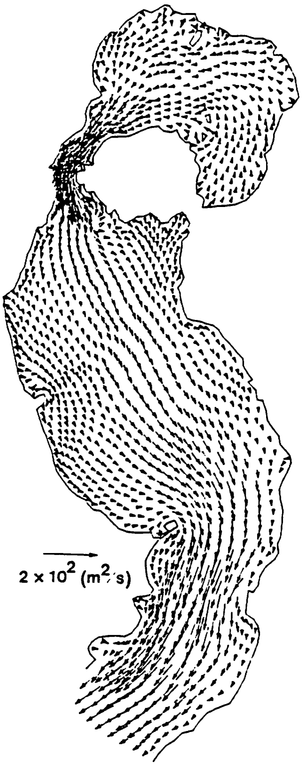


Fig. 5. Distribution of **u** vector at the maximum ebb current.

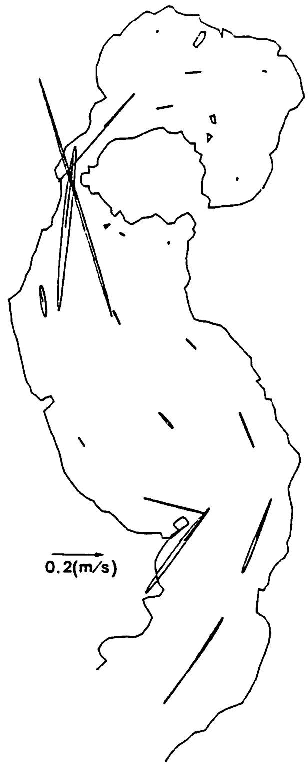


Fig. 6. Hodographs at several points in Kagoshima Bay.

The distribution of the tidal residual flow is presented in Fig. 10. An anti-clockwise and a clockwise vortices appear in the north of the open boundary. In the center of the basin, a large anti-clockwise vortex exists and the water flows northward along the east coast and southward along the west coast. This fact agrees with the oceanographical observations and the distribution of planktons. A clockwise vortex also appears in the Sakurajima Channel (see Fig. 11).

Conclusion

Basing on the direct integral balance of the water mass and the momentum in some a priori defined subdomain, an explicit method is devised for the horizontal two-dimensional tidal flow problems. The method, named as the subdomain finite element method, has the ability of modelling the complicated topography of natural bays and inland seas. From the standpoint of solving the given tidal equations in a bay, SDFEM looks like curious ; the equations must be solved in each isoparametric element subdomain (Fig. 3), instead of the variabie simplex element subdomain (Fig. 4), and the solution in each

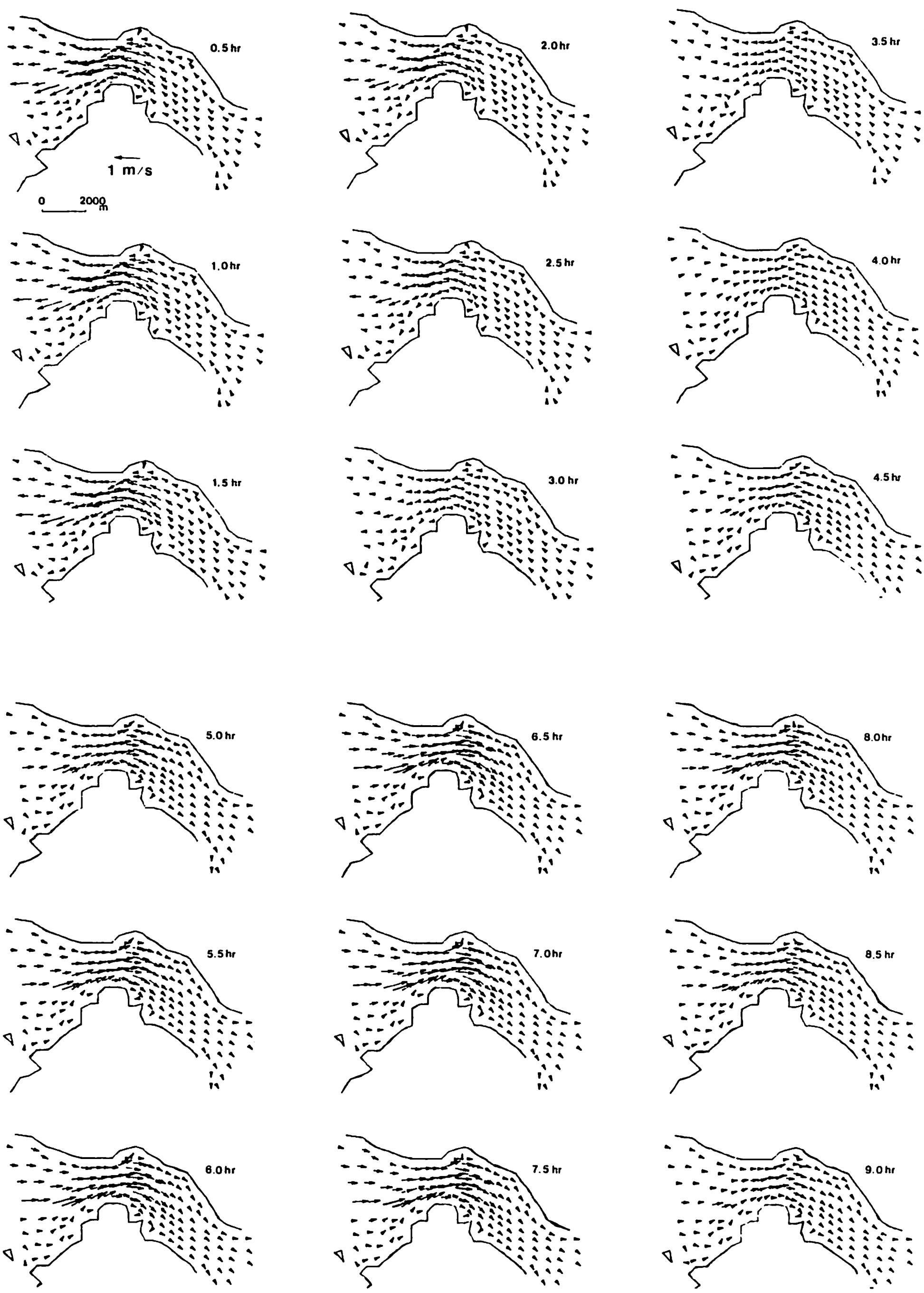
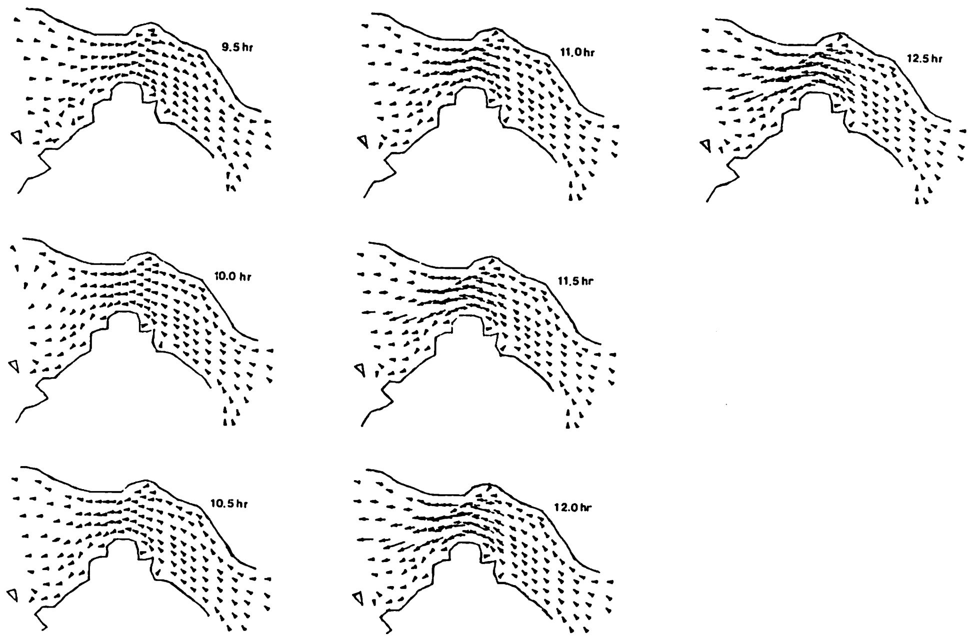


Fig. 7. Distributions of the tidal velocity at every half an hour over one tidal period in the Sakurajima Channel. 0 hour is the time, when the sea level is zero at the open boundary.



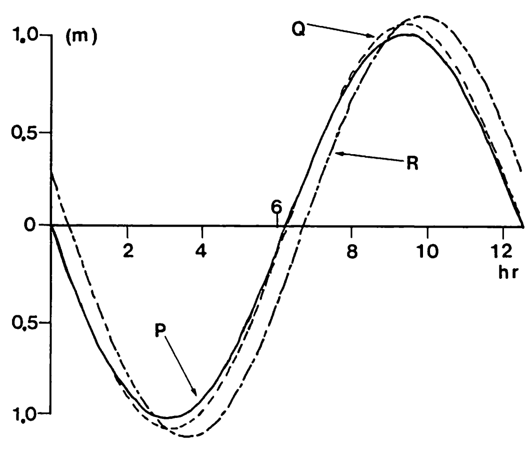


Fig. 8 a. Variation of η at the points P, Q, R in Fig. 2. 0 hour is the time, when the sea level is zero at the open boundary.

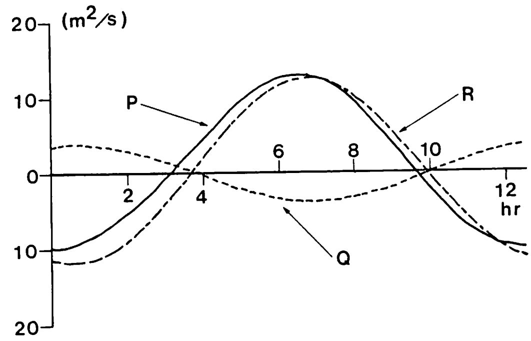


Fig. 8 b. Variation of u_1 at the points P, Q, R in Fig. 2. 0 hour is the time, when the sea level is zero at the open boundary.

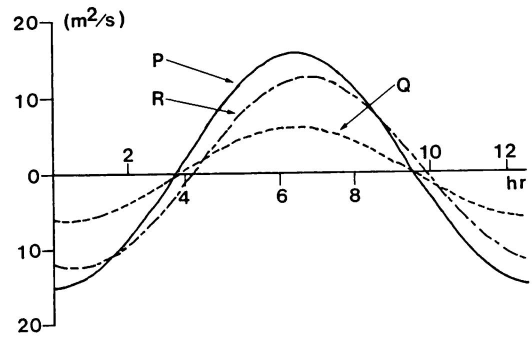


Fig. 8 c. Variation of u_2 at the points P, Q, R in Fig. 2. 0 hour is the time, when the sea level is zero at the open boundary.

subdomain will be summed up to give the one in all the bay. However, if one takes the conservation of the water mass and the momentum as the fundamental priciple, the element subdomain in SDFEM might be allowed.

SDFEM is applied to Kagoshima Bay. The calculated results of the surface elevation and the velocity vector are the natural ones. The phase of the surface elevation delays only through the Sakurajima Cannel. In our case of Eq. (7), the phase difference between the south and the north places of the Sakurajima Channel decreases about 10 minutes from the case of the constant kinetic eddy viscosity $\nu = \nu_0$. In the tidal residual flow, the success of our calculation is to give a large anti-clockwise vortex in the center of the basin.

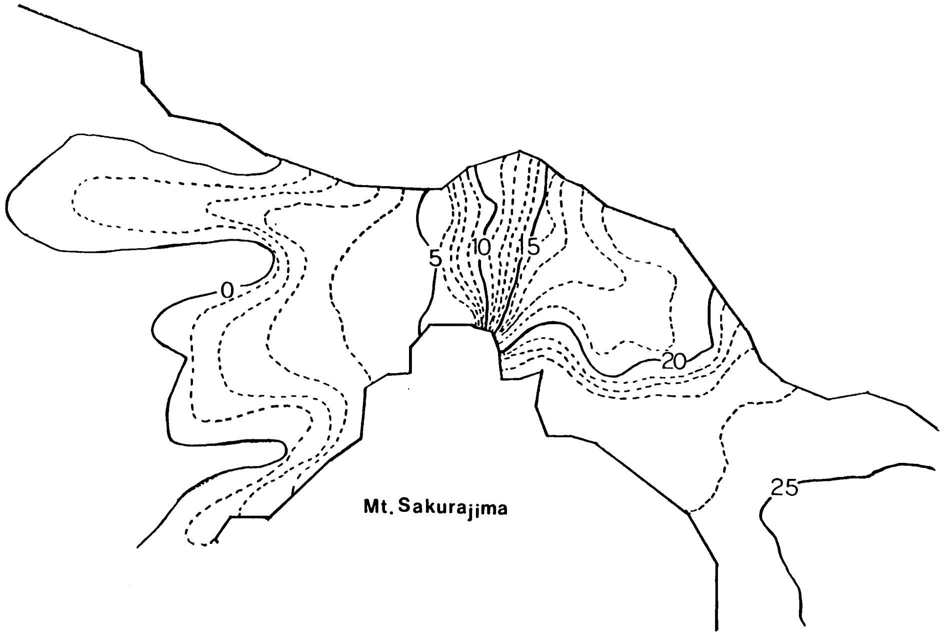


Fig. 9 a. Distribution of the relative phase delay of η in the Sakurajima Channel at the highest sea level. The numbers denote the phase in minutes.

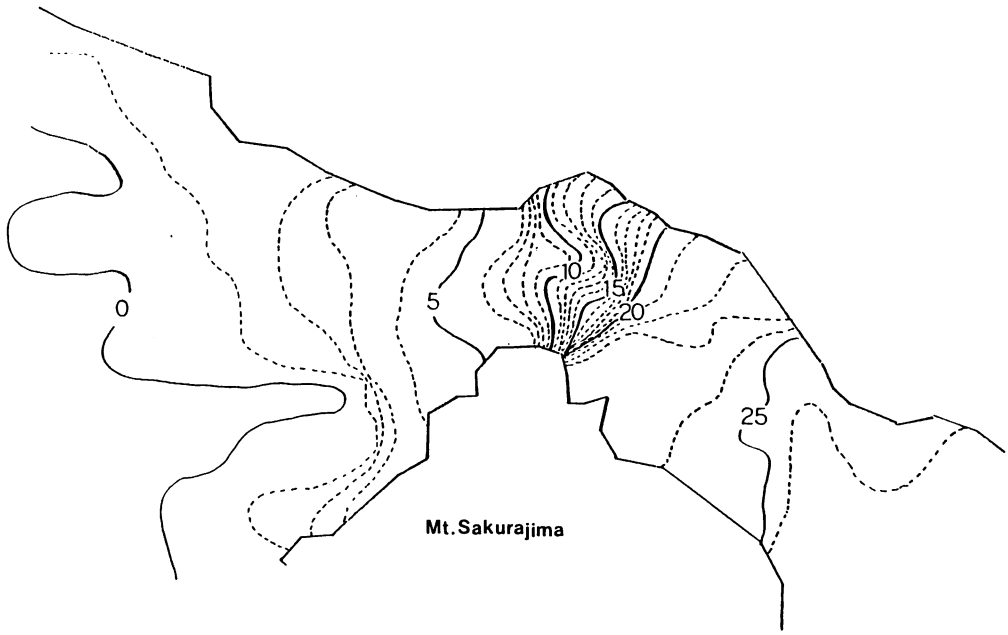


Fig. 9 b. Distribution of the relative phase delay of η in the Sakurajima Channel at the lowest sea level. The numbers denote the phase in minutes.

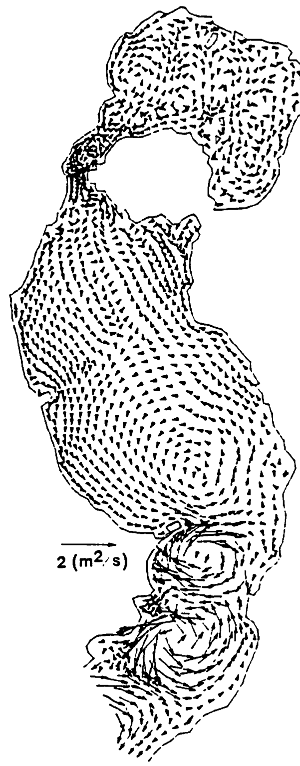


Fig. 10. Distribution of the tidal residual flow.

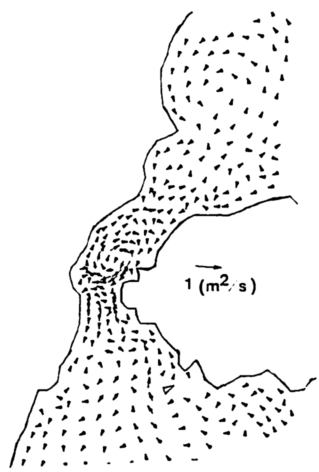


Fig. 11. Distribution of the tidal residual flow magnified around the Sakurajima Channel.

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