Generalized Viviani's Solid

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Abstract

We study solids made from the unit sphere by removing a number n of parallel cylinders of various radii that are externally tangent to each other and internally tangent to the unit sphere. Our main interest is to study extremal properties (the maximum and the minimum) of geometrical characteristics such as surface area, volume, and perimeter length in the space of all of these solids. When $n = 2$, we show that the classical Viviani's solid enjoys an extremal property. When $n = 3$, by restricting the space so that it becomes compact, we show that geometrical charcteristics have extremes when and only when two of radii of cylinders are equal each other.

1 Introduction

Let $S(R)$ be a sphere of radius R and $C(r)$ a cylinder of radius r whose surface is tangent to the surface of the sphere. In the Cartesian coordinates system, for example, they can be represented as

$$x^2 + y^2 + z^2 \leq R^2, \ (x-r)^2 + y^2 \leq r^2$$

respectively. Then we make a solid $S(R) \setminus C(r)$, that is, a solid made from the sphere by removal of the cylinder. Such a solid in a particular case $r = R/2$ was first studied by Viviani in 1692. In this paper, generalizing Viviani's solid, we study the solid made from the sphere by removal of several numbers of cylinders of various radii while axes of cylinders are assumed to be parallel each other.

In the section 2 we study a solid that is made by intersecting a cylinder with the (fixed) unit sphere, and evaluate its several geometrical chracteristics (surface area, volume, and perimeter length). Then we study, as an example, a solid that is made from the unit sphere by removing two cylinders that are parallel and externally tangent. This solid is a slight generalization of the classical Viviani's solid.

In the section 3 we study a solid that is made from the unit sphere by removing 'three' cylinders $C(x), C(y), C(z)$ that are parallel and externally tangent. A triplet of ylinders $C(x), C(y), C(z)$ may be interpreted as a point (x, y, z) in the 3-dimensional Euclidean space. Then we study the shape of the space $\mathbf{S}$ of all of these points. After revealing the structure of the space $\mathbf{S}$, we finally study the maximum and/or the minimum of several geometrical characteristics. It can be shown that these characteristics attain their maximum and/or minimum when and only when two of x, y, z are equal.

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2 Several geometrical characteristics about generalized Viviani's solids

In this section $C(r)$ denotes a cylinder of radius r that passes inside a sphere of unit radius and which is tangent to the surface of the sphere.

2.1 Surface area

Let $S(r)$ be the surface area of the sphere inside the cylinder. It can be computed by

$$S(r) = 4 \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy,$$

where

$$D = \{(x,y) : (x-1+r)^2 + y^2 \le r^2, y \ge 0\}.$$

Since

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \frac{1}{z},$$

it leads to

$$S(r) = 4 \int_{1-2r}^{1} dx \int_0^{\sqrt{r^2-(x-1+r)^2}} \frac{dy}{\sqrt{1-x^2-y^2}}.$$

By change of variable $x = 1 - r + rt$, we have

$$S(r) = 4r \int_{-1}^{1} dt \int_0^{r\sqrt{1-t^2}} \frac{dy}{\sqrt{1-(1-r+rt)^2-y^2}}.$$

Since it is elementary to show that

$$\int_0^b \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{b}{a} \quad (a > b > 0),$$

we have

$$\int_0^{r\sqrt{1-t^2}} \frac{dy}{\sqrt{1-(1-r+rt)^2-y^2}} = \arcsin\frac{r\sqrt{1-t^2}}{\sqrt{1-(1-r+rt)^2}} = \arcsin\sqrt{\frac{1+t}{c+t}},$$

where we write $c = (2-r)/r$. Accordingly we get

$$\begin{aligned} S(r) &= 4r \int_{-1}^{1} \arcsin\sqrt{\frac{1+t}{c+t}}\, dt \\ &= 4r \left[(c+1)\arctan\sqrt{\frac{2}{c-1}} - \sqrt{2(c-1)}\right] = 8\left(\arctan\sqrt{\frac{r}{1-r}} - \sqrt{r(1-r)}\right). \end{aligned}$$

Therefore we obatain the following lemma.

Lemma 1

$$S(r) = 8\left(\arcsin\sqrt{\frac{1}{r}} - \sqrt{r(1-r)}\right).$$

2.2 Volume

Let $V(r)$ be the volume of the sphere inside the cylinder. It can be computed by

$$V(r) = 4\iint_D z\,dxdy = 4\int_{1-2r}^{1} dx \int_0^{\sqrt{r^2-(x-1+r)^2}} \sqrt{1-x^2-y^2}\,dy.$$

By change of variable $x = 1 - r + rt$, we have

$$V(r) = 4r\int_{-1}^{1} dt \int_0^{r\sqrt{1-t^2}} \sqrt{1-(1-r+rt)^2-y^2}\,dy.$$

Now it is elementary to show that

$$\int_0^b \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left(a^2\cdot\arcsin\frac{b}{a} + b\sqrt{a^2-b^2}\right).$$

Accordingly we get $V(r) = I_1 + I_2$, where

$$\begin{aligned} I_1 &= 2r^2\int_{-1}^{1}\sqrt{1-t^2}\cdot\sqrt{(1-(1-r+rt)^2)-r^2(1-t^2)}\,dt \\ I_2 &= 2r\int_{-1}^{1}(1-(1-r+rt)^2)\arcsin\sqrt{\frac{r^2(1-t^2)}{1-(1-r+rt)^2}}\,dt. \end{aligned}$$

First, writing $c = (2-r)/r$ as before, we have

$$\sqrt{(1-(1-r+rt)^2)-r^2(1-t^2)} = r\sqrt{c-1}\sqrt{1-t}.$$

Hence

$$\begin{aligned} I_1 &= 2r^3\sqrt{c-1}\int_{-1}^{1}(1-t)\sqrt{1+t}\,dt \\ &= \frac{32\sqrt{2(c-1)}}{15}r^3 \end{aligned} \tag{1}$$

Next, since

$$\frac{r^2(1-t^2)}{1-(1-r+rt)^2} = \frac{1+t}{c+t},$$

we have

$$\begin{aligned} I_2 &= 2r^3\int_{-1}^{1}(1-t)(c+t)\arcsin\sqrt{\frac{1+t}{c+t}}\,dt \\ &= \left[\frac{(c+1)^3}{3}\arctan\sqrt{\frac{2}{c-1}} - \frac{\sqrt{2(c-1)}}{45}(15c^2+50c-29)\right]r^3 \end{aligned} \tag{2}$$

Consequently, summing up (1) and (2), we get

$$\begin{aligned} V(r) &= \left[\frac{(c+1)^3}{3}\arctan\sqrt{\frac{2}{c-1}} - \frac{\sqrt{2(c-1)}}{9}(3c-5)(c+5)\right]r^3 \\ &= \frac{8}{3}\left[\arctan\sqrt{\frac{r}{1-r}} - \left(1-\frac{4r}{3}\right)(1+3r)\sqrt{r(1-r)}\right]. \end{aligned}$$

Therefore, noting that

$$\arctan\sqrt{\frac{r}{1-r}} = \arcsin\sqrt{r},$$

we obatain the following lemma.

Lemma 2

$$V(r) = \frac{8}{3}\left[\arcsin\sqrt{r} - \left(1 - \frac{4r}{3}\right)(1+3r)\sqrt{r(1-r)}\right].$$

2.3 Perimeter

The intersection of two surfaces of sphere and cylinder forms a spatial curve like a lemniscate. To speak precisely a leaf of the curve can be represented by

$$\begin{pmatrix} x(\theta) \\ y(\theta) \\ z(\theta) \end{pmatrix} = \begin{pmatrix} 1 - r + r\cos\theta \\ r\sin\theta \\ \sqrt{1 - x(\theta)^2 - y(\theta)^2} \end{pmatrix} \quad (0 \leq \theta < 2\pi).$$

Let $L(r)$ be the perimeter length of the curve. Then it can be computed by

$$L(r) = \int_0^{2\pi} ds = 2\int_0^{\pi} ds.$$

Since

$$z(\theta)^2 = 2r(1-r)(1-\cos\theta)$$

and

$$dx = -rd\theta\sin\theta,\ dy = rd\theta\cos\theta,\ dz = \frac{r(1-r)\sin\theta}{z(\theta)}\,d\theta,$$

we have

$$ds^2 = \frac{r}{2}\{(1+r) + (1-r)\cos\theta\}\,d\theta^2.$$

Thus we see

$$L(r) = 2\cdot\sqrt{\frac{r}{2}}\int_0^{\pi}\sqrt{(1+r)+(1-r)\cos\theta}\,d\theta.$$

Now it can be shown that

$$\int_0^{\pi}\sqrt{a + b\cos\theta}\,d\theta = 2\sqrt{a+b}\cdot E(k),$$

where $E(\cdot)$ denotes the complete elliptic integral of the second kind with $k^2 = 2b/(a+b)$. Therefore we obtain the following lemma.

Lemma 3

$$L(r) = 4\sqrt{r}\cdot E(\sqrt{1-r}).$$

2.4 An example of generalized Viviani's solids

Let $C(x), C(y)$ be two cylinders such that they are externally tangent each other, their axes are parallel, and their centers lie on a diameter of the unit sphere. Make a solid that is a part of the sphere outside two cylinders, and let S be its surface area, V its volume, and L its perimeter length. Then we have

$$\begin{aligned} S &= 4\pi - S(x) - S(y), \\ V &= \frac{4\pi}{3} - V(x) - V(y), \\ L &= L(x) + L(y), \end{aligned}$$

where we need to note that $x + y = 1$.

Using Lemma 1 we have

$$S(y) = 8\left(\arctan\sqrt{\frac{y}{1-y}} - \sqrt{y(1-y)}\right) = 8\left(\arctan\sqrt{\frac{1-x}{x}} - \sqrt{x(1-x)}\right).$$

Accordingly

$$S(x) + S(y) = 8\left(\arctan\sqrt{\frac{x}{1-x}} + \arctan\sqrt{\frac{1-x}{x}} - 2\sqrt{x(1-x)}\right).$$

Now note that

$$\arctan t + \arctan\frac{1}{t} = \frac{\pi}{2} \text{ for any } t > 0.$$

Hence it follows

$$S(x) + S(y) = 4\pi - 16\sqrt{x(1-x)},$$

which implies

$$S = 16\sqrt{x(1-x)}.$$

Similarly we obtain

$$V = \frac{128}{9}\{x(1-x)\}^{\frac{3}{2}}$$

and

$$L = 4\left(\sqrt{x}\,E(\sqrt{1-x}) + \sqrt{1-x}\,E(\sqrt{x})\right).$$

Then it can be shown that all of S, V, L have their maximums when $x = y$.

3 Generalized Viviani's solids made by removal of three cylinders

Consider a cylinder of curvature κ that is internally tangent to the surface of the unit sphere. In this section we denote this cylinder by $C(\kappa)$, but sometimes admit to denote a section of the cylinder, that is, a circle, by the same notation.

3.1 Space of mutually tangent three circles

Let $C(1)$ be a fixed circle. Inside it we consider three circles $C(x), C(y), C(z)$ that are externally tangent each other and are internally tangent to $C(1)$. Then the classical Descartes's theorem shows that x, y, z satisfy a quadratic equation

$$Q_1(x, y, z) := 2(1 + x^2 + y^2 + z^2) - (-1 + x + y + z)^2 = 0. \tag{3}$$

We consider the space, which we denote by $\mathbf{S}$, of all the triplets of mutually tangent circles $C(x), C(y), C(z)$. Then, by (3), the space $\mathbf{S}$ can be identified as the set

$$\{(x, y, z) : x > 1, y > 1, z > 1,\ Q_1(x, y, z) = 0\}.$$

In other words the space $\mathbf{S}$ is a part of a quadratic surface.

To determine the shape of $\mathbf{S}$ precisely, we change the coordinates system (x, y, z) to a new system (u, v, w) that are defined by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = u\,\mathbf{p}_1 + v\,\mathbf{p}_2 + (w + \sqrt{3}a)\,\mathbf{p}_3, \tag{4}$$

where

$$\mathbf{p}_1 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \ \mathbf{p}_2 = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}, \ \mathbf{p}_3 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \ a = \frac{2+\sqrt{3}}{\sqrt{3}}.$$

Since vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are orthonormal, change of coordinates system leads to

$$Q_1(x,y,z) = 2u^2 + 2v^2 - (w+2)^2 + 4.$$

Therefore the space $\mathbf{S}$ is a leaf of a hyperboloid

$$\frac{(w+2)^2}{2^2} - \frac{u^2+v^2}{(\sqrt{2})^2} = 1 \tag{5}$$

Now we shall prove that $w \geq 0$. Summing up components ov vectors in (4), we have

$$x + y + z = \sqrt{3}\,(w + \sqrt{3}a). \tag{6}$$

Then, noting that $x > 1, y > 1, z > 1$, we see $w + 2 > 0$. On the other hand, from (5) it follows that

$$\frac{(w+2)^2}{2^2} \geq 1, \ \text{that is, } w(w+4) \geq 0.$$

Accordingly we get $w \geq 0$.

Therefore we obtain

$$\mathbf{S} = \left\{ (u,v,w) : w \geq 0, \frac{(w+2)^2}{2^2} - \frac{u^2+v^2}{(\sqrt{2})^2} = 1 \right\} \tag{7}$$

3.2 Restriction of the space S

Since the space $\mathbf{S}$ is not compact, it is difficult to study maximum-minimum problems in the space. Thus we need restrict the space $\mathbf{S}$. To speak precisely we consider only mutually tangent three circles $C(x), C(y), C(z)$ that are externally tangent to a fixed circle $C(\kappa)$. Then, using Descartes's theorem again, we have

$$Q_2(x,y,z) := 2\,(\kappa^2 + x^2 + y^2 + z^2) - (\kappa + x + y + z)^2 = 0.$$

In the below throughout we only consider this restricted space

$$\{(x,y,z) : x > 1, y > 1, z > 1, \ Q_1(x,y,z) = 0, \ Q_2(x,y,z) = 0\},$$

which we denote by $\mathbf{S}(\kappa)$.

Since $Q_1(x,y,z) = 0$ and $Q_2(x,y,z) = 0$, we can immediately derive

$$x + y + z = \frac{\kappa - 1}{2}. \tag{8}$$

Hence, combining (6) and (8), we see

$$w + \sqrt{3}a = \frac{\kappa - 1}{2\sqrt{3}} \tag{9}$$

Thus, if κ is fixed, a coordinate w is also fixed.

Furthermore (5) implies that

$$u^2 + v^2 = b^2,$$

where

$$b = \sqrt{\frac{w(w+4)}{2}}.$$

Thus coordinates (u, v) lies on a circle of radius b, and they can be represented by a parameter θ as

$$u = b\cos\theta,\ v = b\sin\theta.$$

In summary we see the restricted space is a circle lying on $\mathbf{S}$. Precisely it is described as

$$\mathbf{S}(\kappa) = \left\{ (x,y,z) : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (b\cos\theta)\,\mathbf{p}_1 + (b\sin\theta)\,\mathbf{p}_2 + c\,\mathbf{p}_3 \quad (0 \le \theta < 2\pi) \right\}, \tag{10}$$

where

$$b = \sqrt{\frac{\kappa^2 - 14\kappa + 1}{24}},\ c = \frac{\kappa - 1}{2\sqrt{3}}.$$

3.3 Extremes of geometrical characteristics

Consider generalized Viviani's solids made by removal of three cylinders $C(x), C(y), C(z)$ with $(x,y,z) \in \mathbf{S}(\kappa)$. By (10) curvatures (x,y,z) depend only on one parameter θ. Consequently any geometrical characteristic also depends on θ and thus define a function $F(\theta)$. To write more precisely, by use of lemmes in the section 2, the function $F(\theta)$ is represented by

$$F(\theta) = f(x(\theta)) + f(y(\theta)) + f(z(\theta)) + C,$$

where $f(x)$ stands for $-S(x)$ and $C = 4\pi$ in case of surface area; $f(x)$ for $-V(x)$ and $C = 4\pi/3$ in case of volume; and $f(x)$ for $L(x)$ and $C = 0$ in case of perimeter length.

Lemma 3 *$F(\theta)$ is a periodic function with period $2\pi/3$.*
(Proof) First we show

$$\begin{aligned} x\left(\theta + \frac{2\pi}{3}\right) &= b\cos\left(\theta + \frac{2\pi}{3}\right)\cdot\left(-\frac{1}{\sqrt{2}}\right) + b\sin\left(\theta + \frac{2\pi}{3}\right)\cdot\left(-\frac{1}{\sqrt{6}}\right) + c\cdot\frac{1}{\sqrt{3}} \\ &= b\left(\cos\theta\cos\frac{2\pi}{3} - \sin\theta\sin\frac{2\pi}{3}\right)\cdot\left(-\frac{1}{\sqrt{2}}\right) + b\left(\sin\theta\cos\frac{2\pi}{3} + \cos\theta\sin\frac{2\pi}{3}\right)\cdot\left(-\frac{1}{\sqrt{6}}\right) + c\cdot\frac{1}{\sqrt{3}} \\ &= b\cdot\frac{2}{\sqrt{6}}\sin\theta = z(\theta) \end{aligned}$$

Similarly we can show

$$y\left(\theta + \frac{2\pi}{3}\right) = x(\theta), \quad z\left(\theta + \frac{2\pi}{3}\right) = y(\theta).$$

Therefore. since F is symmetric with respect to (x,y,z), F is periodic.

(Q.E.D.)

To simplify notations we write

$$\mathbf{r} := \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x(\theta) \\ y(\theta) \\ z(\theta) \end{pmatrix}, \quad \mathbf{r}' := \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \frac{d}{d\theta}\begin{pmatrix} x(\theta) \\ y(\theta) \\ z(\theta) \end{pmatrix}.$$

Then we have

$$F'(\theta) = \frac{d}{d\theta}F(\theta) = f'(x)x' + f'(y)y' + f'(z)z'.$$

Lemma 4 *In the interval* $0 \leq \theta \leq \frac{2\pi}{3}$, *the function* $F(\theta)$ *is maximal at* $\theta = \frac{\pi}{2}$ *and minimal at* $\theta = \frac{\pi}{6}$.
(Proof) Since

$$\mathbf{r} = (b\cos\theta)\,\mathbf{p}_1 + (b\sin\theta)\,\mathbf{p}_2 + c\mathbf{p}_3,$$

we see

$$\mathbf{r}' = -(b\sin\theta)\,\mathbf{p}_1 + (b\cos\theta)\,\mathbf{p}_2.$$

Accordingly, if $\theta = \frac{\pi}{2}$, we have $x = y, x' = -y', z' = 0$, which implies $F' = 0$. Similarly, if $\theta = \frac{\pi}{6}$, we have $y = z, x' = 0, y' = -z'$, which also implies $F' = 0$. Therefore we get the conclusion.

(Q.E.D.)

Accordingly we obtain the following theorem.

Theorem *Among all solids* $(x, y, z) \in \mathbf{S}(\kappa)$, *geometrical charcateristics such as surface area, volume, and perimeter length are maximal when*

$$x = y = -\frac{b(\kappa)}{\sqrt{6}} + \frac{c(\kappa)}{\sqrt{3}}, \quad z = \frac{2b(\kappa)}{\sqrt{6}} + \frac{c(\kappa)}{\sqrt{3}},$$

and are minimal when

$$x = -\frac{2b(\kappa)}{\sqrt{6}} + \frac{c(\kappa)}{\sqrt{3}}, \quad y = z = \frac{b(\kappa)}{\sqrt{6}} + \frac{c(\kappa)}{\sqrt{3}},$$

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