Some mean characteristics of Poisson-Voronoi and Poisson-Delaunay tessellations in hyperbolic planes

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1. Introduction

Random tessellations, in particular, Poisson-Voronoi tessellations have interested many mathematicians as well as many researchers in other fields for a long time. As comprehensive references, see Møller (1994), and Stoyan, Kendall and Mecke (1995). However, most of these studies have been concerned with random tessellations in Euclidean spaces. On the other hand, relatively small number of studies have been made in non-Euclidean spaces. For example of these studies we may cite Miles (1971), Santaló and Yañez (1972), and Isokawa (2000). In particular, while Miles (1971) studied Poisson-Voronoi tessellations on 2-dimensional spheres, that is, non-Euclidean planes with positive curvatures, there seem to be no research on Poisson-Voronoi tessellations in non-Euclidean planes with negative curvatures, that is, hyperbolic planes. In this paper we shall investigate Poisson-Voronoi tessellations and their dual, Poisson-Delaunay tessellations, in hyperbolic planes.

Let $\mathbf{H}^2$ be a hyperbolic plane with curvature $(-k^2)$. In $\mathbf{H}^2$ we consider a homogeneous Poisson point process $\mathbf{\Phi}$ with intensity ρ, and construct a Voronoi tessellation $\mathbf{T}$ whose nuclei coincide with points generated by $\mathbf{\Phi}$. In the section 2 we study the Poisson-Voronoi tessellation $\mathbf{T}$, and compute the mean number of vertices and the mean perimeter length of cells of $\mathbf{T}$. In the section 3 we study the Poisson-Delaunay tessellation which is defined by the dual of $\mathbf{T}$. We shall calculate the the mean magnitude of an angle and the mean area of its Delaunay triangles.

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2. Poisson-Voronoi tessellation

In this section we shall compute the mean number of vertices $E(V)$ and the mean perimeter length $E(L)$ of cells of $\mathbf{T}$. Slivnyak's theorem assures that it is sufficient to compute those mean quantities for a typical cell C_0, which is defined as the cell with nucleus at the origin O. We carry out our calculation in a similar manner to that in Meijering (1953). Following the same author, we introduce the concepts of "mathematical" edges and "mathematical" vertices of the cell C_0. A straight line is called to be a mathematical edge of C_0 if it lies equidistant from the nucleus at O and another nucleus. Namely a mathematical edge bisects the line segment which connects the nucleus at O and another nucleus. Similarly a point is called to be a mathematical vertex when it lies equidistant from the nucleus at O and other two nuclei.

Let ω stand for any infinitesimal element of any mathematical edge, or any mathematical vertex. Supposing that ω lies distant r from O, we denote by $P(r)$ the probability that ω is never contained in any other cells than C_0. Then the following lemma will play a crucial role in later arguments.

Lemma 1.

$$P(r) = \exp(-\mu(\cosh kr - 1)),\ where\ \mu = \frac{2\pi\rho}{k^2}.$$

Proof. Let D denotes the disc with center at ω and radius r. As is easily seen, ω is not contained in any other cells than C_0 if and only if any other nuclei other than O never lie in the disc D. Then, since the nuclei of our Voronoi tessellation are generated by a homogeneous Poisson point process with intensity ρ and the area of D equals $\frac{2\pi}{k^2}(\cosh kr - 1)$, we obtain the desired conclusion.

For the mean number of vertices, we can show the following concise result.

Theorem 1.

$$E(V) = 6 \cdot \left(1 + \frac{1}{\mu}\right).$$

Proof. We first consider the mean number of mathematical edges whose distances from O are between z and $z + dz$. Since it is equal to the mean number of nuclei that lie in an annulus with distant $2z$ from O and breadth $2dz$, it equals

$$\rho \cdot d_z \left\{ \frac{2\pi}{k^2} (\cosh(k \cdot 2z) - 1) \right\} = 4\pi\rho \frac{\sinh 2kz}{k} dz . \tag{2.1}$$

Suppose that a mathematical vertex P is the intersection of two mathematical edges l and m, and denote by z and x their distances from O respectively. Let H and K be the feet of perpendiculars from O to l and m respectively, and denote the angle HOK by α. Let C be a circle with center at O and radius r, and denote by β and γ the angles extended by chords which are made by l and m with the circle C respectively. Then hyperbolic trigonometry shows that

$$\cos\beta = \frac{\tanh kz}{\tanh kr}, \quad \cos\gamma = \frac{\tanh kx}{\tanh kr} . \tag{2.2}$$

Furthermore, we can see that the mathematical vertex P lies inside C if and only if

$$|\beta - \gamma| < \alpha < \beta + \gamma . \tag{2.3}$$

Let $v(r)$ be the mean number of mathematical vertices that lie inside C. Then, from (2.1) and (2.3), it follows that

$$\begin{aligned} \upsilon(r) &= \frac{1}{2}\int_0^r 4\pi\rho \frac{\sinh 2kz}{k} dz \\ &\quad \cdot \left(\int_z^r 4\pi\rho \frac{\sinh 2kx}{k} dx \int_{\beta-\gamma}^{\beta+\gamma} \frac{1}{\pi} d\alpha + \int_0^z 4\pi\rho \frac{\sinh 2kx}{k} dx \int_{\gamma-\beta}^{\beta+\gamma} \frac{1}{\pi} d\alpha \right) \\ &= \frac{16\pi\rho^2}{k^2} \int_0^r \sinh 2kz \, dz \left(\int_z^r \sinh 2kx \cdot \gamma + \beta \int_0^z \sinh 2kx \, dx \right) . \end{aligned}$$

Now we change variables from z and x to β and γ by (2.2), and define

$$g(\beta) = \frac{\sin\beta\cos\beta}{\left(1 - t^2\cos^2\beta\right)^2} \tag{2.4}$$

with $t = \tanh kr$.

Then we can rewrite

$$\upsilon(r) = \frac{64\pi\rho^2 t^4}{k^4} \int_0^{\frac{\pi}{2}} g(\beta)\, d\beta \left(\int_0^\beta \gamma g(\gamma)\, d\gamma + \beta \int_\beta^{\frac{\pi}{2}} g(\gamma)\, d\gamma \right) .$$

Now it can be easily seen that

$$\int_0^{\frac{\pi}{2}} g(\beta)\, d\beta \int_0^{\beta} \gamma g(\gamma)\, d\gamma = \int_0^{\frac{\pi}{2}} \beta\, g(\beta)\, d\beta \int_{\beta}^{\frac{\pi}{2}} g(\gamma)\, d\gamma .$$

Consequently we get

$$\upsilon(r) = \frac{128\pi\rho^2 t^4}{k^4} \int_0^{\frac{\pi}{2}} g(\beta)\, d\beta \int_0^{\beta} \gamma g(\gamma)\, d\gamma . \tag{2.5}$$

Now, an elementary calculus shows that

$$\int_0^{\frac{\pi}{2}} g(\beta)\, d\beta \int_0^{\beta} \gamma g(\gamma)\, d\gamma = \frac{\pi}{16t^4} \left\{ 1 - \frac{1 - \frac{3}{2}t^2}{\left(1 - t^2\right)^{\frac{3}{2}}} \right\} . \tag{2.6}$$

From(2.5) and (2.6) it immediately follows that

$$\upsilon(r) = \frac{8\pi^2\rho^2}{k^4} \left\{ 1 - \frac{1 - \frac{3}{2}t^2}{\left(1 - t^2\right)^{\frac{3}{2}}} \right\} \tag{2.7}$$

with $t = \tanh kr$.

Accordingly the mean number of mathematical vertices that lie in anannulus with distant r from O and breadth dr is equal to

$$d\upsilon(r) = \frac{12\pi^2\rho^2}{k^3} \sinh^3 kr dr . \tag{2.8}$$

Now we note that

$$E(V) = \int_0^{\infty} P(r) \cdot d\upsilon(r) ,$$

which we can easily evaluate using Lemma 1 and (2.8). As a consequence the desired result can be obtained.

Next we turn to computation of the mean perimeter length $E(L)$.

Theorem 2.

$$E(L) = \frac{8}{\sqrt{\pi\rho}} \int_0^{\infty} e^{-u} \sqrt{u + \frac{1}{2\mu} u^2}\, du .$$

Proof. We first consider a mathematical edge whose distance from O lie between x and $x + dx$, and denote by $2z$ the length of tits portion that is contained in a circle with center at O and radius r. Hyperbolic trigonometry shows cosh kz = cosh kr/cosh kx. Consequently the length of its portion that is contained in an annulus A with distant r from O and breadth dr is equal to

$$d_r\{2z\} = \frac{2\sin kr\, dr}{\sqrt{\sinh^2 kr - \sinh^2 kx}}.$$

We have already seen that the mean number of mathematical edges whose distances from O are between x and $x + dx$ is given by (2.1), being z replaced by x. Accordingly the mean length of portions of these mathematical edges that are contained in the annulus A are

$$\begin{aligned}&\int_0^r 4\pi\rho \frac{\sinh 2kx}{k} dx \cdot \frac{2\sinh kr\, dr}{\sqrt{\sinh^2 kr - \sinh^2 kx}} \\ &= \frac{16\pi\rho}{k}\sinh kr\, dr \int_0^r \frac{\sinh kx \cosh kx}{\sqrt{\sinh^2 kr - \sinh^2 kx}} dx,\end{aligned}$$

which turns out to be

$$\frac{16\pi\rho}{k^2}\sinh^2 kr\, dr. \tag{2.9}$$

Now Lemma1 states that any infinitesimal element of these mathematical edges becomes that of actual edges with probability P(r). Therefore, using (2.9), we can show that

$$\begin{aligned}E(L) &= \int_0^\infty P(r)\cdot \frac{16\pi\rho}{k^2}\sinh^2 kr\, dr \\ &= \frac{8}{\sqrt{\pi\rho}} \int_0^\infty e^{-u}\sqrt{u + \frac{1}{2\mu}u^2}\, du .\end{aligned}$$

Thus the proof is completed.

3. Poisson-Delaunay tessellation

In this section we shall study the probability distribution of an angle of Delaunay triangle. Let us consider a Delanay triangle OAB. By Slivnyak's theorem we may assume that O is the origin. We put a = OA, b = OB, c = AB, andγ=∠AOB. Furthermore, if it has the circumcenter, we denote its circumradius by R. First we study existence of the circumcenter. For this purpose we introduce the following quantities:

$$\begin{aligned}Q_1 =\ & \sinh^2 ka \sinh^2 kb \sin^2\gamma, \\ Q_2 =\ & 3 + 2(\cosh ka + \cosh kb)\ (\cosh ka \cosh kb - 1) \\ &+ \left(\cosh^2 ka + \cosh^2 kb\right)\ \sin^2\gamma - \cosh^2 ka \cosh^2 kb\left(1 + \cos^2\gamma\right) \\ &+ 2\sinh ka \sinh kb(\cosh ka - 1)\ (\cosh kb - 1)\ \cos\gamma.\end{aligned} \tag{3.1}$$

Lemma 2. *A triangle has the circumcenter if and only if Q2 > 0. If it has the circumcenter, its circumradius is given by*

$$\cosh kR = \frac{\sqrt{Q_1}}{\sqrt{Q_2}} . \tag{3.2}$$

Proof. We set

$$\sigma = \frac{1}{2}\left(\sinh\frac{ka}{2} + \sinh\frac{kb}{2} + \sinh\frac{kc}{2}\right)$$

and

$$\tau = 1 + 4\cdot\frac{\sigma\left(\sigma - \sinh\frac{ka}{2}\right)\left(\sigma - \sinh\frac{kb}{2}\right)\left(\sigma - \sinh\frac{kc}{2}\right)}{\sinh^2\frac{ka}{2}\sinh^2\frac{kb}{2}\sinh^2\frac{kc}{2}} .$$

Then, as is shown in p.118 of Fenchel (1989), a triangle has the circumcenter if only if $\tau > 1$, and moreover, if it has the circumcenter, its circumradius is given by $\tanh kR = 1/\sqrt{\tau}$.

Now note that

$$\begin{aligned}
&16\sigma\left(\sigma - \sinh\frac{ka}{2}\right)\left(\sigma - \sinh\frac{kb}{2}\right)\left(\sigma - \sinh\frac{kc}{2}\right)\\
&= -\sinh^4\frac{ka}{2} - \sinh^4\frac{kb}{2} - \sinh^4\frac{kc}{2}\\
&\quad + 2\sinh^2\frac{ka}{2}\sinh^2\frac{kb}{2} + 2\sinh^2\frac{kb}{2}\sinh^2\frac{kc}{2} + 2\sinh^2\frac{kc}{2}\sinh^2\frac{ka}{2}
\end{aligned}$$

Then, using $\sinh^2\frac{ka}{2} = \frac{1}{2}(\cosh ka - 1)$ and similar relations, we have

$$\tau = \frac{1 - \cosh^2 ka - \cosh^2 kb - \cosh^2 kc + 2\cosh ka\cosh kb\cosh kc}{2(\cosh ka - 1)(\cosh kb - 1)(\cosh kc - 1)} . \tag{3.3}$$

Hence

$$\cosh^2 kR = \frac{1}{1 - \tanh^2 kR} = \frac{1}{1 - 1/\tau} = \frac{\tilde{Q}_1}{\tilde{Q}_2} ,$$

where

$$\begin{aligned}
\tilde{Q}_1 &= 1 - \cosh^2 ka - \cosh^2 kb - \cosh^2 kc + 2\cosh ka\cosh kb\cosh kc,\\
\tilde{Q}_2 &= 3 - 2\cosh ka - 2\cosh kb - 2\cosh kc - \cosh^2 ka - \cosh^2 kb - \cosh^2 kc\\
&\quad + 2\cosh ka\cosh kb + 2\cosh kb\cosh kc + 2\cosh kc\cosh ka.
\end{aligned} \tag{3.4}$$

To (3.4) we apply the cosine formula of hyperbolic trigonometry. As a result, we can see $\tilde{Q}_1 = Q_1$ and $\tilde{Q}_2 = Q_2$ Therefore, since Q_1 is always positive, the proof can be completed.

Next we study a probability density $f(a, b, \gamma)$. Since $\mathbf{H}^2$ has the Riemannian metric

$$ds^2 = dr^2 + \frac{\sinh^2 kr}{k^2} d\theta^2 ,$$

where (γ, θ) denotes polar coordinates, its infinitesimal area element is given by $\dfrac{\sinh kr}{k} dr d\theta.$

Therefore, using Lemma 1 in the preceding section and Lemma 2, we obtain the following lemma.

Lemma 3. *At any point (a, b, γ) for which $Q_2 > 0$,*

$$f(a,b,\gamma) = \frac{1}{I} \cdot \frac{\sinh ka}{k} \cdot \frac{\sinh kb}{k} \cdot \exp\left(-\mu\left(\frac{\sqrt{Q_1}}{\sqrt{Q_2}} - 1\right)\right),$$

where I denotes the normalizing constant and $\mu = 2\pi\rho/k^2$ as in the previous section. Elsewhere $f(a, b, \gamma)$ is identically zero.

From Lemma 3 follows the next Lemma 4, which is concerned with a probability density $f(\gamma)$. To state it, we define

$$J(\lambda) = \int_0^1 k(z,\lambda) \exp\left(-\mu\left(\frac{1}{\sqrt{1-z}} - 1\right)\right) dz \tag{3.5}$$

and

$$k(z,\lambda) = k_1(z,\lambda) + k_2(z,\lambda) + k_3(z,\lambda) , \tag{3.6}$$

where

$$k_1(z,\lambda) = \frac{z}{2(1-z)} \cdot \frac{(2-z)^2 - \lambda^2(8 - 8z + z^2)}{\left(1 - \frac{z(1-\lambda)}{2}\right)^2 \left(1 - \frac{z(1+\lambda)}{2}\right)^2} , \tag{3.7}$$

$$k_2(z,\lambda) = -\frac{\lambda\sqrt{1-\lambda^2}\, z}{8(1-z)^{\frac{3}{2}}} \cdot \left(\arccos \frac{\lambda}{\sqrt{1 - z(1-\lambda^2)}} - \pi\right)$$
$$\cdot \frac{32 - 48z + 4(3 + 2\lambda^2)z^2 - 4(-2 + 3\lambda^2)z^3 - 3(1 - \lambda^2)z^4}{\left(1 - \frac{z(1-\lambda)}{2}\right)^3 \left(1 - \frac{z(1+\lambda)}{2}\right)^3} , \tag{3.8}$$

$$k_3(z,\lambda)=\frac{1}{4\left(1-\lambda^2\right)}\cdot\log\left(1-z\left(1-\lambda^2\right)\right)$$
$$\cdot\frac{-8+4\left(3-3\lambda^2+2\lambda^4\right)z-6\left(1-\lambda^2\right)z^2+\left(1-3\lambda^4+2\lambda^6\right)z^3}{\left(1-\frac{z(1-\lambda)}{2}\right)^3\left(1-\frac{z(1+\lambda)}{2}\right)^3}\ . \tag{3.9}$$

Lemma 4.

$$f(\gamma)=\frac{1}{k^4\cdot I}\cdot J(\cos\gamma)\ .$$

Proof. By Lemma 3 we have

$$f(\gamma)=\frac{1}{I}\iint_{\{(a,b):Q_2>0\}}\frac{\sinh ka}{k}\cdot\frac{\sinh kb}{k}\cdot\exp\left(-\mu\left(\frac{\sqrt{Q_1}}{\sqrt{Q_2}}-1\right)\right)da\,db\ .$$

Here we change variables (a,b) to (x,y) by $x=\tanh\frac{ka}{2}$ and $y=\tanh\frac{kb}{2}$.
Note that, since

$$\cosh ka=\frac{1+x^2}{1-x^2},\sinh ka=\frac{2x}{1-x^2},\cosh kb=\frac{1+y^2}{1-y^2},\sinh kb=\frac{2y}{1-y^2},$$

the expression $\sqrt{Q_1}\,/\sqrt{Q_2}$ reduces to

$$\frac{\sin\gamma}{\sqrt{1-x^2-y^2+2xy\cos\gamma-\cos^2\gamma}}\ .$$

Consequently we have

$$f(\gamma)=\frac{1}{k^4\cdot I}\cdot\tilde{J}(\cos\gamma)\ ,$$

where

$$\tilde{J}(\lambda)=\int\int_{\{(x,y):1-x^2-y^2+2\lambda xy-\lambda^2>0\}}\frac{4xdx}{\left(1-x^2\right)^2}\cdot\frac{4ydy}{\left(1-y^2\right)^2}$$
$$\cdot\exp\left(-\mu\left(\frac{\sqrt{1-\lambda^2}}{\sqrt{1-x^2-y^2+2\lambda xy-\lambda^2}}-1\right)\right). \tag{3.10}$$

By another change of variables (x,y) to (u,v) by $x=(u-v)/\sqrt{2}$ *and* $y=(u+v)/\sqrt{2}$ in (3.10), $\tilde{J}$ can be rewritten as

$$\tilde{J}(\lambda) = \int\int_{\left\{(u,v):\frac{u^2}{1+\lambda}+\frac{v^2}{1-\lambda}<1\right\}} \frac{8\left(u^2-v^2\right)dudv}{\left\{1-\left(u^2+v^2\right)+\left(\frac{u^2-v^2}{2}\right)^2\right\}^2} \cdot \exp\left(-\mu\left(\frac{1}{\sqrt{1-\frac{u^2}{1+\lambda}-\frac{v^2}{1-\lambda}}}-1\right)\right). \tag{3.11}$$

In (3.11) we change variables again from (u,v) to (z, θ) by

$$u=\sqrt{1+\lambda}\sqrt{z}\cos\theta, v=\sqrt{1-\lambda}\sqrt{z}\sin\theta.$$

As a result we have

$$\tilde{J}(\lambda)=\int_0^1 \tilde{k}(z,\lambda)\exp\left(-\mu\left(\frac{1}{\sqrt{1-z}}-1\right)\right)dz,$$

where

$$\tilde{k}(z,\lambda)=8z\sqrt{1-\lambda^2}\int_0^{\theta_0}\frac{\lambda+\cos 2\theta}{\left\{1-z(1+\lambda\cos 2\theta)+\frac{z^2}{4}(\lambda+\cos 2\theta)^2\right\}^2}d\theta \tag{3.12}$$

and

$$0<\theta_0=\arctan\sqrt{\frac{1+\lambda}{1-\lambda}}<\frac{\pi}{2}.$$

Now, by change of variable as $t=\tan\theta$, we have

$$\tilde{k}(z,\lambda)=8z\sqrt{1-\lambda^2}(1-\lambda)\int_0^a\frac{\left(a^2-t^2\right)\left(1+t^2\right)dt}{\left\{\left(1+t^2\right)^2-2b\left(a^2+t^2\right)\left(1+t^2\right)+b^2\left(a^2-t^2\right)^2\right\}^2},$$

where

$$a=\sqrt{\frac{1+\lambda}{1-\lambda}}, \text{ and } b=\frac{z(1-\lambda)}{2}.$$

Since the integrand in the last integral is a rational function of t, we can evaluate it in principle. However this task is so cumbersome that we have carried out it with the aid of computer algebra. As a result it turns out that $\tilde{k}(z,\lambda)$ coincides with $k(z,\lambda)$. Therefore $\tilde{J}(\lambda) = J(\lambda)$, and the proof is completed.

Next we will evaluate the normalizing constant I.

Lemma 5.

$$k^4 \cdot I = 12\pi\left(\frac{1}{\mu^2} + \frac{1}{\mu^3}\right).$$

Proof. Using Lemma 4 we have

$$\begin{aligned} k^4 \cdot I &= \int_0^\pi J(\cos\gamma)\, d\gamma = \int_{-1}^1 J(\lambda)\frac{d\lambda}{\sqrt{1-\lambda^2}} \\ &= \int_{-1}^1 \frac{d\lambda}{\sqrt{1-\lambda^2}} \int_0^1 k(z,\lambda)\exp\left(-\mu\left(\frac{1}{\sqrt{1-z}} - 1\right)\right) dz \\ &= \int_0^1 \exp\left(-\mu\left(\frac{1}{\sqrt{1-z}} - 1\right)\right) dz \int_{-1}^1 k(z,\lambda)\frac{d\lambda}{\sqrt{1-\lambda^2}} \ . \end{aligned}$$

Thus we will first evaluate

$$\int_{-1}^1 k(z,\lambda)\frac{d\lambda}{\sqrt{1-\lambda^2}} \ .$$

With the aid of computer algebra, we can evaluate without difficulty:

$$\int_{-1}^1 k_1(z,\lambda)\frac{d\lambda}{\sqrt{1-\lambda^2}} = 0 \qquad (3,13)$$

and

$$\int_{-1}^1 k_3(z,\lambda)\frac{d\lambda}{\sqrt{1-\lambda^2}} = \frac{4\pi z(8-7z)}{(4-3z)^2(1-z)^{\frac{3}{2}}} \ . \qquad (3.14)$$

On theother hand, in order to evaluate an integral corresponding to $k_2(z,\lambda)$, we introduce a function $h(\lambda)$ by

$$\frac{k_2(z,\lambda)}{\sqrt{1-\lambda^2}} = h(\lambda)\cdot G(\lambda) \ ,$$

where

$$G(\lambda) = \arccos\frac{\lambda}{\sqrt{1-z(1-\lambda^2)}} - \pi \ .$$

Furthermore we denote by $H(\lambda)$ a primitive function $h(\lambda)$.

Then partial integration leads to

$$\int_{-1}^{1} k(z,\lambda)\frac{d\lambda}{\sqrt{1-\lambda^2}} = \int_{-1}^{1} h(\lambda)\cdot G(\lambda)\, d\lambda$$

$$= \left[H(\lambda)\, G(\lambda)\right]_{-1}^{1} - \int_{-1}^{1} H(\lambda)\, G'(\lambda)\, d\lambda .$$

We see that

$$G'(\lambda) = -\frac{\sqrt{1-z}}{\sqrt{1-\lambda^2}\left(1-z\left(1-\lambda^2\right)\right)} \tag{3.15}$$

and

$$\begin{aligned} H(\lambda) = {} & \frac{8}{z(1-z)^{\frac{1}{2}}\left(2-z(1-\lambda)\right)^2} + \frac{8}{z(1-z)^{\frac{1}{2}}\left(2-z(1+\lambda)\right)^2} \\ & + \frac{2(-2+3z)}{z(1-z)^{\frac{3}{2}}\left(2-z(1-\lambda)\right)} + \frac{2(-2+3z)}{z(1-z)^{\frac{3}{2}}\left(2-z(1+\lambda)\right)} . \end{aligned} \tag{3.16}$$

From (3.16) it follows that

$$\left[H(\lambda)\, G(\lambda)\right]_{-1}^{1} = -\frac{\pi(4-z)}{(1-z)^{\frac{5}{2}}} .$$

On the other hand, using (3.15) and (3.16), we can show

$$\int_{-1}^{1} H(\lambda)\, G'(\lambda)\, d\lambda = -\frac{8\pi\left(8-8z+z^2\right)}{(4-3z)^2(1-z)^{\frac{3}{2}}} .$$

Consequently we get

$$\int_{-1}^{1} k_2(z,\lambda)\frac{d\lambda}{\sqrt{1-\lambda^2}} = \frac{\pi z\left(16-12z-z^2\right)}{(4-3z)^2(1-z)^{\frac{5}{2}}} . \tag{3.17}$$

Adding up(3.13), (3.14), and (3.17), we obtain

$$\int_{-1}^{1} k(z,\lambda)\frac{d\lambda}{\sqrt{1-\lambda^2}} = \frac{3\pi z}{(1-z)^{\frac{5}{2}}} .$$

Therefore

$$\begin{aligned} k^4 \cdot I &= \int_0^1 \frac{3\pi z}{(1-z)^{\frac{5}{2}}} \cdot \exp\left(-\mu\left(\frac{1}{\sqrt{1-z}}-1\right)\right) dz \\ &= 6\pi \int_0^{\infty} \left(2x+x^2\right) e^{-\mu x} dx \\ &= 12\pi\left(\frac{1}{\mu^2}+\frac{1}{\mu^3}\right). \end{aligned}$$

Thus the proof is completed.

Combining Lemma 4 and Lemma 5, we obtain the following theorem.

Theorem 3.

$$f(\gamma) = \frac{\mu^3}{12\pi(1+\mu)} \cdot J(\cos\gamma) .$$

The function J is defined by (3.5) in a form of integral. To our regret, this integral seems to be intractable in terms of elementary functions. However we can give an explicit expression for the expectation $E(\gamma)$.

Theorem 4.

$$E(\gamma) = \frac{\pi}{3} \cdot \frac{1}{1+\frac{1}{\mu}}$$

Proof. Using Theorem 3 we have

$$\begin{aligned} E(\gamma) &= \int_0^\pi \gamma f(\gamma)\, d\gamma \\ &= \frac{\mu^3}{12\pi(1+\mu)} \int_0^1 \exp\left(-\mu\left(\frac{1}{\sqrt{1-z}} - 1\right)\right) dz \int_{-1}^1 k(z,\lambda)\ \arccos\lambda \frac{d\lambda}{\sqrt{1-\lambda^2}} . \end{aligned}$$

Thus we will first evaluate

$$\int_{-1}^1 k(z,\lambda)\ \arccos\lambda \frac{d\lambda}{\sqrt{1-\lambda^2}} .$$

With the aid of computer algebra, we can evaluate without difficlity:

$$\int_{-1}^1 k_1(z,\lambda)\ \arccos\lambda \frac{d\lambda}{\sqrt{1-\lambda^2}} = 0 \tag{3.18}$$

and

$$\int_{-1}^1 k_3(z,\lambda)\ \arccos\lambda \frac{d\lambda}{\sqrt{1-\lambda^2}} = \frac{2\pi^2 z(8-7z)}{(4-3z)^2(1-z)^{\frac{3}{2}}} . \tag{3.19}$$

On the other hand,

$$\begin{aligned}
\int_{-1}^{1} k_2(z,\lambda)\ \arccos\lambda \frac{d\lambda}{\sqrt{1-\lambda^2}} &= \int_{-1}^{1} h(\lambda)\, G(\lambda)\ \arccos\lambda\, d\lambda \\
&= \int_{-1}^{1} h(\lambda)\left(G(\lambda)+\frac{\pi}{2}-\frac{\pi}{2}\right)\left(\arccos\lambda-\frac{\pi}{2}+\frac{\pi}{2}\right) d\lambda \\
&= \int_{-1}^{1} h(\lambda)\left(G(\lambda)+\frac{\pi}{2}\right)\left(\arccos\lambda-\frac{\pi}{2}\right) d\lambda \\
&\quad +\frac{\pi}{2}\int_{-1}^{1} h(\lambda)\left(G(\lambda)+\frac{\pi}{2}\right) d\lambda-\frac{\pi}{2}\int_{-1}^{1} h(\lambda)\left(\arccos\lambda-\frac{\pi}{2}\right) d\lambda \\
&\quad -\left(\frac{\pi}{2}\right)^2\int_{-1}^{1} h(\lambda)\, d\lambda .
\end{aligned}$$

Now we observe that the function h is an odd function of λ and, on the other hand, the function $\left(G(\lambda)+\frac{\pi}{2}\right)\left(\arccos\lambda-\frac{\pi}{2}\right)$ is an even function. Consequently

$$\begin{aligned}
&\int_{-1}^{1} k_2(z,\lambda)\ \arccos\lambda \frac{d\lambda}{\sqrt{1-\lambda^2}} \\
&\quad =\frac{\pi}{2}\int_{-1}^{1} h(\lambda)\left(G(\lambda)+\frac{\pi}{2}\right) d\lambda-\frac{\pi}{2}\int_{-1}^{1} h(\lambda)\left(\arccos\lambda-\frac{\pi}{2}\right) d\lambda \\
&\quad =\frac{\pi}{2}\int_{-1}^{1} h(\lambda)\, G(\lambda)\, d\lambda-\frac{\pi}{2}\int_{-1}^{1} h(\lambda)\ \arccos\lambda\, d\lambda .
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_{-1}^{1} k_2(z,\lambda)\ \arccos\lambda \frac{d\lambda}{\sqrt{1-\lambda^2}} \\
&\quad =\frac{\pi}{2}\int_{-1}^{1} k_2(z,\lambda)\frac{d\lambda}{\sqrt{1-\lambda^2}}-\frac{\pi}{2}\int_{-1}^{1} h(\lambda)\ \arccos\lambda\, d\lambda . \qquad (3.20)
\end{aligned}$$

Here we can evaluate again with the help of computer

$$\int_{-1}^{1} h(\lambda)\ \arccos\lambda\, d\lambda=\frac{\pi\left(4-z-4\sqrt{1-z}\right)}{(1-z)^{\frac{5}{2}}} . \qquad (3.21)$$

Accordingly, substituting (3.17) and (3.21) into (3.20), we get

$$\int_{-1}^{1} k_2(z,\lambda)\ \arccos\lambda \frac{d\lambda}{\sqrt{1-\lambda^2}}=\frac{4\pi^2\left(8-8z+z^2\right)}{(4-3z)^2(1-z)^{\frac{3}{2}}}+\frac{2\pi^2}{(1-z)^2} . \qquad (3.22)$$

Then, adding up (3.18), (3.19), and (3.22), we obtain

$$\int_{-1}^{1} k(z,\lambda)\ \arccos\lambda \frac{d\lambda}{\sqrt{1-\lambda^2}} = 2\pi^2\left(\frac{1}{(1-z)^2} - \frac{1}{(1-z)^{\frac{3}{2}}}\right).$$

Therefore

$$\begin{aligned} E(\gamma) &= \frac{\mu^3}{12\pi(1+\mu)}\int_0^1 2\pi^2\left(\frac{1}{(1-z)^2} - \frac{1}{(1-z)^{\frac{3}{2}}}\right)\cdot \exp\left(-\mu\left(\frac{1}{\sqrt{1-z}} - 1\right)\right) dz \\ &= \frac{\mu^3}{12\pi(1+\mu)}\int_0^\infty 4\pi^2 x\, e^{-\mu x}\, dx \\ &= \frac{\pi\mu}{3(1+\mu)}\ . \end{aligned}$$

Thus we have completed the proof.

At first sight Theorem 4 above and Theorem 1 in the preceding section are equivalent, that is, deduced from each other. Heuristically we may expect $E(V)\cdot E(\gamma) = 2\pi$. However I can not prove this simple relation before we have computed both $E(V)$ and $E(\gamma)$ individually.

From Theorem 4 and the Gauss-Bonnet formula immediately follows the following corollary.

Corollary 1. *The expectation of sum of three angles of a Delaunay triangle is equal to* $\dfrac{\pi\mu}{1+\mu}$. *And the expectation of area of a Delaunay triangle is equal to* $\dfrac{\pi}{k^2+2\pi\rho}$.

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