# Recursive Least-Squares Wiener Fixed-Point Smoother with Uncertain Observations for Colored Observation Noise in Linear Discrete-Time Stochastic Systems

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## Abstract

This paper proposes recursive least-squares (RLS) Wiener fixed-point smoothing and filtering algorithms with uncertain observations for colored observation noise in linear discrete-time stochastic systems. The observation equation is given by $y(k) = \gamma(k)z(k) + v_c(k)$, $z(k) = Hx(k)$, where $\{\gamma(k)\}$ is a binary switching sequence with conditional probability, which satisfies (3). The estimators require the following information. (1) The system matrix $\Phi$ for the state vector $x(k)$. (2) The observation matrix $H$. (3) The variance $K(k,k)$ of the state vector $x(k)$. (4) The variance $K_c(k,k)$ of the colored observation noise. (5) The system matrix $\Phi_c$ for the colored observation noise $v_c(k)$. (6) The probability $p(k) = P\{\gamma(k) = 1\}$ that the signal exists in the uncertain observation equation and the $(2,2)$ element $[P(k|j)]_{2,2}$ of the conditional probability of $\gamma(k)$, given $\gamma(j)$, $1 \le j < k$.

**Keywords**: Uncertain observations, RLS Wiener fixed-point smoother, Conditional probability, Discrete-time stochastic systems

## 1. Introduction

The estimation problem given uncertain observations has been an important research in the area of detection and estimation problems in communication systems [1]. Nahi [2], assuming that the state-space model is given, proposes the RLS estimation method with uncertain observations, when the uncertainty is modeled in terms of independent random variables, and the probability that the signal exists in each observation is available. By uncertain observations it is meant that some observations do not contain the signal and consist only of observation noise. In Hadidi and Schwartz [3], the work of Nahi is extended to the case where the variables modeling the uncertainty are not

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necessarily independent.

In the above researches, it is assumed that the state-space model for the signal is given. However, to use the state-space model, the state-space model must be modeled and its modeling errors might cause the degradation of estimation accuracy. In [4], the RLS Wiener fixed-point smoothing and filtering algorithms are derived, based on the invariant imbedding method, from uncertain observations with the uncertainty modeled by independent random variables. In the RLS Wiener estimators, the system matrix $\Phi$, the observation matrix $\mathrm{H}$, the variance $\mathrm{K(k,k)}$ of the state vector $\mathrm{x(k)}$, the variance $\mathrm{R(k)}$ of the observation noise $\mathrm{v(k)}$ and the observed values $\mathrm{y(k)}$ are used. Also, in [5], based on the innovation approach, the RLS Wiener fixed-point smoother and filter are proposed in linear discrete-time stochastic systems. Here, the observation equation is given by $\mathrm{y(k)=\gamma(k)z(k)+v(k)}$, $\mathrm{z(k)=Hx(k)}$, where $\{\gamma(\mathrm{k})\}$ is a binary switching sequence with conditional probability, which satisfies (3). The innovation process is given by $\nu(\mathrm{s})=\mathrm{y(s)}-\hat{y}(s,s-1)$, $\hat{y}(s,s-1)=P_{2,2}(s)H\Phi\hat{x}(s-1,s-1)$ in terms of the (2,2) element $[\mathrm{P(k|j)}]_{2,2}$ of the conditional probability of $\gamma(\mathrm{k})$, given $\gamma(\mathrm{j})$. This expression for the innovation process is shown in [5], [6]. Similarly, in Nakamori et. al. [7], the RLS Wiener prediction algorithm is proposed.

In this paper, with the same assumptions for the observation equation as in [5], the algorithms for the RLS Wiener fixed-point smoother and filter are derived based on the invariant imbedding method. Namely, the observation equation is given by $\mathrm{y(k)}=\gamma(\mathrm{k})\mathrm{z(k)}+v_c(k)$, $\mathrm{z(k)=Hx(k)}$, where $\{\gamma(\mathrm{k})\}$ is a binary switching sequence with conditional probability, which satisfies (3). The estimators require the following information. (1) The system matrix $\Phi$ concerned with the state vector $\mathrm{x(k)}$. (2) The observation matrix $\mathrm{H}$. (3) The variance $\mathrm{K(k,k)}$ of the state vector $\mathrm{x(k)}$. (4) The variance $K_c(k,k)$ of the colored observation noise. (5) The system matrix $\Phi_c$ concerned with the colored observation noise $v_c(k)$. (6) The probability $\mathrm{p(k)}=\mathrm{P}\{\gamma(\mathrm{k})=1\}$ that the signal exists in the uncertain observation equation and the (2,2) element $[\mathrm{P(k|j)}]_{2,2}$ of the conditional probability of $\gamma(\mathrm{k})$, given $\gamma(\mathrm{j})$, $1\le \mathrm{j}<k$. The RLS Wiener fixed-point smoothing and filtering algorithms are proposed in Theorem 1 and its proof is shown in the Appendix in details.

## 2. Problem formulation

Let an observation equation be given by

$$\mathrm{y(k)}=\gamma(\mathrm{k})\mathrm{z(k)}+v_c(k),\ \mathrm{z(k)=Hx(k)}, \tag{1}$$

where z(k) is a signal, x(k) the $n \times 1$ zero-mean state vector and H is the $m \times n$ observation matrix.

- The sequence $\{v_c(k)\}$ is colored noise with its mean zero and the variance of $v_c(k)$ is $K_c(k,k)$, that is,

$$K_c(k+1,k+1) = \Phi_c K_c(k,k)\Phi_c^T + R_u(k). \qquad (2)$$
$$v_c(k+1) = \Phi_c v_c(k) + u(k),\ E[u(k)u^T(s)] = R_u(k)\delta_K(k-s).$$

Here, $R_u(k)$ denotes the input variance of white noise u(k). For the wide-sense stationary-stochastic systems, from the relationship $K_c(k+1,k+1) = K_c(k,k)$, $R_u(k)$ is calculated by $R_u(k) = K_c(k,k) - \Phi_c K_c(k,k)\Phi_c^T$. Also, we assume that the signal $z(\cdot)$ is uncorrelated with the colored observation noise $v_c(\cdot)$.

- The random sequence $\{\gamma(k)\}$, which describes the uncertainty in the observations, has the following stochastic properties (Hadidi and Schwartz (1979)):
  (P-1) $\gamma(k)$ is a discrete-time random variable which takes on the values 0 or 1 with $P\{\gamma(k) = 1\} = p(k)$. So, p(k) represents the probability that observed value y(k) contains the signal z(k), and we will assume that this probability is nonzero.
  (P-2) The noise $\{\gamma(k)\}$ is a sequence of random variables with initial probability vector $(1 - p(0), p(0))^T$ and conditional probability matrix $P(k|j)$. The (2,2) element of the conditional probability matrix of $\gamma(k)$ given $\gamma(j)$, is independent of j, for $j < k$, that is

$$[P(k|j)]_{2,2} = \frac{E[\gamma(j)\gamma(k)]}{P\{\gamma(j)=1\}} = P_{2,2}(k),\ j = 0, \cdots, k-1. \qquad (3)$$

- The state process $\{x(k)\}$ and the sequences $\{\gamma(k)\}$ and $\{u(k)\}$ are mutually independent.

Let us introduce the system matrix $\Phi$ in the state-space model for the state vector x(k) and the variance $K(s,s)$ of the state vector x(s). Then the autocovariance function $K_z(k,s)$ of the signal z(k) is factorized as

$$K_z(k,s) = HK(k,s)H^T,$$
$$K(k,s) = A(k)B^T(s),\ A(k) = \Phi^k,\ B^T(s) = \Phi^{-s}K(s,s),\ 0 \le s \le k. \qquad (4)$$

Let the fixed-point smoothing estimate $\hat{x}(k,L)$, at the fixed point k, of x(k) be given by

$$\hat{x}(k,L)=\sum_{i=1}^{L}h(k,i,L)y(i) \tag{5}$$

as a linear transformation of the observed values $y(i)$, $1\le i\le L$. Let us consider least-squares fixed-point smoothing problem, which minimizes the criterion

$$J=E[(x(k)-\hat{x}(k,L))^T(x(k)-\hat{x}(k,L))]. \tag{6}$$

The optimum impulse response function $h(k,s,L)$, which minimizes the cost function (6), satisfies the Wiener-Hopf equation

$$E[x(k)y(s)]=\sum_{i=1}^{L}h(k,i,L)E[y(i)y(s)] \tag{7}$$

in terms of the orthogonal projection lemma [8]

$$x(k)-\hat{x}(k,L)\perp y(i),\ 1\le i\le L. \tag{8}$$

From $P\{\gamma(k)=1\}=p(k)$, the left hand side of (7) is written as

$$E[x(k)y(s)]=K(k,s)H^Tp(s). \tag{9}$$

Let $E_\gamma[\cdot]$ denote the statistical expectation with respect to $\gamma(\cdot)$. Then, from the observation equation (1) and the covariance function (2) for colored observation noise $v_c(k)$, $E[y(i)y(s)]$ is reduced to

$$E[y(i)y^T(s)]=E_\gamma[\gamma(i)\gamma(s)]HK(i,s)H^T+K_c(i,s), \tag{10}$$
$$K_c(i,s)=\Phi_cK_c(i,s)\Phi_c^T+\Phi_cK_{vu}(i,s)+K_{vu}(i,s)\Phi_c^T+R_u(i)\delta_K(i-s),$$
$$K_c(k,s)=A_c(k)B_c^T(s),\ 0\le s\le k.$$

Substituting (9) and (10) into (7), we have

$$h(k,s,L)R_u(s)=p(s)K(k,s)H^T-\sum_{i=1}^{L}h(k,i,L)\{E_\gamma[\gamma(i)\gamma(s)]HK(i,s)H^T+\Phi_cK_c(i,s)\Phi_c^T+$$
$$\Phi_cK_{vu}(i,s)+K_{uv}(i,s)\Phi_c^T\},$$
$$K_{uv}(i,s)=E[u(i)v_c^T(s)],\ K_{vu}(i,s)=E[v_c(i)u^T(s)]. \tag{11}$$

In section 3, the RLS Wiener fixed-point smoothing and filtering algorithms are presented in linear discrete-time stochastic systems.

## 3. RLS Wiener fixed-point smoothing and filtering algorithms

In [5], [6], based on the innovation approach, in the case of the white observation noise, the algorithms for the fixed-point smoothing and filtering estimates are proposed. The innovation process is expressed as

$$\nu(s) = y(s) - \hat{y}(s, s-1),\ \hat{y}(s, s-1) = P_{2,2}(s)H\Phi\hat{x}(s-1, s-1).$$

Theorem 1, under the preliminary assumptions in section 2, proposes the RLS Wiener algorithms for the fixed-point smoothing and filtering estimates of the signal z(k) and the state vector x(k). The algorithms are derived, starting with (11), by iterative use of the invariant imbedding method.

**Theorem 1**. Let us consider the observation equation (1). Let the probability P(k) and the (2,2) element $P_{2,2}(k)$ of the conditional probability matrix P(k|j) be given. Let the system matrix Φ, the observation matrix H, the autovariance function K(s, s) of the state vector x(s), the variance R(k) of the white observation noise v(k) and the observed value y(k) be given. Then the RLS Wiener algorithms for the fixed-point smoothing estimate $\hat{z}(k, L)$ of the signal z(k) and the fixed-point smoothing estimate $\hat{x}(k, L)$ of the state vector x(k), at the fixed point k, consist of (12)-(35).

Fixed-point smoothing estimate of the signal z(k) at the fixed point k: $\hat{z}(k, L)$

$$\hat{z}(k, L) = H\hat{x}(k, L) \tag{12}$$

Fixed-point smoothing estimate of the state vector x(k) at the fixed point k: $\hat{x}(k, L)$

$$\hat{x}(k, L) = \hat{x}(k, L-1) + h(k, L, L)(y(L) - P_{2,2}(L)H\Phi l_1(L-1) - (\Phi_c)^2 l_2(L-1) - \Phi_c l_3(L-1)) \tag{13}$$

$$l_1(L) = \Phi l_1(L-1) + G_1(L, L)\left(y(L) - P_{2,2}(L)H\Phi l_1(L-1) - (\Phi_c)^2 l_2(L-1) - \Phi_c l_3(L-1)\right),$$
$$l_1(0) = 0 \tag{14}$$

$$l_2(L) = \Phi_c l_2(L-1) + G_2(L, L)\left(y(L) - P_{2,2}(L)H\Phi l_1(L-1) - (\Phi_c)^2 l_2(L-1) - \Phi_c l_3(L-1)\right),$$
$$l_2(0) = 0 \tag{15}$$

$$l_3(L) = \Phi_c l_3(L-1) + G_3(L, L)\left(y(L) - P_{2,2}(L)H\Phi \bar{x}_1(L-1) - (\Phi_c)^2 \bar{x}_2(L-1) - \Phi_c \bar{x}_3(L-1)\right),$$
$$l_3(0) = 0 \tag{16}$$

Smoother gain: h(k, L, L)

$$\mathrm{h}(\mathrm{k}, \mathrm{L}, \mathrm{L}) = (p(L)K(k, k)(\Phi^T)^{L-k}H^T - P_{2,2}(L)f_4(k, L-1)\Phi^T H^T$$
$$-f_2(k, L-1)\Phi_c^T - f_3(k, L-1)\Phi_c^T)\bar{R}(L)^{-1} \tag{17}$$

$$\bar{R}(L) = R_u(L) + p(L)HK(L,L)H^T - P_{2,2}(L)\big(P_{2,2}(L)H\Phi\bar{S}_{11}(L-1)\Phi^T + \Phi_c^2\bar{S}_{12}^T(L-1)\Phi^T + \Phi_c\bar{S}_{13}^T(L-1)\Phi^T\big)H^T + \Phi_c K_c(L,L)\Phi_c^T - \big(P_{2,2}(L)H\Phi S_{12}(L-1)\Phi_c^T + \Phi_c^2 S_{22}(L-1)\Phi_c^T + \Phi_c S_{32}(L-1)\Phi_c^T\big)\Phi_c^T + \mathrm{R_u(L)} - \big(P_{2,2}(L)H\Phi S_{13}(L-1)\Phi_c^T + \Phi_c^2 S_{23}(L-1)\Phi_c^T + \Phi_c S_{33}(L-1)\Phi_c^T\big) \quad (18)$$

$$\mathrm{G_1(L,L)} = \big(\mathrm{p(L)K(L,L)H^T} - \mathrm{P_{2,2}(L)}\Phi_c\bar{S}_{11}(L-1)\Phi^T H^T - \Phi \mathrm{S_{12}(L-1)}(\Phi_c^T)^2 - \Phi S_{12}(L-1)\Phi_c^T\big)\bar{R}(L)^{-1} \quad (19)$$

$$\mathrm{G_2(L,L)} = \big(K_c(L,L)\Phi_c^T - \mathrm{P_{2,2}(L)}\Phi_c\bar{S}_{12}^T(L-1)\Phi^T H^T - \Phi_c \mathrm{S_{22}(L-1)}(\Phi_c^{\mathrm{T}})^2 - \Phi_c S_{23}(L-1)\Phi_c^T\big)\bar{R}(L)^{-1} \quad (20)$$

$$\mathrm{G_3(L,L)} = \big(\mathrm{R_u(L)} - \mathrm{P_{2,2}(L)}\Phi_c\bar{S}_{12}^T(L-1)\Phi^T H^T - \Phi_c \mathrm{S_{22}(L-1)}(\Phi_c^{\mathrm{T}})^2 - \Phi_c S_{23}(L-1)\Phi_c^T\big)\bar{R}(L)^{-1} \quad (21)$$

$$\mathrm{f_2(k,L)} = \mathrm{f_2(k,L-1)}\Phi_c^{\mathrm{T}} + \mathrm{h(k,L,L)}\Phi_c \mathrm{K_c(L,L)} - \mathrm{h(k,L,L)}\big(P_{2,2}(L)H\Phi \mathrm{S_{12}(L-1)}\Phi_c^{\mathrm{T}} + \Phi_c^2 \mathrm{S_{22}(L-1)}\Phi_c^{\mathrm{T}} + \Phi_c \mathrm{S_{32}(L-1)}\Phi_c^{\mathrm{T}}\big), \mathrm{f_2(k,k)} = \mathrm{S_{12}(k)} \quad (22)$$

$$\mathrm{f_3(k,L)} = \mathrm{f_3(k,L-1)}\Phi_c^{\mathrm{T}} + \mathrm{h(k,L,L)}R_u\mathrm{(L)} - \mathrm{h(k,L,L)}\big(P_{2,2}(L)H\Phi \mathrm{S_{13}(L-1)}\Phi_c^{\mathrm{T}} + \Phi_c^2 \mathrm{S_{23}(L-1)}\Phi_c^{\mathrm{T}} + \Phi_c \mathrm{S_{33}(L-1)}\Phi_c^{\mathrm{T}}\big), \mathrm{f_3(k,k)} = \mathrm{S_{13}(k)} \quad (23)$$

$$\mathrm{f_4(k,L)} = \mathrm{f_4(k,L-1)}\Phi^T + \mathrm{p(L)h(k,L,L)HK(L,L)} - \mathrm{h(k,L,L)}\big(P_{2,2}(L)H\Phi \mathrm{\bar{S}_{11}(L-1)}\Phi^T + \Phi_c^2 \mathrm{\bar{S}_{12}^T(L-1)}\Phi^T + \Phi_c \mathrm{\bar{S}_{13}^T(L-1)}\Phi^T\big), \mathrm{f_4(k,k)} = \mathrm{\bar{S}_{11}(k)} \quad (24)$$

Filtering estimate $\mathrm{\hat{x}(k,k)}$ of the state vector $\mathrm{x(k)}$

$$\mathrm{\hat{x}(k,k)} = \mathrm{\hat{x}(k-1,l-1)} + \mathrm{h(k,k,k)}\big(\mathrm{y(k)} - P_{2,2}(k)H\Phi l_1(k-1,k-1) - (\Phi_c)^2 l_2(k-1,k-1) - \Phi_c l_3(k-1,k-1)\big), \mathrm{\hat{x}(0,0)} = 0 \quad (25)$$

Filter gain: $\mathrm{h(k,k,k)}$

$$\mathrm{h(k,k,k)} = \big(\mathrm{p(k)K(k,k)}H^T - \mathrm{P_{2,2}(k)}\Phi\mathrm{\bar{S}_{11}(k-1)}\Phi^T\mathrm{H^T} - \Phi\mathrm{S_{12}(k-1)}(\Phi_c^{\mathrm{T}})^2 - \Phi\mathrm{S_{13}(k-1)}\Phi_c^{\mathrm{T}}\big)\bar{R}(k)^{-1} \quad (26)$$

$$\mathrm{\bar{S}_{11}(k)} = \mathrm{p(L)G_1(k,k)HK_x(k,k)} + \Phi\mathrm{\bar{S}_{11}(k-1)}\Phi^T - \mathrm{G_1(k,k)}\big(\mathrm{P_{2,2}(k)H}\Phi\mathrm{\bar{S}_{11}(k-1)}\Phi^T + (\Phi_c)^2\mathrm{\bar{S}_{12}^T(k-1)}\Phi^T + \Phi_c\mathrm{\bar{S}_{13}^T(k-1)}\Phi^T\big),\ \mathrm{\bar{S}_{11}(0)} = 0 \quad (27)$$

$$\mathrm{\bar{S}_{12}^T(k)} = \mathrm{p(k)G_2(k,k)HK(k,k)} + \Phi_c\mathrm{\bar{S}_{12}^T(k-1)}\Phi^T - \mathrm{G_2(k,k)}\big(\mathrm{P_{2,2}(k)H}\Phi\mathrm{\bar{S}_{11}(k-1)}\Phi^T + (\Phi_c)^2\mathrm{\bar{S}_{12}^T(k-1)}\Phi^T + \Phi_c\mathrm{\bar{S}_{13}^T(k-1)}\Phi^T\big),\ \mathrm{\bar{S}_{12}^T(0)} = 0 \quad (28)$$

$$\mathrm{\bar{S}_{13}^T(k)} = \mathrm{p(k)G_3(k,k)HK(k,k)} + \Phi_c\mathrm{\bar{S}_{13}^T(k-1)}\Phi^T - \mathrm{G_3(k,k)}\big(\mathrm{P_{2,2}(k)H}\Phi\mathrm{\bar{S}_{11}(k-1)}\Phi^T + (\Phi_c)^2\mathrm{\bar{S}_{12}^T(k-1)}\Phi^T + \Phi_c\mathrm{\bar{S}_{13}^T(k-1)}\Phi^T\big),\ \mathrm{\bar{S}_{13}^T(0)} = 0 \quad (29)$$

$\mathrm{S_{12}(k)} =$

$$\mathrm{G}_1(\mathrm{k},\mathrm{k})\Phi_c\mathrm{K}_\mathrm{c}(\mathrm{k},\mathrm{k}) + \Phi\mathrm{S}_{12}(k-1)\Phi_c^\mathrm{T} - \mathrm{G}_1(\mathrm{k},\mathrm{k})\big(\mathrm{P}_{2,2}(\mathrm{k})\mathrm{H}\Phi\mathrm{S}_{12}(k-1)\Phi_c^\mathrm{T} + (\Phi_c)^2\mathrm{S}_{22}(k-1)\Phi_c^\mathrm{T} + \Phi_\mathrm{c}\mathrm{S}_{32}(k-1)\Phi_c^\mathrm{T}\big),\ \mathrm{S}_{12}(0) = 0 \quad (30)$$

$$\mathrm{S}_{22}(\mathrm{k}) = \mathrm{G}_2(\mathrm{k},\mathrm{k})\Phi_c\mathrm{K}_\mathrm{c}(\mathrm{k},\mathrm{k}) + \Phi_\mathrm{c}\mathrm{S}_{22}(k-1)\Phi_c^\mathrm{T} - \mathrm{G}_2(\mathrm{k},\mathrm{k})\big(\mathrm{P}_{2,2}(\mathrm{k})\mathrm{H}\Phi\mathrm{S}_{12}(k-1)\Phi_c^\mathrm{T} + (\Phi_c)^2\mathrm{S}_{22}(k-1)\Phi_c^\mathrm{T} + \Phi_\mathrm{c}\mathrm{S}_{32}(k-1)\Phi_c^\mathrm{T}\big),\ \mathrm{S}_{22}(0) = 0 \quad (31)$$

$$\mathrm{S}_{32}(\mathrm{k}) = \mathrm{G}_3(\mathrm{k},\mathrm{k})\Phi_c\mathrm{K}_\mathrm{c}(\mathrm{k},\mathrm{k}) + \Phi_\mathrm{c}\mathrm{S}_{32}(k-1)\Phi_c^\mathrm{T} - \mathrm{G}_3(\mathrm{k},\mathrm{k})\big(\mathrm{P}_{2,2}(\mathrm{k})\mathrm{H}\Phi\mathrm{S}_{12}(k-1)\Phi_c^\mathrm{T} + (\Phi_c)^2\mathrm{S}_{22}(k-1)\Phi_c^\mathrm{T} + \Phi_\mathrm{c}\mathrm{S}_{32}(k-1)\Phi_c^\mathrm{T}\big),\ \mathrm{S}_{32}(0) = 0 \quad (32)$$

$$\mathrm{S}_{13}(\mathrm{k}) = \mathrm{G}_1(\mathrm{k},\mathrm{k})\mathrm{R}_\mathrm{u}(\mathrm{k}) + \Phi\mathrm{S}_{13}(k-1)\Phi_c^\mathrm{T} - \mathrm{G}_1(\mathrm{k},\mathrm{k})\big(\mathrm{P}_{2,2}(\mathrm{k})\mathrm{H}\Phi\mathrm{S}_{13}(k-1)\Phi_c^\mathrm{T} + (\Phi_c)^2\mathrm{S}_{23}(k-1)\Phi_c^\mathrm{T} + \Phi_\mathrm{c}\mathrm{S}_{33}(k-1)\Phi_c^\mathrm{T}\big),\ \mathrm{S}_{13}(0) = 0 \quad (33)$$

$$\mathrm{S}_{23}(\mathrm{k}) = \mathrm{G}_2(\mathrm{k},\mathrm{k})\mathrm{R}_\mathrm{u}(\mathrm{k}) + \Phi_\mathrm{c}\mathrm{S}_{23}(k-1)\Phi_c^\mathrm{T} - \mathrm{G}_2(\mathrm{k},\mathrm{k})\big(\mathrm{P}_{2,2}(\mathrm{k})\mathrm{H}\Phi\mathrm{S}_{13}(k-1)\Phi_c^\mathrm{T} + (\Phi_c)^2\mathrm{S}_{23}(k-1)\Phi_c^\mathrm{T} + \Phi_\mathrm{c}\mathrm{S}_{33}(k-1)\Phi_c^\mathrm{T}\big),\ \mathrm{S}_{23}(0) = 0 \quad (34)$$

$$\mathrm{S}_{33}(\mathrm{k}) = \mathrm{G}_3(\mathrm{k},\mathrm{k})\mathrm{R}_\mathrm{u}(\mathrm{k}) + \Phi_\mathrm{c}\mathrm{S}_{33}(k-1)\Phi_c^\mathrm{T} - \mathrm{G}_3(\mathrm{k},\mathrm{k})\big(\mathrm{P}_{2,2}(\mathrm{k})\mathrm{H}\Phi\mathrm{S}_{13}(k-1)\Phi_c^\mathrm{T} + (\Phi_c)^2\mathrm{S}_{23}(k-1)\Phi_c^\mathrm{T} + \Phi_\mathrm{c}\mathrm{S}_{33}(k-1)\Phi_c^\mathrm{T}\big),\ \mathrm{S}_{33}(0) = 0 \quad (35)$$

Proof of **Theorem 1** is deferred to the **Appendix**.

From Theorem 1, it is found that the innovation process $\nu(\mathrm{k})$ is represented by

$$\nu(\mathrm{k}) = y(k) - P_{2,2}(k)H\Phi l_1(k-1) - (\Phi_c)^2 l_2(k-1) - \Phi_c l_3(k-1). \quad (36)$$

## 4. A numerical simulation example

In order to show the estimation characteristic of the RLS Wiener fixed-point smoothing algorithm proposed in Theorem 1, we consider to estimate a scalar signal $\mathrm{z(k)}$ whose autocovariance function $K_z(m)$ is given as follows [9].

$$\begin{aligned} &K_z(0) = \sigma^2, \\ &K_z(m) = \sigma^2\{\alpha_1(\alpha_2^2-1)\alpha_1^m/[(\alpha_2-\alpha_1)(\alpha_2\alpha_1+1)] \\ &\quad -\alpha_2(\alpha_1^2-1)\alpha_2^m/[(\alpha_2-\alpha_1)(\alpha_2\alpha_1+1)],\ 0 < m, \end{aligned} \quad (37)$$

$\alpha_1, \alpha_2 = (-a_1 \pm \sqrt{a_1^2 - 4a_2})/2,\ a_1 = -0.1,\ a_2 = -0.8,\ \sigma = 0.5.$

The covariance function (37) corresponds to a signal process generated by a second-order AR model. Then, according to [4], the observation vector H, the variance $\mathrm{K(k,k) = K(0)}$ of the state vector $\mathrm{x(k)}$ and the system matrix $\Phi$ in the state equation are as follows:

$$\mathrm{H} = [1 \quad 0],\ \mathrm{K(k,k)} = \begin{bmatrix} K_z(0) & K_z(1) \\ K_z(1) & K_z(0) \end{bmatrix},\ \Phi = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix},$$

$$K_z(0) = 0.25,\ K_z(1) = 0.125. \tag{38}$$

As in [7], we consider that the signal z(k) is transmitted through one of two channels, each characterized by its observation equation as follows:

Channel 1: $\mathrm{y(k)} = \mathrm{z(k)} + v_c(\mathrm{k})$,
Channel 2: $\mathrm{y(k)} = \mathrm{U(k)z(k)} + v_c(\mathrm{k}),\ \Phi_c = 0.91$,

where $v_c(\mathrm{k})$ is a colored observation noise and $\{\mathrm{U(k)}\}$ is a sequence of independent random variables taking values 0 or 1 with $\mathrm{P\{U(k) = 1\}} = \breve{p} = 0.8$, for all k.

We suppose that channel 1 is chosen at random with probability $1 - \mathrm{q} = 0.7$ and, hence, channel 2 is selected with probability $\mathrm{q} = 0.3$. Then, the observation equation is described by

$$\mathrm{y(k)} = \gamma(\mathrm{k})\mathrm{z(k)} + \mathrm{v(k)}, \tag{39}$$

where $\gamma(\mathrm{k}) = (1 - \alpha)1 + \alpha\mathrm{U(k)}$ and $\alpha$ is a random variable, independent of $\{\mathrm{U(k)}\}$, taking values 0 or 1 with $\mathrm{P}\{\alpha = 1\} = \mathrm{q} = 0.3$. It can be shown that $\{\gamma(\mathrm{k})\}$ is a sequence of random variables, which take values 0 or 1 with $\mathrm{p(k)} = \mathrm{P}\{\gamma(\mathrm{k}) = 1\} = \mathrm{P}\{\alpha = 1, \mathrm{U(k)} = 1\} + \mathrm{P}\{\alpha = 0\} = \breve{p}q + (1 - q) = 0.94$, for all k, and conditional probability matrix

$$\mathrm{P(k|j)} = \begin{bmatrix} 1 - \breve{p} & \breve{p} \\ \dfrac{q\breve{p}(1 - \breve{p})}{1 - q(1 - \breve{p})} & \dfrac{1 - q(1 - \breve{p}^2)}{1 - q(1 - \breve{p})} \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & 0.8 \\ 0.0510638 & 0.9489362 \end{bmatrix},$$

for all $\mathrm{k, j} = 0,1,\cdots,\mathrm{k} - 1$. From (3), it is clear that $[\mathrm{P(k|j)}]_{2,2} = P_{2,2}(k) = 0.9489362$, for all $\mathrm{k, j} = 0,1,\cdots,\mathrm{k} - 1$.

Substituting H, K(k,k), Φ, $K_c(k,k)$, $\Phi_c$, p(k) and $P_{2,2}(k)$ into the estimation algorithms of Theorem 1, we can calculate the fixed-point smoothing and filtering estimates of the signal recursively.

Fig.1 illustrates the colored observation noise process for the values of the input noise variance, $R_u(k) = 0.1^2,\ 0.15^2,\ 0.2^2$. As $R_u(k)$ becomes large, it is seen that the variance of the colored observation noise process tends to be large. Fig.2 illustrates the sequences of the fixed-point smoothing estimate $\hat{z}(k, k + 5)$ and the filtering estimate $\hat{z}(k, k)$ of the signal z(k) for the input noise variance $R_u(k) = 0.1^2$. Fig.3 illustrates the mean-square values (MSVs) of the filtering and fixed-point smoothing errors in the certain and uncertain observation cases for the values of the input noise variance as $R_u(k) = 0.1^2,\ 0.15^2,\ 0.2^2$. It is found that the estimation accuracy of the fixed-point smoother is better than the filter for each input noise variance both for the uncertain and certain observation cases. Also, the estimation accuracy for the certain observed value sequence is better than that for the uncertain observation sequence in each input noise variance.

Here, the certain observations correspond to the relationship $\mathrm{p(k)} = P_{2,2}(k)$. The MSVs of the fixed-point smoothing errors are evaluated by $\sum_{i=1}^{2000}(z(i) - \hat{z}(i, i+L))^2/2000$, $\mathrm{L} = 1,2,\cdots,5$. The case of $\mathrm{L} = 0$ corresponds to the calculation of the MSV of the filtering errors.

For references, the autoregressive (AR) model, which generates the signal process, is given by

$$\mathrm{z(k+1)} - a_1 z(k) - a_2 z(k-1) + w(k+1),\ \mathrm{E[w(k)w(s)]} = \sigma^2 \delta_K(k-s). \qquad (40)$$

## 5. Conclusions

Under the preliminary assumptions of section 2, for the observation equation (1) with additive colored noise, this paper, by iterative use of the invariant imbedding method, has proposed the RLS Wiener algorithms for the fixed-point smoothing and filtering estimates of the signal z(k) and the state vector x(k). The fixed-point smoothing and filtering algorithms take into accounts of the stochastic properties of the random variables $\{\gamma(\mathrm{k})\}$ in the observation equation (1) such as the probability $\mathrm{p(k)} = \mathrm{P}\{\gamma(\mathrm{k}) = 1\}$, that the signal exists in the uncertain observation equation, and the (2.2) element $[\mathrm{P(k|j)}]_{2,2}$ of the conditional probability of $\gamma(\mathrm{k})$, given $\gamma(\mathrm{j})$, $\mathrm{j} < \mathrm{k}$.

A numerical simulation example in section 4 shows that the fixed-point smoothing and filtering algorithms proposed in this paper are feasible.

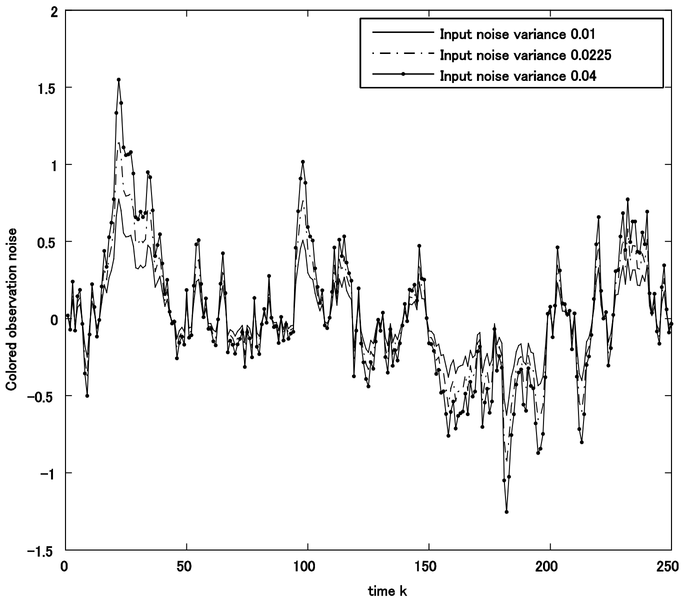


Fig.1 Colored observation noise process for the values of the input noise variance, $R_u(k) = 0.1^2,\ 0.15^2,\ 0.2^2$.

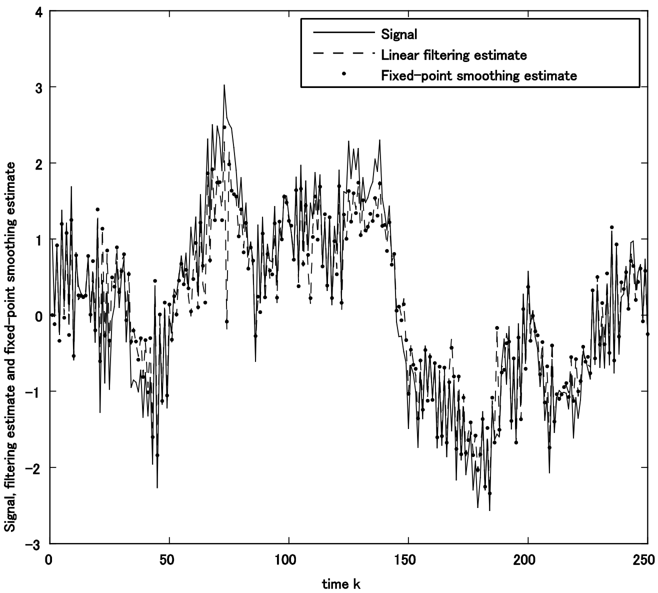


Fig.2 Fixed-point smoothing estimate $\hat{z}(k, k+5)$ of the signal z(k) for the input noise variance $R_u(k) = 0.1^2$.

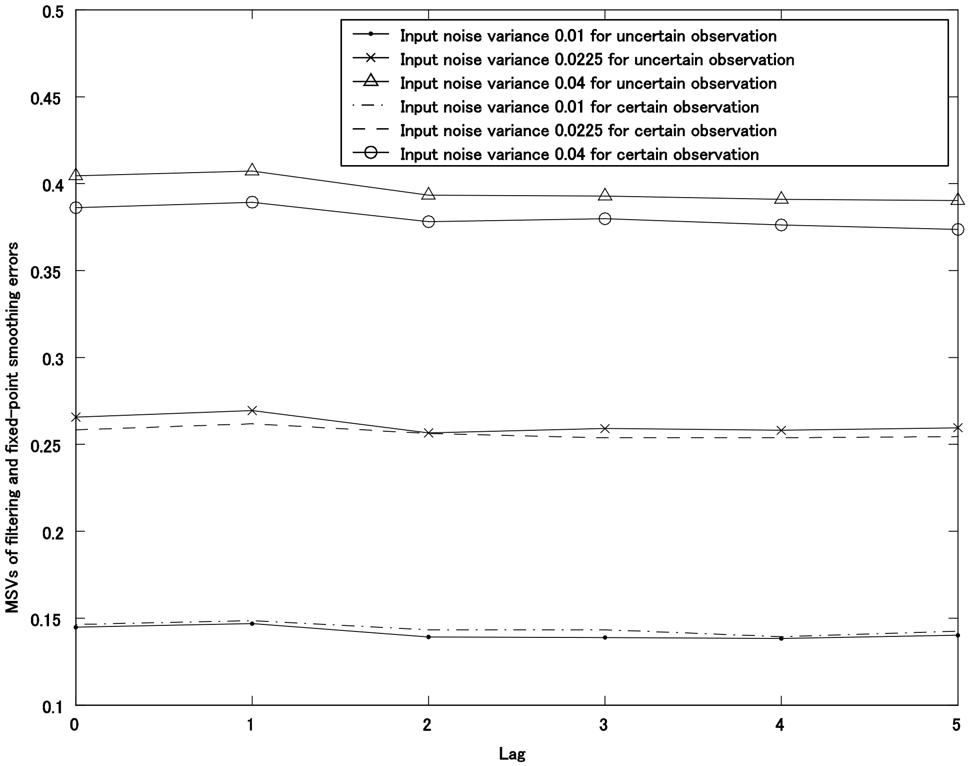


Fig.3 Mean-square values of the filtering and fixed-point smoothing errors for the certain and uncertain observation cases when the values of the input noise variance are $R_u(k) = 0.1^2,\ 0.15^2,\ 0.2^2$.

## Appendix A. Proof of Theorem 1

Subtracting the equation obtained by putting $\mathrm{L} = \mathrm{L} - 1$ in (11) from (11), we have

$$\left(\mathrm{h(k,s,L)} - h(k,s,L-1)\right)R_u(s) = -h(k,L,L)\{E_\gamma[\gamma(L)\gamma(s)]HK(L,s)H^T + \Phi_c K_c(L,s)\Phi_c^T + \Phi_c K_{vu}(L,s) + K_{uv}(L,s)\Phi_c^T\} - \sum_{i=1}^{L-1}(h(k,i,L) - h(k,i,L-1))\{E_\gamma[\gamma(i)\gamma(s)]HK(i,s)H^T + \Phi_c K_c(i,s)\Phi_c^T + \Phi_c K_{vu}(i,s) + K_{uv}(i,s)\Phi_c^T\}. \quad \text{(A-1)}$$

From $E_\gamma[\gamma(L)\gamma(s)] = P_{2,2}(L)p(s)$, $K_{uv}(L,s) = 0$, $K_{vu}(L,s) = \Phi_c^{L-s-1}R_u(s)$, we rewrite (A-1) as

$$\left(\mathrm{h(k,s,L)} - h(k,s,L-1)\right)R_u(s) = -h(k,L,L)\{P_{2,2}(L)p(s)HK(L,s)H^T + \Phi_c K_c(L,s)\Phi_c^T + \Phi_c \Phi_c^{L-s-1}R_u(s)\} - \sum_{i=1}^{L-1}(h(k,i,L) - h(k,i,L-1))\{E_\gamma[\gamma(i)\gamma(s)]HK(i,s)H^T + \Phi_c K_c(i,s)\Phi_c^T + \Phi_c K_{vu}(i,s) + K_{uv}(i,s)\Phi_c^T\}. \quad \text{(A-2)}$$

Let us introduce following equations.

$$J_1(s,L-1)R_u(s) = p(s)\Phi^{-s}K(s,s)H^T - \sum_{i=1}^{L-1} J_1(i,L-1)\{E_\gamma[\gamma(i)\gamma(s)]HK(i,s)H^T + \Phi_c K_c(i,s)\Phi_c^T + \Phi_c K_{vu}(i,s) + K_{uv}(i,s)\Phi_c^T\} \quad \text{(A-3)}$$

$$J_2(s,L-1)R_u(s) = \Phi_c^{-s}K_c(s,s)\Phi_c^T - \sum_{i=1}^{L-1} J_2(i,L-1)\{E_\gamma[\gamma(i)\gamma(s)]HK(i,s)H^T + \Phi_c K_c(i,s)\Phi_c^T + \Phi_c K_{vu}(i,s) + K_{uv}(i,s)\Phi_c^T\} \quad \text{(A-4)}$$

$$J_3(s,L-1)R_u(s) = \Phi_c^{-s}R_u(s) - \sum_{i=1}^{L-1} J_3(i,L-1)\{E_\gamma[\gamma(i)\gamma(s)]HK(i,s)H^T + \Phi_c K_c(i,s)\Phi_c^T + \Phi_c K_{vu}(i,s) + K_{uv}(i,s)\Phi_c^T\} \quad \text{(A-5)}$$

From (A-2) with (A-3)-(A-5), we obtain

$$\mathrm{h(k,s,L)} - h(k,s,L-1) = -h(k,L,L)(P_{2,2}(L)H\Phi^L J_1(s,L-1) + \Phi_c^{L+1}J_2(s,L-1) + \Phi_c^L J_3(s,L-1)). \quad \text{(A-6)}$$

Subtracting the equation obtained by putting $\mathrm{L = L-1}$ in (A-3) from (A-3), we have

$$(J_1(s,L) - J_1(s,L-1))R_u(s) = -J_1(L,L)\{E_\gamma[\gamma(L)\gamma(s)]HK(L,s)H^T + \Phi_c K_c(L,s)\Phi_c^T + \Phi_c K_{vu}(L,s) + K_{uv}(L,s)\Phi_c^T\} - \sum_{i=1}^{L-1} J_1(i,L-1)\{E_\gamma[\gamma(i)\gamma(s)]HK(i,s)H^T + \Phi_c K_c(i,s)\Phi_c^T + \Phi_c K_{vu}(i,s) + K_{uv}(i,s)\Phi_c^T\}. \quad \text{(A-7)}$$

From (A-7) with (A-3)-(A-5), we obtain

$$J_1(s,L) - J_1(s,L-1) = -J_1(L,L)(P_{2,2}(L)H\Phi^L J_1(s,L-1) + \Phi_c^{L+1}J_2(s,L-1) + \Phi_c^L J_3(s,L-1)). \quad \text{(A-8)}$$

Similarly, we obtain

$$J_2(s,L) - J_2(s,L-1) = -J_2(L,L)(P_{2,2}(L)H\Phi^L J_1(s,L-1) + \Phi_c^{L+1}J_2(s,L-1) + \Phi_c^L J_3(s,L-1)). \quad \text{(A-9)}$$

$$J_3(s,L) - J_3(s,L-1) = -J_3(L,L)(P_{2,2}(L)H\Phi^L J_1(s,L-1) + \Phi_c^{L+1}J_2(s,L-1) + \Phi_c^L J_3(s,L-1)). \quad \text{(A-10)}$$

By putting $\mathrm{s \to t}$ in (A-3), we get

$$\begin{aligned} J_1(L-1,L-1)R_u(L-1) \\ &= p(L-1)\Phi^{-(L-1)}K(L-1,L-1)H^T \\ &- \sum_{i=1}^{L-1} J_1(i,L-1)\{E_\gamma[\gamma(i)\gamma(L-1)]HK(i,L-1)H^T + \Phi_c K_c(i,L-1)\Phi_c^T \\ &+ \Phi_c K_{vu}(i,L-1) + K_{uv}(i,L-1)\Phi_c^T\} \end{aligned}$$

$= p(L-1)\Phi^{-(L-1)}K(L-1,L-1)H^T - \sum_{i=1}^{L-1} J_1(i,L-1)\{E_\gamma[\gamma(i)\gamma(L-1)]HB(i)A^T(L-1)H^T + \Phi_c B_c(i) A_c^T(L-1)\Phi_c^T + B_{uv}(i)(\Phi_c^T)^{L-2}\Phi_c^T\}$. (A-11)

Here, we used the relationships

$$K_{vu}(i,L-1) = 0,$$

$$\begin{aligned} K_{uv}(i,L-1) &= E[u(i)v_c^T(L-1)] = R_u(i)(\Phi_c^T)^{-i}(\Phi_c^T)^{L-2} \\ &= B_{uv}(i)(\Phi_c^T)^{L-2},\ B_{uv}(i) = R_u(i)(\Phi_c^T)^{-i},\ 1 \le i \le L-1. \end{aligned} \quad \text{(A-12)}$$

Introducing functions

$r_{11}(L-1) = \sum_{i=1}^{L-1} E_\gamma[\gamma(i)\gamma(L-1)]\, J_1(i,L-1)HB(i)$, (A-13)

$r_{12}(L-1) = \sum_{i=1}^{L-1} J_1(i,L-1)\, \Phi_c B_c(i)$, (A-14)

$r_{13}(L-1) = \sum_{i=1}^{L-1} J_1(i,L-1)\, B_{uv}(i)$, (A-15)

we rewrite (A-11) as

$J_1(L-1,L-1)R_u(L-1) = p(L-1)\Phi^{-(L-1)}K(L-1,L-1)H^T - r_{11}(L-1)A^T(L-1)H^T - r_{12}(L-1)(\Phi_c^T)^L - r_{13}(L-1)(\Phi_c^T)^{L-1}$. (A-16)

From (A-13), we have

$$\begin{aligned} r_{11}(L-1) &= E_\gamma[\gamma(L-1)\gamma(L-1)]J_1(L-1,L-1)HB(L-1) \\ &+ \sum_{i=1}^{L-2} E_\gamma[\gamma(i)\gamma(L-1)]\, J_1(i,L-1)HB(i) \\ &= \mathrm{p}(\mathrm{L}-1)J_1(L-1,L-1)HB(L-1) + \textstyle\sum_{i=1}^{L-2} P_{2,2}(L-1)p(i)\, J_1(i,L-1)HB(i). \end{aligned}$$

By use of (A-8) and introducing functions

$\bar{r}_{11}(L-1) = \sum_{i=1}^{L-1} p(i)\, J_1(i,L-1)HB(i)$, (A-17)

$\bar{r}_{12}^T(L-1) = \sum_{i=1}^{L-1} p(i)\, J_2(i,L-1)HB(i)$, (A-18)

$\bar{r}_{13}^T(L-1) = \sum_{i=1}^{L-1} p(i)\, J_3(i,L-1)HB(i)$, (A-19) we get

$r_{11}(L-1) = \mathrm{p(L-1)}J_1(L-1, L-1)HB(L-1) + P_{2,2}(L-1)\{\bar{r}_{11}(L-2) - J_1(L-1, L-1)[P_{2,2}(L-1)H\Phi^{L-1}\bar{r}_{11}(L-2) + \Phi_c^L\bar{r}_{12}^T(L-2) + \Phi_c^{L-1}\bar{r}_{13}^T(L-2)]\}$, $r_{11}(0) = 0.$ (A-20)

In a similar fashion, from (A-17) and (A-8), we derive

$\bar{r}_{11}(L-1) = \mathrm{p(L-1)}J_1(L-1, L-1)HB(L-1) + \bar{r}_{11}(L-2) - J_1(L-1, L-1)[P_{2,2}(L-1)H\Phi^{L-1}\bar{r}_{11}(L-2) + \Phi_c^L\bar{r}_{12}^T(L-2) + \Phi_c^{L-1}\bar{r}_{13}^T(L-2)]\}$, $\bar{r}_{11}(0) = 0.$ (A-21)

From (A-18) and (A-9), we derive

$\bar{r}_{12}^T(L-1) = \mathrm{p(L-1)}J_2(L-1, L-1)HB(L-1) + \bar{r}_{12}^T(L-2) - J_2(L-1, L-1)[P_{2,2}(L-1)H\Phi^{L-1}\bar{r}_{11}(L-2) + \Phi_c^L\bar{r}_{12}^T(L-2) + \Phi_c^{L-1}\bar{r}_{13}^T(L-2)]\}$, $\bar{r}_{12}^T(0) = 0.$ (A-22)

From (A-19) and (A-10), we derive

$\bar{r}_{13}^T(L-1) = \mathrm{p(L-1)}J_3(L-1, L-1)HB(L-1) + \bar{r}_{13}^T(L-2) - J_3(L-1, L-1)[P_{2,2}(L-1)H\Phi^{L-1}\bar{r}_{11}(L-2) + \Phi_c^L\bar{r}_{12}^T(L-2) + \Phi_c^{L-1}\bar{r}_{13}^T(L-2)]\}$, $\bar{r}_{13}^T(0) = 0.$ (A-23)

Similarly, from (A-14) and (A-8), we obtain

$r_{12}(L-1) = J_1(L-1, L-1)\Phi_c B_c(L-1) + r_{12}(L-2) - J_1(L-1, L-1)[P_{2,2}(L-1)H\Phi^{L-1}r_{12}(L-2) + \Phi_c^L r_{22}(L-2) + \Phi_c^{L-1}r_{32}(L-2)$, $r_{12}(0) = 0.$ (A-24)

Here, we introduced functions

$r_{22}(L-1) = \sum_{i=1}^{L-1} J_2(i, L-1)\,\Phi_c B_c(i),$ (A-25)

$r_{32}(L-1) = \sum_{i=1}^{L-1} J_3(i, L-1)\,\Phi_c B_c(i).$ (A-26)

From (A-15) and (A-8), we get

$r_{13}(L-1) = J_1(L-1, L-1)B_{uv}(L-1) + r_{13}(L-2) - J_1(L-1, L-1)[P_{2,2}(L-1)H\Phi^{L-1}r_{13}(L-2) + \Phi_c^L r_{23}(L-2) + \Phi_c^{L-1}r_{33}(L-2)$, $r_{13}(0) = 0.$ (A-27)

Here, we introduced the functions

$r_{23}(L-1) = \sum_{i=1}^{L-1} J_2(i, L-1)\,B_{uv}(i),$ (A-28)

$r_{33}(L-1) = \sum_{i=1}^{L-1} J_3(i, L-1)\,B_{uv}(i).$ (A-29)

Now, putting $s \to L-1$ in (A-4), similarly to the derivation of (A-16), we get

$J_2(L-1,L-1)R_u(L-1)=\Phi_c^{-(L-1)}K_c(L-1,L-1)\Phi_c^T-r_{21}(L-1)A^T(L-1)H^T-r_{22}(L-1)A_c^T(L-1)\Phi_c^T-r_{23}(L-1)(\Phi_c^T)^{L-1}.$ (A-30)

Here, we introduced the function

$r_{21}(L-1)=\sum_{i=1}^{L-1}E_\gamma[\gamma(i)\gamma(L-1)]J_2(i,L-1)HB(i).$ (A-31)

From (A-31) with (A-9), we get

$r_{21}(L-1)=\mathrm{p}(\mathrm{L}-1)J_2(L-1,L-1)HB(L-1)+P_{2,2}(L-1)\{\bar{r}_{12}^T(L-2)-J_2(L-1,L-1)[P_{2,2}(L-1)H\Phi^{L-1}\bar{r}_{11}(L-2)+\Phi_c^L\bar{r}_{12}^T(L-2)+\Phi_c^{L-1}\bar{r}_{13}^T(L-2)]\},\ r_{21}(0)=0.$ (A-32)

From (A-25) with (A-9), we get

$r_{22}(L-1)=J_2(L-1,L-1)\Phi_cB_c(L-1)+r_{22}(L-2)-J_2(L-1,L-1)[P_{2,2}(L-1)H\Phi^{L-1}r_{12}(L-2)+\Phi_c^Lr_{22}(L-2)+\Phi_c^{L-1}r_{32}(L-2),\ r_{22}(0)=0.$ (A-33)

From (A-28) with (A-9), we get

$r_{23}(L-1)=J_2(L-1,L-1)B_{uv}(L-1)+r_{23}(L-2)-J_2(L-1,L-1)[P_{2,2}(L-1)H\Phi^{L-1}r_{13}(L-2)+\Phi_c^Lr_{23}(L-2)+\Phi_c^{L-1}r_{33}(L-2),\ r_{23}(0)=0.$ (A-34)

By putting $s\to t$ in (A-4), similarly to the derivation of (A-30), we get

$J_3(L-1,L-1)R_u(L-1)=\Phi_c^{-(L-1)}R_u(L-1)-r_{31}(L-1)A^T(L-1)H^T-r_{32}(L-1)A_c^T(L-1)\Phi_c^T-r_{33}(L-1)(\Phi_c^T)^{L-1}.$ (A-35)

Here, we introduced the function

$r_{31}(L-1)=\sum_{i=1}^{L-1}E_\gamma[\gamma(i)\gamma(L-1)]J_3(i,L-1)HB(i).$ (A-36)

From (A-36) with (A-10), we get

$r_{31}(L-1)=\mathrm{p}(\mathrm{L}-1)J_3(L-1,L-1)HB(L-1)+P_{2,2}(L-1)\{\bar{r}_{13}^T(L-2)-J_3(L-1,L-1)[P_{2,2}(L-1)H\Phi^{L-1}\bar{r}_{11}(L-2)+\Phi_c^L\bar{r}_{12}^T(L-2)+\Phi_c^{L-1}\bar{r}_{13}^T(L-2)]\},\ r_{31}(0)=0.$ (A-37)

From (A-26) with (A-10), we get

$r_{32}(L-1)=J_3(L-1,L-1)\Phi_cB_c(L-1)+r_{32}(L-2)-J_3(L-1,L-1)[P_{2,2}(L-1)H\Phi^{L-1}r_{12}(L-2)+\Phi_c^Lr_{22}(L-2)+\Phi_c^{L-1}r_{32}(L-2),\ r_{32}(0)=0.$ (A-38)

From (A-29) with (A-10), we get

$r_{33}(L-1)=$
$J_3(L-1,L-1)B_{uv}(L-1)+r_{33}(L-2)-J_3(L-1,L-1)[P_{2,2}(L-1)H\Phi^{L-1}r_{13}(L-2)+$
$\Phi_c^L r_{23}(L-2)+\Phi_c^{L-1}r_{33}(L-2),\ r_{33}(0)=0.$ (A-39)

Substituting (A-20), (A-24) and (A-27) into (A-16), after some manipulations, we get

$J_1(\mathrm{L}-1,\mathrm{L}-1)=\left(\mathrm{p}(\mathrm{L}-1)\Phi^{-(L-1)}\mathrm{K}(\mathrm{L}-1,\mathrm{L}-1)\mathrm{H}^{\mathrm{T}}-\mathrm{P}_{2,2}(\mathrm{L}-1)\bar{r}_{11}(L-2)A^T(L-1)H^T-\right.$
$\left.r_{12}(\mathrm{L}-2)(\Phi_c^T)^L-r_{12}(L-2)(\Phi_c^T)^{L-1}\right)\bar{R}(L-1)^{-1}.$
(A-40)

Here, by introducing the following functions, $\bar{R}(L)$ is given by (18).

$\bar{S}_{11}(L-1)=\Phi^{L-1}\bar{r}_{11}(L-1)A^T(L-1),\ \bar{S}_{12}^T(L-1)=\Phi_c^{L-1}\bar{r}_{12}^T(L-1)A^T(L-1),$

$\bar{S}_{13}^T(L-1)=\Phi_c^{L-1}\bar{r}_{13}^T(L-1)A^T(L-1),\ S_{12}(L-1)=\Phi^{L-1}r_{12}(L-1)(\Phi_c^T)^{L-1},$

$S_{22}(L-1)=\Phi_c^{L-1}r_{22}(L-1)(\Phi_c^T)^{L-1},\ S_{32}(L-1)=\Phi_c^{L-1}r_{32}(L-1)(\Phi_c^T)^{L-1},$

$S_{13}(L-1)=\Phi^{L-1}r_{13}(L-1)(\Phi_c^T)^{L-1},\ S_{23}(L-1)=\Phi_c^{L-1}r_{23}(L-1)(\Phi_c^T)^{L-1},$

$S_{33}(L-1)=\Phi_c^{L-1}r_{33}(L-1)(\Phi_c^T)^{L-1}.$ (A-41)

Similarly, substituting (A-32), (A-33) and (A-34) into (A-30), after some manipulations, we get

$J_2(L-1,L-1)=\left(\Phi_c^{-(L-1)}K_c(L-1,L-1)\Phi_c^T-\mathrm{P}_{2,2}(\mathrm{L}-1)\bar{r}_{12}^T(L-2)A^T(L-1)H^T-\right.$

$\left.r_{22}(\mathrm{L}-2)(\Phi_c^T)^L-r_{23}(L-2)(\Phi_c^T)^{L-1}\right)\bar{R}(L-1)^{-1}.$ (A-42)

Substituting (A-26), (A-29) and (A-36) into (A-35), after some manipulations, we get

$J_3(L-1,L-1)=\left(\Phi_c^{-(L-1)}R_u(L-1)-\mathrm{P}_{2,2}(\mathrm{L}-1)\bar{r}_{13}^T(L-2)A^T(L-1)H^T-r_{32}(\mathrm{L}-\right.$

$\left.2)(\Phi_c^T)^L-r_{33}(L-2)(\Phi_c^T)^{L-1}\right)\bar{R}(L-1)^{-1}.$ (A-43)

Let us introduce functions $G_1(L-1,L-1)=\Phi^{L-1}J_1(L-1,L-1)$, $G_2(L-1,L-1)=\Phi_c^{L-1}J_2(L-1,L-1)$ and $G_3(L-1,L-1)=\Phi_c^{L-1}J_3(L-1,L-1)$. From (A-40), we see that

$G_1(L-1,L-1)=\left(\mathrm{p}(\mathrm{L}-1)\mathrm{K}(\mathrm{L}-1,\mathrm{L}-1)\mathrm{H}^{\mathrm{T}}-\mathrm{P}_{2,2}(\mathrm{L}-1)\bar{S}_{11}(L-2)\Phi^T H^T-\Phi S_{12}(\mathrm{L}-\right.$
$\left.2)(\Phi_c^T)^2-\Phi S_{12}(L-2)\Phi_c^T\right)\bar{R}(L-1)^{-1}.$

(A-44)

Similarly, from (A-42) and (A-43), we get

$$G_2(L-1,L-1) = \big(K_c(L-1,L-1)\Phi_c^T - \mathrm{P}_{2,2}(\mathrm{L}-1)\Phi_c\bar{S}_{12}^T(L-2)\Phi^T H^T - \Phi_c S_{22}(\mathrm{L}-2)(\Phi_c^T)^2 - \Phi_c S_{23}(L-2)\Phi_c^T\big)\bar{R}(L-1)^{-1}, \quad \text{(A-45)}$$

$$G_3(L-1,L-1) = \big(R_u(L-1) - \mathrm{P}_{2,2}(\mathrm{L}-1)\Phi_c\bar{S}_{13}^T(L-2)\Phi^T H^T - \Phi_c S_{32}(\mathrm{L}-2)(\Phi_c^T)^2 - \Phi_c S_{33}(L-2)\Phi_c^T\big)\bar{R}(L-1)^{-1}. \quad \text{(A-46)}$$

From (A-41) with (A-21), we get

$$\bar{S}_{11}(L-1) = \mathrm{p}(\mathrm{L}-1)G_1(L-1,L-1)HK(L-1,L-1) + \Phi\bar{S}_{11}(L-2)\Phi^T - G_1(L-1,L-1)[P_{2,2}(L-1)H\Phi\bar{S}_{11}(L-2)\Phi^T + \Phi_c^2\bar{S}_{12}^T(L-2)\Phi^T + \Phi_c\bar{S}_{13}^T(L-2)\Phi^T],\ \bar{S}_{11}(0) = 0. \quad \text{(A-47)}$$

From (A-41) with (A-22), we get

$$\bar{S}_{12}^T(L-1) = \mathrm{p}(\mathrm{L}-1)G_2(L-1,L-1)HK(L-1,L-1) + \Phi_c\bar{S}_{12}^T(L-2)\Phi^T - G_2(L-1,L-1)[P_{2,2}(L-1)H\Phi\bar{S}_{11}(L-2)\Phi^T + \Phi_c^2\bar{S}_{12}^T(L-2)\Phi^T + \Phi_c\bar{S}_{13}^T(L-2)\Phi^T],\ \bar{S}_{12}^T(0) = 0. \quad \text{(A-48)}$$

From (A-41) with (A-23), we get

$$\bar{S}_{13}^T(L-1) = \mathrm{p}(\mathrm{L}-1)G_3(L-1,L-1)HK(L-1,L-1) + \Phi_c\bar{S}_{13}^T(L-2)\Phi^T - G_3(L-1,L-1)[P_{2,2}(L-1)H\Phi\bar{S}_{11}(L-2)\Phi^T + \Phi_c^2\bar{S}_{12}^T(L-2)\Phi^T + \Phi_c\bar{S}_{13}^T(L-2)\Phi^T],\ \bar{S}_{13}^T(0) = 0. \quad \text{(A-49)}$$

Now, let us introduce a function

$$\mathrm{S}_{11}(\mathrm{L}-1) = \Phi^{L-1}\mathrm{r}_{11}(\mathrm{L}-1)(\Phi^{\mathrm{T}})^{\mathrm{L}-1}. \quad \text{(A.50)}$$

From (A-50) with (A-20), we get

$$S_{11}(L-1) = \mathrm{p}(\mathrm{L}-1)G_1(L-1,L-1)HK(L-1,L-1) + P_{2,2}(L-1)\Phi\bar{S}_{11}(L-2)\Phi^T - P_{2,2}(L-1)G_1(L-1,L-1)[P_{2,2}(L-1)H\Phi\bar{S}_{11}(L-2)\Phi^T + \Phi_c^2\bar{S}_{12}^T(L-2)\Phi^T + \Phi_c\bar{S}_{13}^T(L-2)\Phi^T],\ S_{11}(0) = 0. \quad \text{(A-51)}$$

From (A-41) with (A-24), we get

$$S_{12}(L-1) = G_1(L-1,L-1)\Phi_c K_c(L-1,L-1) + \Phi\mathrm{S}_{12}(L-2)\Phi_c^T - G_1(L-1,L-1)[P_{2,2}(L-1)H\Phi\mathrm{S}_{12}(L-2)\Phi_c^T + \Phi_c^2\mathrm{S}_{22}(L-2)\Phi_c^T + \Phi_c\mathrm{S}_{32}(L-2)\Phi_c^T),\ \mathrm{S}_{12}(0) = 0. \quad \text{(A-52)}$$

From (A-41) with (A-27), we get

$$S_{13}(L-1)=G_1(L-1,L-1)\mathrm{R}_u(L-1)+\Phi\mathrm{S}_{13}(L-2)\Phi_c^T-G_1(L-1,L-1)[P_{2,2}(L-1)H\Phi\mathrm{S}_{13}(L-2)\Phi_c^T+(\Phi_\mathrm{c})^2\mathrm{S}_{23}(L-2)\Phi_c^T+\Phi_c\mathrm{S}_{33}(L-2)\Phi_c^T),\ \mathrm{S}_{13}(0)=0. \quad \text{(A-53)}$$

From (A-41) with (A-33), we get

$$S_{22}(L-1)=G_2(L-1,L-1)\Phi_cK_c(L-1,L-1)+\Phi_c\mathrm{S}_{22}(L-2)\Phi_c^T-G_2(L-1,L-1)[P_{2,2}(L-1)H\Phi\mathrm{S}_{12}(L-2)\Phi_c^T+(\Phi_\mathrm{c})^2\mathrm{S}_{22}(L-2)\Phi_c^T+\Phi_c\mathrm{S}_{33}(L-2)\Phi_c^T),\ \mathrm{S}_{22}(0)=0. \quad \text{(A-54)}$$

From (A-41) with (A-34), we get

$$S_{23}(L-1)=G_2(L-1,L-1)\mathrm{R}_u(L-1)+\Phi_c\mathrm{S}_{23}(L-2)\Phi_c^T-G_2(L-1,L-1)[P_{2,2}(L-1)H\Phi\mathrm{S}_{13}(L-2)\Phi_c^T+(\Phi_\mathrm{c})^2\mathrm{S}_{23}(L-2)\Phi_c^T+\Phi_c\mathrm{S}_{33}(L-2)\Phi_c^T),\ \mathrm{S}_{23}(0)=0. \quad \text{(A-55)}$$

From (A-41) with (A-38), we get

$$S_{32}(L-1)=G_3(L-1,L-1)\Phi_cK_c(L-1,L-1)+\Phi_c\mathrm{S}_{32}(L-2)\Phi_c^T-G_3(L-1,L-1)[P_{2,2}(L-1)H\Phi\mathrm{S}_{12}(L-2)\Phi_c^T+(\Phi_\mathrm{c})^2\mathrm{S}_{22}(L-2)\Phi_c^T+\Phi_c\mathrm{S}_{32}(L-2)\Phi_c^T),\ \mathrm{S}_{32}(0)=0. \quad \text{(A-56)}$$

From (A-41) with (A-38), we get

$$S_{32}(L-1)=G_3(L-1,L-1)\Phi_cK_c(L-1,L-1)+\Phi_c\mathrm{S}_{32}(L-2)\Phi_c^T-G_3(L-1,L-1)[P_{2,2}(L-1)H\Phi\mathrm{S}_{12}(L-2)\Phi_c^T+(\Phi_\mathrm{c})^2\mathrm{S}_{22}(L-2)\Phi_c^T+\Phi_c\mathrm{S}_{32}(L-2)\Phi_c^T),\ \mathrm{S}_{32}(0)=0. \quad \text{(A-57)}$$

From (A-41) with (A-39), we get

$$S_{33}(L-1)=G_3(L-1,L-1)\mathrm{R}_u(L-1)+\Phi_c\mathrm{S}_{33}(L-2)\Phi_c^T-G_3(L-1,L-1)[P_{2,2}(L-1)H\Phi\mathrm{S}_{13}(L-2)\Phi_c^T+(\Phi_\mathrm{c})^2\mathrm{S}_{23}(L-2)\Phi_c^T+\Phi_c\mathrm{S}_{33}(L-2)\Phi_c^T),\ \mathrm{S}_{33}(0)=0. \quad \text{(A-58)}$$

Now, the fixed-point smoothing estimate of x(k) is given by (5). Subtracting $\hat{x}(k,L-1)$ from $\hat{x}(k,L)$ and using (A-6), we obtain

$$\hat{x}(k,L)-\hat{x}(k,L-1)=h(k,L,L)(y(L)-P_{2,2}(L)H\Phi^Le_1(L-1)-\Phi_c^{L+1}e_2(L-1)-\Phi_c^Le_3(L-1)). \quad \text{(A-59)}$$

Here, we introduced the following equations.

$$e_1(L) = \sum_{i=1}^{L} J_1(i,L)y(i) \tag{A-60}$$

$$e_2(L) = \sum_{i=1}^{L} J_2(i,L)y(i) \tag{A-61}$$

$$e_3(L) = \sum_{i=1}^{L} J_3(i,L)y(i) \tag{A-62}$$

Subtracting $e_1(L-1)$ from $e_1(L)$ and using (A-8), we obtain

$$e_1(L) - e_1(L-1) = J_1(L,L)(y(L) - P_{2,2}(L)H\Phi^L e_1(L-1) - \Phi_c^{L+1} e_2(L-1) - \Phi_c^L e_3(L-1)),$$
$$e_1(0) = 0. \tag{A-63}$$

Subtracting $e_2(L-1)$ from $e_2(L)$ and using (A-9), we obtain

$$e_2(L) - e_2(L-1) = J_2(L,L)(y(L) - P_{2,2}(L)H\Phi^L e_1(L-1) - \Phi_c^{L+1} e_2(L-1) - \Phi_c^L e_3(L-1)),$$
$$e_2(0) = 0. \tag{A-64}$$

Subtracting $e_3(L-1)$ from $e_3(L)$ and using (A-10), we obtain

$$e_3(L) - e_3(L-1) = J_3(L,L)(y(L) - P_{2,2}(L)H\Phi^L e_1(L-1) - \Phi_c^{L+1} e_2(L-1) - \Phi_c^L e_3(L-1)),$$
$$e_3(0) = 0. \tag{A-65}$$

Let us introduce functions $l_1(L) = \Phi^L e_1(L)$, $l_2(L) = \Phi^L e_2(L)$ and $l_3(L) = \Phi^L e_3(L)$. From (A-63)-(A-65), we obtain

$$l_1(L) = \Phi l_1(L-1) + G_1(L,L)(y(L) - P_{2,2}(L)H\Phi l_1(L-1) - (\Phi_c)^2 l_2(L-1) - \Phi_c l_3(L-1)),$$
$$l_1(0) = 0, \tag{A-66}$$

$$l_2(L) = \Phi l_2(L-1) + G_2(L,L)(y(L) - P_{2,2}(L)H\Phi l_1(L-1) - (\Phi_c)^2 l_2(L-1) - \Phi_c l_3(L-1)),$$
$$l_2(0) = 0, \tag{A-67}$$

$$l_3(L) = \Phi l_3(L-1) + G_3(L,L)(y(L) - P_{2,2}(L)H\Phi l_1(L-1) - (\Phi_c)^2 l_2(L-1) - \Phi_c l_3(L-1)),$$
$$l_3(0) = 0. \tag{A-68}$$

By the way, putting $\mathrm{s} \to \mathrm{L}$ in (11) and using the relationships $K_{vu}(i,L) = 0$, $K_{uv}(i,L) = R_u^T(\mathrm{i})(\Phi_c^T)^{-i}(\Phi_c^T)^{L-1}$ with (4) and (10), we get

$$\mathrm{h}(\mathrm{k},\mathrm{L},\mathrm{L})R_u(L) =$$
$$p(L)K(k,k)(\Phi^T)^{L-k}H^T - F_1(k,L)(\Phi^T)^L H^T - F_2(k,L)A_c^T(L)\Phi_c^T - F_3(k,L)(\Phi_c^T)^L. \tag{A-69}$$

Here, we introduced

$$F_1(k,L) = \sum_{i=1}^{L} E_\gamma\{\gamma(i)\gamma(L)\}h(k,i,L)B(i), \tag{A-70}$$

$$F_2(k,L) = \sum_{i=1}^{L} h(k,i,L)\Phi_c B_c(i), \tag{A-71}$$

$$F_3(k,L) = \sum_{i=1}^{L} h(k,i,L) R_u^T(i) (\Phi_c^T)^{-i}. \tag{A-72}$$

Also, introducing

$$\mathrm{f_1(k,L) = F_1(k,L)A^T(L)}, \tag{A-73}$$

$$\mathrm{f_2(k,L) = F_2(k,L)A_c^T(L)}, \tag{A-74}$$

$$\mathrm{f_3(k,L) = F_3(k,L)}\Phi_c^{\mathrm{T}}\mathrm{(L)}, \tag{A-75}$$

we can rewrite (A-69) as

$$\mathrm{h(k,L,L)}R_u(L) = p(L)K(k,k)(\Phi^T)^{L-k}H^T - f_1(k,L)H^T - f_2(k,L)\Phi_c^T - f_3(k,L). \tag{A-76}$$

From (A-70) and (A-6), we get

$$\mathrm{F_1(k,L) = p(L)h(k,L,L)HB(L)} + P_{2,2}(L)\mathrm{F_4(k,L-1)} - P_{2,2}(L)h(k,L,L)(P_{2,2}(L)H\Phi^L \bar{\mathrm{r}}_{11}(L-1) + \Phi_c^{L+1}\bar{\mathrm{r}}_{12}^{\mathrm{T}}(L-1) + \Phi_c^{\mathrm{L}}\bar{\mathrm{r}}_{13}^{\mathrm{T}}(L-1)). \tag{A-77}$$

Here, we introduced the function

$$F_4(k,L) = \sum_{i=1}^{L} p(i) h(k,i,L) HB(i). \tag{A-78}$$

From (A-71) and (A-6), we get

$$\mathrm{F_2(k,L) = h(k,L,L)}\Phi_c B_c\mathrm{(L) + F_2(k,L-1)} - h(k,L,L)(P_{2,2}(L)H\Phi^L r_{12}(L-1) + \Phi_c^{L+1} r_{22}(L-1) + \Phi_c^{\mathrm{L}} r_{32}(L-1)). \tag{A-79}$$

From (A-72) and (A-6), we get

$$\mathrm{F_3(k,L) = h(k,L,L)}B_{uv}\mathrm{(L) + F_3(k,L-1)} - h(k,L,L)(P_{2,2}(L)H\Phi^L r_{13}(L-1) + \Phi_c^{L+1} r_{23}(L-1) + \Phi_c^{\mathrm{L}} r_{33}(L-1)). \tag{A-80}$$

From (A-78) and (A-6), we get

$$\mathrm{F_4(k,L) = p(L)h(k,L,L)HB(L) + F_4(k,L-1)} - h(k,L,L)(P_{2,2}(L)H\Phi^L \bar{\mathrm{r}}_{11}(L-1) + \Phi_c^{L+1}\bar{\mathrm{r}}_{12}^{\mathrm{T}}(L-1) + \Phi_c^{\mathrm{L}}\bar{\mathrm{r}}_{13}^{\mathrm{T}}(L-1)). \tag{A-81}$$

Now, let us introduce a function

$$f_4(k,L) = F_4(k,L)A^T(L). \tag{A-82}$$

From (A-73) with (A-77), we get

$$f_1\mathrm{(k,L) = p(L)h(k,L,L)HK(L,L)} + P_{2,2}(L)\mathrm{f_4(k,L-1)}\Phi^T - P_{2,2}(L)h(k,L,L)(P_{2,2}(L)H\Phi\bar{\mathrm{S}}_{11}(L-1)\Phi^T + \Phi_c^2\bar{\mathrm{S}}_{12}^{\mathrm{T}}(L-1)\Phi^T + \Phi_c\bar{\mathrm{S}}_{13}^{\mathrm{T}}(L-1)\Phi^T). \tag{A-83}$$

From (A-74) with (A-79), we get

$$f_2(\mathrm{k,L}) = \mathrm{h(k,L,L)H}K_c(\mathrm{L,L}) + \mathrm{f_2(k,L-1)}\Phi_c^T - h(k,L,L)(P_{2,2}(L)H\Phi S_{12}(L-1)\Phi_c^T + \Phi_c^2 S_{22}(L-1)\Phi_c^T + \Phi_c S_{32}(L-1)\Phi_c^T). \quad \text{(A-84)}$$

From (A-75) with (A-80), we get

$$f_3(\mathrm{k,L}) = \mathrm{h(k,L,L)}R_u^T(\mathrm{L}) + \mathrm{f_3(k,L-1)}\Phi_c^T - h(k,L,L)(P_{2,2}(L)H\Phi S_{13}(L-1)\Phi_c^T + \Phi_c^2 S_{23}(L-1)\Phi_c^T + \Phi_c S_{33}(L-1)\Phi_c^T). \quad \text{(A-85)}$$

From (A-82) with (A-81), we get

$$\mathrm{f_4(k,L) = p(L)h(k,L,L)HK(L,L) + f_4(k,L-1)}\Phi^T - h(k,L,L)(P_{2,2}(L)H\Phi\bar{S}_{11}(L-1)\Phi^T + \Phi_c^2\bar{\mathrm{S}}_{12}^{\mathrm{T}}(L-1)\Phi^T + \Phi_c\bar{S}_{13}^{\mathrm{T}}(L-1)\Phi^T). \quad \text{(A-86)}$$

In (11) putting $\mathrm{L} \to \mathrm{k}$, we have

$$\mathrm{h(k,s,k)}R_u(s) = p(s)K(k,k)H^T - \sum_{i=1}^{k} h(k,i,L)\{E_\gamma[\gamma(i)\gamma(k)]HK(i,k)H^T + \Phi_c K_c(i,k)\Phi_c^T + \Phi_c K_{vu}(i,k) + K_{uv}(i,k)\Phi_c^T\}. \quad \text{(A-87)}$$

From (A-87) with (A-3), we obtain

$$\mathrm{h(k,s,k) = A(k)J_1(s,k)}. \quad \text{(A-88)}$$

Initial conditions of $f_1(\mathrm{k,L})$, $f_2(\mathrm{k,L})$, $f_3(\mathrm{k,L})$ and $f_4(\mathrm{k,L})$ at $\mathrm{L = k}$ are as follows.

$$f_1(k,k) = F_1(k,k)A^T(k) = \sum_{i=1}^{k} E_\gamma\{\gamma(i)\gamma(k)\}A(k)J(k,i)B(i)A^T(k) = A(k)r_{11}(k)A^T(k) = S_{11}(k) \quad \text{(A-89)}$$

Here, we used (A-13).

$$f_2(k,k) = F_2(k,k)A_c^T(k) = \sum_{i=1}^{L} h(k,i,k)\Phi_c B_c(i)A_c^T(k) = A(k)r_{12}(k)A_c^T(k) = S_{12}(k) \quad \text{(A-90)}$$

Similarly,

$$f_3(k,k) = F_3(k,k)(\Phi_c^T)^L = \sum_{i=1}^{k} h(k,i,k)R_u^T(i)(\Phi_c^T)^{-i}(\Phi_c^T)^k = A(k)r_{13}(k)(\Phi_c^T)^k = S_{13}(k). \quad \text{(A-91)}$$

Finally,

$$f_4(k,k) = F_4(k,k)A^T(k) = \sum_{i=1}^{k} p(i)h(k,i,k)HB(i)A^T(k) = A(k)\bar{\mathrm{r}}_{11}(k)A^T(k) = \bar{\mathrm{S}}_{11}(k). \quad \text{(A-92)}$$

Substituting (A-83)-(A-86) into (A-76), we obtain (17) and (18) after some manipulations.

By the way, from (A-59), the fixed-point smoothing estimate is updated by

$$\hat{x}(k,L)=\hat{x}(k,L-1)+h(k,L,L)(y(L)-P_{2,2}(L)H\Phi l_1(L-1)-(\Phi_c)^2 l_2(L-1)-\Phi_c l_3(L-1)) \quad . \tag{A-93}$$

Also, from (5), the filtering estimate is given by

$$\hat{x}(k,k)=\sum_{i=1}^{L} h(k,i,k)y(i). \tag{A-94}$$

From (A-88) with (A-60), we have

$$\hat{x}(k,k)=\sum_{i=1}^{L} \mathrm{A(k)J_1(i,k)}y(i)=\mathrm{A(k)}e_1\mathrm{(k)}. \tag{A-95}$$

Substituting (A-63) into (A-95) and using (A-88), we obtain

$$\hat{x}(k,k)=A(k)e_1(k-1)+h(k,k,k)(\,y(k)-P_{2,2}(k)H\Phi^k e_1(k-1)-\Phi_c^{k+1}e_2(k-1)-\Phi_c^k e_3(k-1))=\Phi\hat{x}(k-1,k-1)+h(k,k,k)(y(L)-P_{2,2}(k)H\Phi l_1(k-1)-(\Phi_c)^2 l_2(k-1)-\Phi_c l_3(k-1)),\ \ \hat{x}(k,k)=0.$$

(Q.E.D.)

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