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QUASI POLYTOPES AND FINITE TOPOLOGICAL SPACES

By

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§ 1. Introduction

We have investigated finite topological spaces and those simplicial structures in [1] and [2]. In [2] we have adopted the concept, called eigen values, which was important for the characterization of finite T_0-spaces. Moreover, in [1] we have showed that there is an equivalent correspondence that associates with each finite T_0-space a space called a partially polytopes.

In this note we shall consider applications of the conception. We shall state that for a space called a quasi polytope the eigen values also may be defined, and we shall consider to extend each quasi polytope to a partially polytope inclued it.

§ 2. Quasi polytopes.

In an Euclidean space E^N, a set of properly joined open simplexes is said to be a *quasi simplicial complex* here (see [1]).

The geometric carrier of a star-finite quasi simplicial complex K is called a *quasi polytope* and is denoted by the symbol $|K|$. But, in this note quasi complexes and quasi polytopes shall always be assumed to be finite quasi complexes and finite quase polytopes, respectively.

A topological space X that is homeomorphic to a quasi polytope $|K|$ is called a *triangulated space* and the quasi complex K is a *triangulation* of the space X. Next, let K be a quasi simplicial complex, then a set

$$Cl(K) = \{s \mid \exists \sigma \in K: s < \sigma\}$$

is a simplicial complex, which is said to be *induced* by K.

DEFINITION 1. Let $f: K \to L$ be a mapping of a quasi simplicial complex K to a quasi simplicial complex L. Then f is *simplicial* if there is simplical mapping $f_0: Cl(K) \to Cl(L)$ such that $f = f_0 | K$.

DEFINITION 2. Two quasi simplicial complexes K and L are said to be *isomorphic* each other if there is a bijective simplicial mapping φ of one onto the other such that the inverse mapping φ^{-1} also is simplicial, and in such a case we denote by $K \approx L$.

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Evidently we have

$$K \approx L \Rightarrow |K| \simeq |L| .$$

A triangulation K of a quasi polytope X is said to be *minimal* if the cardinality of K is minimum in the cardinalities of all triangulations of X.

From now on, we shall consider only quasi polytopes satisfying the following property: *if K and L are two minimal triangulations of such a quasi polytope X, then $K \approx L$.*

Now, let X be a quasi polytope, K be a minimal trangulation of X, and K^0 be the set of vertices belonging to $Cl(K)$. Set

$$K^0 = \{v_1, v_2, \cdots, v_k\} ,$$

and

$$K = \{\sigma_1, \sigma_2, \cdots, \sigma_l\} ,$$

and for K we define a (k, l) matrix $A=[a_{ij}]$ as follows:

$$\begin{aligned} a_{ij} &= 1 \quad \text{if} \quad v_i < \sigma_j , \\ &= 0 \text{ otherwse} . \end{aligned}$$

Such a matrix is said to be a *p-matrix* of K.

DEFINITION 3. Let X be a quasi polytope, K be a minimal triangulation of X, and A be a p-matrix of K. Then the characteristic polynomial $f(x)$ of AA', that is,

$$f(x) = |xE - AA'|$$

is *the polynomial of the space X*. And the eigen values of AA' is said to be *the eigen values of the space X*.

In a matrix $AA'=[c_{ij}]$, c_{ij} is the number of open simplexes belonging to K which have vertices v_i and v_j as their faces. Because of

$$c_{ij} = \sum_{r=1}^{k} a_{ir} a_{jr} ,$$

and

$$a_{ir} a_{jr} = 1 \Leftrightarrow (a_{ir} = 1 \text{ and } a_{jr} = 1) \Leftrightarrow (v_i < \sigma_r \text{ and } v_j < \sigma_r) .$$

The following theorem results easily from the matrix theory.

THEOREM 1. *Let X be a quasi polytope and K be a minimal triangulation of X. Then the eigen values of X have the following properties:*

(1) *the number of the eigen values is equal to the number of vertices of open simplexes belonging to K;*

(2) *each eigen value is a non-negative real number;*

(3) *if there are rational roots, they are non-negative integers.*

PROOF. For (1), let A be a p-matrix of K, and suppose that the number of open simplexes belonging to K is n. Then AA' is a (n, n) square matrix. whence (1) implies.

For (2), AA' is a non-negative Hermition matrix. Hence each eigen value is a non-negative real number.

For (3), the polynomial of X is

$$f(x) = |xE - AA'| = x^n - \cdots + (-1)^n |AA'|$$

and if $f(x)=0$ has rational roots, then they are non-negative integers, since $|AA'|$ is a non-negative integer.

Here, we shall show the eigen values or the characteristic polynomials of simple spaces.

(1) The eigen values of the n-closed ball are $n+1$ integers:

$$2^{n-1}, 2^{n-1}, \cdots, 2^{n-1}, (n+2)2^{n-1}.$$

In fact, a minimal triangulation of the n-closed ball is the closure K_1 of a n-simplex. Let A_1 be a p-matrix of K_1, and set $A_1A_1'=[c_{ij}]$. Each c_{ij} is the number of open simplex belongiog to K, then

$$c_{ii} = {}_nC_0 + {}_nC_1 + \cdots + {}_nC_n = 2^n,$$

and for $i \neq j$

$$c_{ij} = {}_{(n-1)}C_0 + {}_{(n-1)}C_1 + \cdots + {}_{(n-1)}C_{(n-1)} = 2^{n-1},$$

where ${}_mC_r$ is the total number of combinations of m elements taken r elements at a time. Hence

$$A_1A_1' = \begin{bmatrix} 2^n & & & \\ & 2^n & 2^{n-1} & \\ & & \ddots & \\ & 2^{n-1} & & 2^n \end{bmatrix},$$

that is, the diagonal elements are all 2^n, and the other elements are all 2^{n-1}. So that the eigen values of the space are

$$2^{n-1}, 2^{n-1}, \cdots, 2^{n-1}, (n+2)2^{n-1}.$$

(2) The eigen values of n-open ball are $n+1$ integers:

$$0, 0, \cdots, n+1.$$

In fact, a minimal triangulation of a n-open ball is a quasi simplical complex K_2 consisting of one n-open simplex. Let A_2 be a p-matrix, then elements of A_2A_2' are all 1. Hence the eigen values are $n+1$ elements, $O, O, \cdots, O, n+1$.

(3) The eigen values of the $(n-1)$-sphere are $n+1$ integers which are the difference between (1) and (2).

$$2^{n-1}, 2^{n-1}, \cdots, 2^{n-1}, (n+2)2^{n-1} - (n+1).$$

In fact, a minimal triangulation of a $(n-1)$-sphere is clearly $K_3 = K_1 - K_2$. Now, let A_3 be a p-matrix of K_3 and setting $A_3A_3' = [c_{ij}]$,

$$c_{ij} = 2^n - 1,$$

$$c_{ij} = 2^{n-1} - 1 \quad (\text{for } i \neq j),$$

so that $A_3A_3'=A_1A_1'-A_2A_2'$. Hence, the eigen values are $n+1$ elements, 2^{n-1}, $2^{n-1},\cdots,$ 2^{n-1}, $(n+2)2^{n-1}-(n+1)$.

(4) The eigen values of the Torus are 7 elements,

$$10,10,\cdots,10,31\,.$$

(5) The eigen values of the Möbius band are 6 elements,

$$5,5,\cdots,5,23\,.$$

(6) The eigen values of the projective plane are 6 elements,

$$8,8,\cdots,8,26\,.$$

§ 3. Regularization of quasi polytopes.

A finite quasi complex K is said to be a *n-partially simplicial complex* if K satisfies the following conditions:

(1) if σ, $\tau\in K$, $dim\,(\sigma\cap\tau)=k$, then a set $\{\eta\in K\,|\,\eta<\sigma\cap\tau\}$ has $k+1$ elements,

(2) let K^0 be the set of vertices of open simplexes belonging to K, then K^0 has n elements.

The geometric carrier of a partially simplicial complex is called a *partially polytope.* There exists an equivalent correspondence that assigns to each partially polytope a finite T_0-space.

Here, for each quasi polytope we shall consider a minimal dimensional partially polytope which will be called a regularization.

Now, let X be a quasi polytope, K be a minimal triangulation of X, and K^0 be a set of vertices belonging to $Cl(K)$. Set

$$K=\{\sigma_1,\sigma_2,\cdots,\sigma_l\}\,,$$

and

$$K^0=\{v_1,v_2,\cdots,v_k\}\,.$$

For $v_p\in K^0$, let

$$V_p=\{\sigma_i\in K\,|\,\sigma_i>v_p\}\,, \tag{3.1}$$

and put

$$T_1=\{V_1,V_2,\cdots,V_k\}\,.$$

Then we define an ordering on T_1 as follows: $V_i<V_j$ if and only if (1) or (2) implies,

$$\begin{aligned}&(1)\quad V_i\neq V_j \text{ and } V_i\subset V_j;\\ &(2)\quad V_i=V_j \text{ and } i<j\,.\end{aligned} \tag{3.2}$$

Next, for $\sigma_j\in K$ let

$$W_j=\cap\,\{V_r\in T_1\,|\,V_r\ni\sigma_j\}\,, \tag{3.3}$$

where, if there is more than two elements of (3.1) which are equal to W_j, then W_j is defined to be the smallest element of them with respected to the ordering (3.2). Here, putting

$$T = T_1 \cup \{W_1, W_2, \cdots, W_l\}\ ,$$

we define an ordering on T as follows: for U_1, $U_2 \in T$ such that $U_1 \neq U_2$, $U_1 < U_2$ if and only if (1) or (2) or (3) below, implies

(1) U_1, $U_2 \in T_1$ and $U_1 < U_2$ with respected to (3.2);

(2) U_1, $U_2 \notin T_1$ and $U_1 \subset U_2$;

(3) one is in T_1, say $U_1 \in T_1$ and the other is not, say $U_2 \notin T_1$, and $U_1 \subset U_2$ under the comments of (3.3).

Then $(T, \leq)$ is a partially ordered set, so that a finite T_0-space is defined on X. If $L(T)$ is the simplical presentation of T (see [1]), then $\tilde{X} = |L(T)|$ is said to be a *regularization* of X.

Moreover, if to each $U_p \in T$ we assign an simplex

$$\tilde{\sigma} = \langle \{U \in T \mid U > U_p\} \rangle$$

and consider the collection $\tilde{K}$ of such simplexes, then we have

$$\tilde{K} \approx L(T)$$

(see [1]). Especially for $\sigma_i \in K$ let

$$\tilde{\sigma}_i = \langle \{U \in T \mid U > W_i\} \rangle\ .$$

Given any quasi simplicial complex M, we may define an ordering $\lhd$ on M by

$$\sigma \lhd \tau \Leftrightarrow \sigma > \tau\ ,$$

then $(M, \lhd)$ is a partially ordered set. So the finite T_0-space is defined on M and is denoted by $F(M)$. Then we define ϕ: $|M| \to F(M)$ as follows: if $x \in |M|$, then there is a unique simplex $\sigma \in M$ such that $x \in \sigma$. So put

$$\phi(x) = \sigma\ .$$

Then ϕ is a weak homotopy equivalence (see [1]).

Now a mapping f: $Cl(K) \to Cl(\tilde{K})$ is defined as follows: if $\tau \in Cl(K)$, $\tau = \langle v_1, v_2 \cdots v_m \rangle$, then let $f(\tau)$ be an open simplex constructed by vertices

$$\{V_1, V_2, \cdots, V_m\} \cup \{U \in T \mid \exists \sigma \in K: \sigma < \tau\ ,\ U < \tilde{\sigma}\}\ .$$

Then we have

$$\left.\begin{array}{ll} (a) & f(v_i) = V_i\ (i = 1,2, \cdots, k); \\ (b) & f(\sigma_j) = \tilde{\sigma}_j\ (j = 1,2, \cdots, l); \\ (c) & f \text{ is injective;} \\ (d) & \sigma < \tau \Leftrightarrow f(\sigma) < f(\tau)\ . \end{array}\right. \qquad (3.4)$$

In fact, (*a*) and (*b*) follow immediately from the definition.

For (*c*), if σ, $\tau \in Cl(K)$, $\sigma \neq \tau$, then we may assume that there is $v_i \in K^0$ such that $v_i < \tau$ and $v_i \not< \sigma$. Clearly $U_i < f(\sigma)$, and while, if $\rho \in K$, $\rho < \sigma$, then $v_i \not< \rho$, so that $U_i \not< \tilde{\rho}$ by the definition of $\tilde{\rho}$ $(\rho \in K)$. Hence $U_i \not< f(\sigma)$, thus $f(\tau) \neq f(\sigma)$.

For (d), $\sigma<\tau \Rightarrow f(\sigma)<f(\tau)$ follows from the definition of f. The converse follows from the proof of (c).

Next, define $h: K\to\tilde{K}$ by $h=f|k$, then $\sigma<\tau \Rightarrow h(\sigma)<h(\tau)$, hence h is a quasi simplicial mapping.

Let $Cl(K_1)$ and $Cl(\tilde{K}_1)$ be first subdivisions of $Cl(K)$ and $Cl(\tilde{K})$, respectively. And f_1: $Cl(K_1)\to Cl(\tilde{K}_1)$ is defined as follows: for $\langle b(\tau_1)\,b(\tau_2)\cdots b(\tau_m)\rangle \in Cl(K_1)$ (where $b(\tau_j)$ is the barycenter of τ_j), let

$$f(\langle b(\tau_1)b(\tau_2)\cdots b(\tau_m)\rangle)=\langle b(\tilde{\tau}_1)b(\tilde{\tau}_2)\cdots b(\tilde{\tau}_m)\rangle$$

where $\tilde{\tau}_i=f(\tau_i)$. Then from (3.4) f_1 is bijective and f_1, f_1^{-1} are simplicial, so that

$$Cl(K_1)\approx f_1(Cl(K))\,.$$

Next, the first barycentric subdivision M_1 of a quasi simplicial complex M is defined by

$$K_1=\{\langle b(\tau_1)\cdots\cdot b(\tau_j)\rangle\in Cl(K_1)\,|\,\tau_1<\cdots<\tau_j, \tau_j\in K\}\,,$$

Now, define h_1: $K_1\to Cl(K_1)$ by $h_1=f_1|K_1$, then $h_1(K_1)\subset\tilde{K}_1$. Because, if $\langle b(\tau_1)\cdots b(\tau_j)\rangle\in K_1$ then $\tau_1<\cdots<\tau_j$ and $\tau_j\in K$. So from (3.4), $\tilde{\tau}_1<\cdots<\tilde{\tau}_j$ and $\tilde{\tau}_j\in\tilde{K}$, hence $\langle b(\tilde{\tau}_1)\cdots b(\tilde{\tau}_j)\rangle\in\tilde{K}_1$ thus $h_1(K_1)\subset\tilde{K}_1$. From (3.4)

$$K_1\approx h_1(K_1)\,.$$

Therefore h_1: $|K_1|\to|\tilde{K}_1|$ is an embedding.

THEOREM 2. *Let X be a quasi polytope, $\tilde{X}$ be a regularization of X, and K, $\tilde{K}$ be minimal triangulation X, $\tilde{X}$, respectively, and let h: $K\to\tilde{K}$ be a canonical quasi simplicial mapping defined by $h=f|k$. Then h induces the simplicial mapping h_1: $K_1\to\tilde{K}_1$ which satisfy the following conditions*:

(1) *h_1: $X=|K_1|\to\tilde{X}-|\tilde{K}_1|$ is an embedding*;

(2) *h: $F(K)\to F(\tilde{K})$ is an embedding such that $h(F(K))$ is dense in $F(\tilde{K})$*;

(3) *let ϕ: $|K|\to F(K)$ and $\tilde{\phi}$: $|\tilde{K}|\to F(\tilde{K})$ be two weak homotopy equivalence defined as the above, then $h\circ\phi=\tilde{\phi}\circ h$.*

PROOF. (1) has been already proved.

For (2), of h: $K\to\tilde{K}$,

$$\sigma_i<\sigma_j\Leftrightarrow h(\sigma_i)<h(\sigma_j)\,.$$

Hence, of h: $F(K)\to F(\tilde{K})$.

$$\sigma_i\rhd\sigma_j\Leftrightarrow h(\sigma_i)\rhd h(\sigma_j)\,.$$

Thus h is a homeomorphism of $F(K)$ to $f(F(K))$. On the other hand, using the previous symbols, if $\tau\in F(\tilde{K})-h(F(K))$, then from the definition of $\tilde{K}$ there is $V_i\in T_1$ such that $\tau=\langle\{U\in T\,|\,U>V_i\}\rangle$. Now, taking σ_j such that $\sigma_j\in V_i$,

$$\tilde{\sigma}_j=\langle\{\,U\in T\,|\,U>W_j\}\rangle$$

where $W_j=\cap\{V\in T_1|\,V>\sigma_j\}$. So $V_i\supset W_j$, that is $V_i>W_j$, hence, $\tilde{\sigma}_j>\tau$, whence

$\tilde{\sigma}_j \lhd \tau$. So, in the space $F(\tilde{K})$ the minimal basic neighborhood of τ contains $\tilde{\sigma}_j$, thus $h(F(K))$ is a dense subspace in $F(\tilde{K})$.

For (3), if $x \in |K| = |K_1|$, there is a unique simplex $\sigma_i \in K$ containing x. Then

$$h(\phi(x)) = h(\sigma_i) = \tilde{\sigma}_i\,.$$

While, there is a unique simplex $\langle b(\sigma_1)\cdots b(\sigma_i)\rangle \in K_1$ containing x. Then

$$h(x) \in \langle b(\tilde{\sigma}_1)\cdots b(\tilde{\sigma}_i)\rangle \subset |h(K_1)| \subset |\tilde{K}_1|\,.$$

So that

$$\tilde{\phi}(h(x)) = \tilde{\phi}(\tilde{\sigma}_i) = \tilde{\sigma}_i\,.$$

Therefore

$$h \circ \phi = \tilde{\phi} \circ h_1\,.$$

References

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[2] M. Shiraki: On finite topological spaces II. This Journal, Vol. **2**, (1969), 1–15.