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著者	ATSUMI Tsuyoshi
journal or publication title	鹿児島大学理学部紀要. 数学・物理学・化学
volume	16
page range	23-26
別言語のタイトル	スタイナー システムの会合数行列の固有空間について
URL	<a href="http://hdl.handle.net/10232/00012465">http://hdl.handle.net/10232/00012465</a>

# A NOTE ON THE EIGENSPACES FOR THE ADJACENCY MATRICES OF THE STEINER SYSTEMS

By

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(Received September 7, 1983)

## 1. Introduction and Summary

Let  $v, k, \lambda$  and  $t$  be positive integers with  $v \geq k \geq t$  and let  $[t/2]$  be  $s$ . Let  $P$  be a  $v$ -set  $\{\alpha_1, \dots, \alpha_v\}$  and  $\mathfrak{B}$  a  $\lambda_0$ -set  $\{B_1, \dots, B_{\lambda_0}\}$  (a collection of  $k$ -subsets of  $P$ ). The elements of  $P$  will be called points and the elements of  $\mathfrak{B}$  will be called blocks.  $(P, \mathfrak{B})$  is defined to be a  $t$ - $(v, k, \lambda)$  design if each  $t$ -subset of  $P$  is contained in exactly  $\lambda$  blocks. Let  $P_{(h)}$  be a set of  $h$ -subsets of  $P$  and  $P_h(i) (i=1, \dots, \binom{v}{h})$  the elements of  $P_{(h)}$ . Let  $M_h (h=0, \dots, k)$  be the incidence matrix for  $(P, \mathfrak{B})$  defined by

$$M_h(P_h(i), B_j) = \begin{cases} 1 & \text{if } P_h(i) \subseteq B_j \\ 0 & \text{otherwise.} \end{cases}$$

In order to state Yoshizawa's Proposition [5], we need the following row vector  $\alpha_i (i=1, \dots, v)$ .  $\alpha_i = (\beta_i(1), \dots, \beta_i(\lambda_0))$ , the  $j$ -th coordinate  $\beta_i(j) = \begin{cases} 1 & \text{if } \alpha_i \in B_j \\ 0 & \text{otherwise.} \end{cases}$  He proved that the real vector space spanned by  $\alpha_i - \alpha_j (1 \leq i, j \leq v)$  is an eigenspace for  $M_l^T M_l (l=0, \dots, t-1)$ .

In this note we shall prove two propositions one of which is an extension of Yoshizawa's.

PROPOSITION 1.  $V_h = W_{h,h} M_h$  (see Cameron [1]) is an eigenspace for  $M_l^T M_l (l=0, \dots, t-h)$ .

PROPOSITION 2. Let  $(P, \mathfrak{B})$  be a  $t$ - $(v, k, 1)$  design (i.e., Steiner system) such that for any block  $B$  of  $\mathfrak{B}$ , the set of points outside  $B$ , with blocks of the form  $B' - (B \cap B')$  where  $B'$  is a block with  $|B \cap B'| = t-1$  is a  $2$ - $(v-k, k-t+1, c)$  design for some integer  $c$ . Then  $W_{2,2} M_2$  is an eigenspace for the adjacency matrices of the Steiner system  $(P, \mathfrak{B})$ .

We shall use the same notations as in [1] and [3].

## 2. Proof of Proposition 1

For  $i, j, n=0, \dots, [v/2]$ , define  $C_{ij}^n$  to be the matrix with rows indexed by the  $i$ -subsets of  $P$ , columns indexed by the  $j$ -subsets of  $P$ , and with  $(P_i(a), P_j(b))$  entry equal to 1 if  $|P_i(a) \cap P_j(b)| = n$ , and 0 otherwise.

We proceed in several steps to investigate  $(w_h M_h) M_l^T M_l$  for  $w_h \in W_{h,h}$ . Note that  $h$

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$\leq l$  by Peterson's Theorem [3].

Step 1. We show that

$$(1) \quad M_h M_l^T = \sum_{j=0}^h \lambda_{h+l-j} C_{hl}^j.$$

$M_h M_l^T$  will be a matrix with rows indexed by the  $h$ -subsets of  $P$  and columns indexed by the  $l$ -subsets of  $P$ . If  $P_h(a)$  is an  $h$ -subset of  $P$  and  $P_l(b)$  is an  $l$ -subset of  $P$ , then  $(P_h(a), P_l(b))$  entry will be equal to number of blocks of  $\mathfrak{B}$  which contain  $P_h(a) \cup P_l(b)$ . Thus, if  $|P_h(a) \cap P_l(b)| = j$ , the  $(P_h(a), P_l(b))$  entry is  $\lambda_{h+l-j}$ .

Then

$$M_h M_l^T = \sum_{j=0}^h \lambda_{h+l-j} C_{hl}^j.$$

For  $m, i$ , define  $M_m^i$  to be the matrix with rows indexed by the  $m$ -subsets of  $P$ , columns indexed by the blocks of  $\mathfrak{B}$ , and with  $(P_m(a), B_b)$  entry equal to 1 if  $|P_m(a) \cap B_b| = i$ , and 0 otherwise. Note  $M_i = M_i^i$ .

Step 2. We show that

$$(2) \quad C_{hl}^j M_l = \sum_{p=j}^h \binom{p}{j} \binom{k-p}{l-j} M_h^p.$$

$C_{hl}^j M_l$  will be a matrix with rows indexed by the  $h$ -subsets of  $P$  and columns indexed by the blocks of  $\mathfrak{B}$ . If  $P_h(a)$  is an  $h$ -subset of  $P$  and  $B_b$  is a block, then  $(P_h(a), B_b)$  entry will be equal to the number of  $l$ -subsets of  $P$  which are contained by  $B_b$  and meet  $P_h(a)$  in exactly  $j$  points. Thus, if  $|P_h(a) \cap B_b| = p$ , then  $(P_h(a), B_b)$  entry is

$$\binom{p}{j} \binom{k-j}{l-j}.$$

$$C_{hl}^j M_l = \sum_{p=j}^h \binom{p}{j} \binom{k-p}{l-j} M_h^p.$$

Step 3. We show the following

LEMMA 1. If  $w_h \in W_{h, h}$ , then  $w_h M_h^p = (-1)^{h-p} \binom{h}{p} w_h M_h$ .

Proof. Let the  $P_h(i)$ -coordinate of  $w_h$  be  $\gamma\{a_1, \dots, a_h\}$ . Let the  $i$ -th coordinate of  $w_h M_h^p$  be  $n_B$  and let the  $i$ -th coordinate of  $w_h M_h^{p+1}$  be  $n'_B$ . We have

$$\begin{aligned} n_B &= \sum'_{\{a_1, \dots, a_p\}} \sum''_{\{a_{p+1}, \dots, a_h\}} \gamma\{a_1, \dots, a_p, a_{p+1}, \dots, a_h\} \\ &= \sum'_{\{a_1, \dots, a_p\}} \frac{1}{h-p} \sum'''_{\{a_{p+1}, \dots, a_{h-1}\}} \sum''''_{a_h} \gamma\{a_1, \dots, a_p, a_{p+1}, \dots, a_{h-1}, a_h\}, \end{aligned}$$

where the sum  $\sum'$  is taken over all  $p$ -subsets of  $B$  and the sum  $\sum''$  is taken over all  $(h-p)$ -subsets of  $\mathcal{Q}-B$  and the sum  $\sum'''$  is taken over all  $(h-p-1)$ -subsets of  $\mathcal{Q}-B$  and the sum  $\sum''''$  is taken over all elements of  $\mathcal{Q}-B \cup \{a_{p+1}, \dots, a_{h-1}\}$ . Since  $W_{h, h} = \{w \in W_h \mid w B_{h-1}^T = 0\}$ , it follows that for any  $\{a_1, \dots, a_{h-1}\} \in P_{(h-1)}$ ,  $\sum_{a_h} \gamma\{a_1, \dots, a_{h-1}, a_h\} = 0$ , where the sum  $\sum$  is taken over all elements of  $\mathcal{Q}-\{a_1, \dots, a_{h-1}\}$ . So

$$n_B = -\frac{1}{h-p} \sum'''_{\{a_{p+1}, \dots, a_{h-1}\}} \sum'_{\{a_1, \dots, a_p\}} \sum^\circ_{a_h} \gamma\{a_1, \dots, a_h\},$$

where the sum  $\sum^\circ$  is taken over all elements of  $B - \{a_1, \dots, a_p\}$ . By easy calculation we have

$$n_B = -\frac{p+1}{h-p} n'_B.$$

It follows that

$$w_h M_h^p = -\frac{p+1}{h-p} w_h M_h^{p+1} = (-1)^{h-p} \binom{h}{p} w_h M_h.$$

This completes the proof of Lemma 1.

Step 4. By (1), (2) and Lemma 1, we have

$$(w_h M_h) M_l^T M_l = \sum_{j=0}^h \lambda_{h+l-j} \sum_{p=j}^h \binom{p}{j} \binom{k-p}{l-j} (-1)^{h-p} \binom{h}{p} w_h M_h \text{ for } w_h \in W_{h,h}.$$

In order to rewrite the above expression we need the following

LEMMA 2. 
$$\sum_{p=j}^h \binom{p}{j} \binom{k-p}{l-j} (-1)^{h-p} \binom{h}{p} = (-1)^{h-j} \binom{h}{j} \binom{k-h}{l-h}.$$

In this proof we use the combinatorial identity (see Riordan [4]).

$$(3) \quad \binom{n-p}{m-p} = \sum_{k=0}^n (-1)^k \binom{n-k}{m} \binom{p}{k}.$$

Since  $\binom{p}{j} \binom{h}{p} = \binom{h}{j} \binom{h-j}{p-j}$ , we have

$$(-1)^{h-j} \sum_{p=j}^h \binom{p}{j} \binom{k-p}{l-j} (-1)^{h-p} \binom{h}{p} = (-1)^{h-j} \binom{h}{j} \sum_{p=j}^h \binom{h-j}{p-j} \binom{k-p}{l-j} (-1)^{h-p}.$$

If we set  $r = p - j$ , then we have

$$(-1)^{h-j} \sum_{p=j}^h \binom{p}{j} \binom{h-p}{l-j} (-1)^{h-p} \binom{h}{p} = \binom{h}{j} \sum_{r=0}^{h-j} \binom{h-j}{r} \binom{k-j-r}{l-j} (-1)^r.$$

By (3), the right hand side of the above equation is equal to  $\binom{h}{j} \binom{k-h}{l-h}$ . We complete the proof of Lemma 2.

Now it follows that  $(w_h M_h) M_l^T M_l = \sum_{j=0}^h \lambda_{h+l-j} (-1)^{h-j} \binom{h}{j} \binom{k-h}{l-h} w_h M_h$  for  $w_h \in W_{h,h}$ .

This completes the proof of Proposition 1.

*Remark.* Easily we can prove Proposition 1 by Peterson's Result [3]. Peterson's Result is the following.

RESULT. If  $w_h \in W_{h,h}$  and  $h < l$ , then  $w_h C_{hl}^j = (-1)^{h-j} \binom{h}{j} w_h B_{h,l}$ .

But it seems to me that this method does not apply to the proof of Proposition 2.

### 3. Application of Proposition 1 to Steiner systems

Let  $(P, \mathfrak{B})$  be a  $t$ -( $v, k, 1$ ) design (Steiner system  $S(t, k, v)$ ). Define  $A_i$  to be the adjacency matrix, corresponding to  $i$ : That is,  $A_i$  is the matrix with rows and columns indexed by blocks, and with  $(B_a, B_b)$  entry equal to 1 if  $|B_a \cap B_b| = i$  and 0 otherwise. Let  $I$  be the  $(\lambda_0, \lambda_0)$  identity matrix. Then we have

LEMMA 3. 
$$M_l^T M_l = \sum_{h=l}^{t-1} \binom{h}{l} A_h + \binom{k}{l} I, \text{ for } l=0, \dots, t-1.$$

Proof. See Cameron [1].

Now let us prove Proposition 2. Note that

$$M_2 A_{t-1} = (\lambda_{t-1} - 1) \binom{k-2}{t-3} M_2^2 + \binom{k-1}{t-2} M_2^1 + c M_2^0.$$

By Lemma 1,  $w_2 M_2^i = (-1)^{2-i} \binom{2}{i} w_2 M_2$  for  $w_2 \in W_{2,2}$ . It follows that  $W_{2,2} M_2$  is an eigenspace for  $A_{t-1}$ . Proposition 1 shows that  $W_{2,2} M_2$  is an eigenspace for  $M_l^T M_1$  ( $l=0, \dots, t-2$ ). Using Lemma 3 we obtain that  $W_{2,2} M_2$  is an eigenspace for  $A_i$  ( $i=0, \dots, t-1$ ). Thus we complete the proof of Proposition 2.

*Remark.* Let us give an example of the Steiner systems satisfying the assumption of Proposition 2.

The example (Cameron [2]) is the non-degenerate Steiner system  $S(t, k, v)$  such that the sets,  $B' - B$ , where  $B$  is a fixed block and  $B'$  is any block with  $|B \cap B'| = t-1$ , form a Steiner system  $S(t-1, k-t+1, v-k)$  on the points outside  $B$ , for all  $B$ .

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