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ON THE CONDITION THAT A RANDERS SPACE BE CONFORMALLY FLAT

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Abstract

The purpose of the present paper is to find a conformally invariant linear connection of a Finsler space with (α, β)-metric, and to give the condition that a Randers space be conformally flat in the tensorial form expressed in terms of the given metric. Especially, we determine all two-dimensional conformally flat Randers spaces.

1. Introduction

On an n-dimensional differentiable manifold M, we consider an (α, β)-metric $L(\alpha, \beta)$, where α is a Riemannian metric and β is a non-zero 1-form on M (Matsumoto [6, Definition 30.1]). We put

$$\alpha = (a_{ij}(x)y^i y^j)^{1/2}, \quad \beta = b_i(x)y^i. \tag{1.1}$$

Let $\left\{ {i \atop j\,k} \right\}$ be the Christoffel symbols constructed from a_{ij}, and ∇_k and $R_h{}^i{}_{jk}$ denote the covariant differentiation and the curvature tensor with respect to $\left\{ {i \atop j\,k} \right\}$ respectively.

In his paper [5], Kikuchi treated Finsler spaces with (α, β)-metric of several types, and obtained the condition that such a space of each type be locally Minkowski. In the case of a Randers space (M, L), where $L = \alpha + \beta$ (Randers [8]), the condition is given as follows:

Theorem 1.1. (Kikuchi [5]) *A Randers space is locally Minkowski if and only if* $R_h{}^i{}_{jk} = 0$ *and* $\nabla_k b_i = 0$ *are satisfied.*

In the present paper, corresponding to Theorem 1.1, we shall give the condition that a Randers space be conformally flat (Theorem 3.1). We shall here define as follows:

Definition 1.1. A Finsler space (M, L) is said to be *conformally flat*, if for any point p of M there exist a local coordinate neighbourhood (U, x) containing p and a function $\sigma(x)$ on U such that $e^{\sigma}L$ is a locally Minkowski metric.

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Briefly speaking, "a Finsler space (M, L) is conformally flat" means that L is locally conformal to a locally Minkowski metric. The condition that a general Finsler space be conformally flat has been given by many authors from various standpoints (e.g., Hashiguchi [1, Theorem 4.8], Hashiguchi-Ichijyō [2, Theorem C], Ichijyō [4, Theorem 7.1], Matsumoto [7, §2, Theorem]). In each case such a space is characterized by the existence of some geometrical object (e.g., the so-called Wagner connection in [2]). The condition given in the present paper is concerned with a Randers space only, but it should be noted that it is invariantly expressed in terms of the given Randers metric itself.

In order to use Theorem 1.1 for our purpose, in the next section we shall find a conformally invariant linear connection $M_j{}^i{}_k$ for a Randers space. The result is stated in the tensorial form as the condition imposed on $M_j{}^i{}_k$. This connection is, however, defined for a Finsler space with general (α, β)-metric (Theorem 2.1), so it will be also useful for the study of Finsler spaces with (α, β)-metric of other types.

If a Randers space (M, L) is conformally flat, where $L = \alpha + \beta$, so is the corresponding Riemannian space (M, α) (Theorem 4.1). Paying attention to this fact, in the last section we shall give a local expression to L of a conformally flat Randers space (M, L). This is given in the form (4.8) around any point of M and tells us various examples. Especially, in the two-dimensional case, we can determine all conformally flat Randers spaces (Theorem 4.3).

2. The conformally invariant linear connection

Let (M, L) be a Finsler space with (α, β)-metric $L = L(\alpha, \beta)$, where α and β are expressed as (1.1). A point of M and a tangent vector at the point are denoted by $x = (x^i)$ and $y = (y^i)$ respectively. We put $\partial_k = \partial/\partial x^k$, $(a^{ij}) = (a_{ij})^{-1}$, $b^i = a^{ir}b_r$, and $b = (b_r b^r)^{1/2}$.

By a conformal change

$$L = L(\alpha, \beta) \to \tilde{L} = e^{\sigma(x)} L(\alpha, \beta), \tag{2.1}$$

we have also an (α, β)-metric $\tilde{L} = L(\tilde{\alpha}, \tilde{\beta})$, where $\tilde{\alpha} = e^\sigma \alpha$, $\tilde{\beta} = e^\sigma \beta$. Putting $\tilde{\alpha} = (\tilde{a}_{ij}(x) y^i y^j)^{1/2}$, $\tilde{\beta} = \tilde{b}_i(x) y^i$, we have $\tilde{a}_{ij} = e^{2\sigma} a_{ij}$, $\tilde{b}_i = e^\sigma b_i$.

We shall find a linear connection on M which is invariant under the change (2.1). The Christoffel symbols $\widetilde{\{{}_j{}^i{}_k\}}$ constructed from $\tilde{a}_{ij}$ are written as

$$\widetilde{\{{}_j{}^i{}_k\}} = \{{}_j{}^i{}_k\} + \delta_j{}^i \sigma_k + \delta_k{}^i \sigma_j - \sigma^i a_{jk}, \tag{2.2}$$

where $\sigma_k = \partial_k \sigma$, $\sigma^i = a^{ir} \sigma_r$. Thus we have

$$\tilde{\nabla}_k \tilde{b}_j = e^\sigma (\nabla_k b_j - b_k \sigma_j + b_r \sigma^r a_{jk}). \tag{2.3}$$

Eliminating $b_r \sigma^r$ from (2.3) and putting

$$M_j = (1/b^2)\{b^r \nabla_r b_j - (\nabla_r b^r) b_j/(n-1)\}, \tag{2.4}$$

we have

(2.5) $$\sigma_j = M_j - \tilde{M}_j.$$

Substituting (2.5) in (2.2) and putting

(2.6) $$M_j{}^i{}_k = \left\{ {}_j{}^i{}_k \right\} + \delta_j{}^i M_k + \delta_k{}^i M_j - M^i a_{jk},$$

where $M^i = a^{ir} M_r$, we have

(2.7) $$\tilde{M}_j{}^i{}_k = M_j{}^i{}_k.$$

$M_j{}^i{}_k$ is a symmetric linear connection on M. Thus we have shown

Theorem 2.1. *In a Finsler space with* (α, β)*-metric there exists a conformally invariant symmetric linear connection* $M_j{}^i{}_k$.

We shall call the linear connection $M_j{}^i{}_k$ given by (2.6) the *conformally invariant linear connection* of an (α, β)-metric. We denote its curvature tensor by $M_h{}^i{}_{jk}$. We have from (2.7)

(2.8) $$\tilde{M}_h{}^i{}_{jk} = M_h{}^i{}_{jk}.$$

Especially, we have from (2.4) and (2.6)

Proposition 2.1. *In the case of* $\nabla_k b_j = 0$, *we have* $M_j = 0$, $M_j{}^i{}_k = \left\{ {}_j{}^i{}_k \right\}$, *and* $M_h{}^i{}_{jk} = R_h{}^i{}_{jk}$.

3. The condition that a Randers space be conformally flat

Let (M, L) be a Randers space, where $L = \alpha + \beta$, and we shall consider a conformal change (2.1) of L. Then we have a Randers metric $\tilde{L} = e^\sigma L = \tilde{\alpha} + \tilde{\beta}$, where $\tilde{\alpha} = e^\sigma \alpha$, $\tilde{\beta} = e^\sigma \beta$. If $(M, \tilde{L})$ is locally Minkowski, we have by Theorem 1.1 applied to $(M, \tilde{L})$

(3.1) $$\tilde{R}_h{}^i{}_{jk} = 0, \quad \tilde{\nabla}_k \tilde{b}_j = 0.$$

From $(3.1)_2$ we can apply Proposition 2.1 to $(M, \tilde{L})$, so we have $\tilde{M}_j = 0$ and $\tilde{M}_h{}^i{}_{jk} = \tilde{R}_h{}^i{}_{jk}$. Thus we have from (2.8) and $(3.1)_1$

(3.2) $$M_h{}^i{}_{jk} = 0.$$

Then, since (2.5) becomes $M_j = \sigma_j (= \partial_j \sigma)$, the covariant vector field M_j is locally gradient:

(3.3) $$\nabla_k M_j = \nabla_j M_k,$$

and we have from (2.3)

(3.4) $$\nabla_k b_j = b_k M_j - b_r M^r a_{jk}.$$

Since (3.2), (3.3), (3.4) are expressed in the tensorial form in terms of the given metric, we have these conditions even if (M, L) is not globally but locally conformal to a locally Minkowski space.

Conversely, let a Randers metric $L=\alpha+\beta$ satisfy the conditions (3.2), (3.3), (3.4). The condition (3.3) gives us that for any point p of M there exist a local coordinate neighbourhood (U, x) containing p and a function $\sigma(x)$ on U such that $\partial_j\sigma = M_j$. By using this σ we consider the conformal change $L \to \tilde{L} = e^{\sigma}L$ on each U. Then from (2.5) we have $\tilde{M}_j = 0$, which yields $\tilde{M}_j{}^i{}_k = \{\widetilde{{}^{\,i}_{j\,k}}\}$ and so $\tilde{M}_h{}^i{}_{jk} = \tilde{R}_h{}^i{}_{jk}$. Hence $\tilde{R}_h{}^i{}_{jk} = 0$ follows from (2.8) and the condition (3.2). Using the condition (3.4), from (2.3) we have $\tilde{\nabla}_k\tilde{b}_j = 0$. Thus we have proved

Theorem 3.1. *A Randers space is conformally flat if and only if the conditions* (3.2), (3.3) *and* (3.4) *are satisfied.*

For later use, we shall give another expression of the condition (3.4). From (3.4) we have

$$b^r \nabla_k b_r = 0. \tag{3.5}$$

Thus from (2.4) we have

$$b_r M^r = -(\nabla_r b^r)/(n-1). \tag{3.6}$$

Substituting (2.4) and (3.6) in (3.4) we have

$$\nabla_k b_j = (1/b^2)\{b_k b^r \nabla_r b_j + (\nabla_r b^r)(b^2 a_{jk} - b_j b_k)/(n-1)\}. \tag{3.7}$$

Conversely, the condition (3.4) follows from (3.5) and (3.7). As is shown in Ichijyō [3], (3.5) is equivalent to $\nabla_k(b_r b^r) = 0$, which means that b is constant when M is connected. Thus we have

Theorem 3.2. *A Randers space is conformally flat if and only if the conditions* (3.2), (3.3), (3.5) *and* (3.7) *are satisfied. Then* $b = (b_r b^r)^{1/2}$ *is constant if the underlying manifold is connected.*

Now, the condition (3.2) is explicitly concerned with $M_j{}^i{}_k$. The conditions (3.3) and (3.4) are also expressed in terms of $M_j{}^i{}_k$. Let $\overset{m}{\nabla}_k$ denote the covariant differentiation with respect to $M_j{}^i{}_k$. Since $M_j{}^i{}_k$ is symmetric, (3.3) is equivalent to

$$\overset{m}{\nabla}_k M_j = \overset{m}{\nabla}_j M_k. \tag{3.8}$$

On the other hand, we have

$$\overset{m}{\nabla}_k a_{ij} = -2M_k a_{ij}, \tag{3.9}$$

(3.10) $$\overset{m}{\nabla}_k b_j = \nabla_k b_j - b_k M_j - b_j M_k + b_r M^r a_{jk}.$$

(3.9) shows that $M_j{}^i{}_k$ is recurrent for a_{ij}, and (3.10) shows that (3.4) is expressed in the corresponding recurrent form:

(3.11) $$\overset{m}{\nabla}_k b_j = - M_k b_j.$$

Thus we have

Theorem 3.3. *A Randers space is conformally flat if and only if the conditions* (3.2), (3.8) *and* (3.11) *are satisfied.*

4. Examples

Let (M, L) be a conformally flat Randers space, where $L = \alpha + \beta$, and we shall give a local expression to L. Since L is locally conformal to a locally Minkowski metric, for any point p of M there exist a local coordinate neighbourhood (U, x) containing p and a function $\sigma(x)$ on U such that $\tilde{L} = e^\sigma L (= \tilde{\alpha} + \tilde{\beta})$ is locally Minkowski. Due to Theorem 1.1 we have $(3,1)_1$ for the Riemannian metric $\tilde{\alpha} = e^\sigma \alpha$, so $\tilde{\alpha}$ is locally Euclidean. Thus we have

Theorem 4.1. *If a Randers space* $(M, \alpha + \beta)$ *is conformally flat, the corresponding Riemannian space* (M, α) *is also conformally flat.*

Hence for any point p of M there exist a connected local coordinate neighbourhood (U, x) containing p and a non-zero function $a(x)$ on U such that α is written in the form $\alpha = a(x)(\delta_{ij} y^i y^j)^{1/2}$. Then, putting $c_i = b_i/a$ we can express L on U in the form

(4.1) $$L = aL^*,$$

where

(4.2) $$L^* = (\delta_{ij} y^i y^j)^{1/2} + c_i(x) y^i.$$

Since (M, L) is conformally flat if and only if so is each (U, L^*), we shall consider the condition that the Randers space (U, L^*) is conformally flat.

Since $a_{ij} = \delta_{ij}$, $a^{ij} = \delta^{ij}$, and $b_i = c_i$ in (U, L^*), in the formulas stated before we can put $b_i = b^i = c_i$, $b = c = (\delta^{ij} c_i c_j)^{1/2}$, and $\{{}^{\,i}_{j\,k}\} = 0$, $\nabla_k = \partial_k$. We also put $c_{jk} = \partial_k c_j$ and $\sum = \sum_{r=1}^n$. Then, M_j and $M_j{}^i{}_k$ are expressed as

(4.3) $$M_j = (1/c^2)\{\textstyle\sum c_r c_{jr} - (\sum c_{rr}) c_j/(n-1)\},$$

(4.4) $$M_j{}^i{}_k = \delta_j{}^i M_k + \delta_k{}^i M_j - \delta_{jk} M_i,$$

and the conditions (3.3), (3.5) and (3.7) are expressed as

$$\partial_k M_j = \partial_j M_k, \tag{4.5}$$

$$\sum c_r c_{rk} = 0, \tag{4.6}$$

$$c_{jk} = (1/c^2)\{c_k \sum c_r c_{jr} + (\sum c_{rr})(c^2 \delta_{jk} - c_j c_k)/(n-1)\} \tag{4.7}$$

respectively. (4.5) means that M_k is locally gradient, and since U is connected, (4.6) shows that c is constant on U. Thus Theorem 3.2 is restated as

Theorem 4.2. *A Randers space (M, L) is conformally flat if and only if M is covered by a system of local coordinate neighbourhoods $\{(U, x)\}$ such that, in each U, the fundamental function L is expressed in the form*

$$L = a(x)\{(\delta_{ij} y^i y^j)^{1/2} + c_i(x) y^i\}, \tag{4.8}$$

where $a(x)$ and $c_i(x)$ are a function and a covariant vector field on U respectively, and c_i satisfies the following conditions:
(1) *$c = (\delta^{ij} c_i c_j)^{1/2}$ is constant, and* (4.7) *is satisfied,*
(2) *$M_j(x)$ given by* (4.3) *satisfies* (4.5),
(3) *the curvature tensor $M_h{}^i{}_{jk}$ of the linear connection $M_j{}^i{}_k$ given by* (4.4) *vanishes.*

The case where each c_i is constant gives a trivial example, but (4.8) has a rather wide variety. We shall suggest this by considering the two-dimensional case. By checking each case of the values that j and k take, it is shown that (4.7) follows from (4.6) as a result of the condition that c is constant.

We put

$$c_1 = c \cos\theta, \quad c_2 = c \sin\theta, \tag{4.9}$$

where θ is given by

$$\theta = \tan^{-1}(c_2/c_1), \quad \text{or} \quad \theta = \cot^{-1}(c_1/c_2). \tag{4.10}$$

If we write down (4.3) for $j = 1, 2$ explicitly, we have $M_1 = (c_2 c_{12} - c_1 c_{22})/c^2$, $M_2 = (c_1 c_{21} - c_2 c_{11})/c^2$, which are expressed as

$$M_1 = -\theta_2, \quad M_2 = \theta_1, \tag{4.11}$$

where $\theta_j = \partial_j \theta$. Thus, M_j satisfies (4.5) if and only if θ is harmonic:

$$\partial_1 \partial_1 \theta + \partial_2 \partial_2 \theta = 0. \tag{4.12}$$

The coefficients $M_j{}^i{}_k$ given by (4.4) become as follows:

$$\begin{aligned} &M_1{}^1{}_2 = M_2{}^1{}_1 = -M_1{}^2{}_1 = M_2{}^2{}_2 = \theta_1, \\ &M_1{}^1{}_1 = -M_2{}^1{}_2 = M_1{}^2{}_2 = M_2{}^2{}_1 = -\theta_2, \end{aligned} \tag{4.13}$$

from which it is directly shown that the curvature tensor $M_{h\ jk}^{\ i}$ vanishes under the condition (4.12). Thus we have shown

Theorem 4.3. *A two-dimensional Randers space (M, L) is conformally flat if and only if M is covered by a system of local coordinate neighbourhoods $\{(U, (x^1, x^2))\}$ such that, in each U, the fundamental function L is expressed in the form*

(4.14) $$L = a(x)\{((y^1)^2 + (y^2)^2)^{1/2} + (c\cos\theta)y^1 + (c\sin\theta)y^2\},$$

where $a(x)$ is a function, c is constant and $\theta(x)$ is a harmonic function. Then, we can locally take a function $\sigma(x)$ such that $\partial_1\sigma = -\partial_2\theta$, $\partial_2\sigma = \partial_1\theta$, so $\tilde{L} = (e^\sigma/a)L$ is locally Minkowski.

The latter statement of Theorem 4.3 follows from the proof of Theorem 3.1. A two-dimensional Riemannian space is always conformally flat, but Theorem 4.3 shows that there are two-dimensional Randers spaces which are not conformally flat.

Now, on a well-known underlying manifold we shall give a non-trivial example of a Randers space whcih is globally conformal to a locally Minkowski space. On a sphere S^2 it is impossible to introduce a global Randers metric, because there is not a global non-zero vector field b_i on S^2.

Ler $H^2 = \{(x^1, x^2) | x^2 > 0\}$ be the upper half-plane in the x^1x^2-plane. If we put $\theta(x) = x^1$, $a(x) = k/x^2$ in (4.14), where k is a positive constant, we can take $\sigma(x) = x^2$, and then

(4.15) $$L = (k/x^2)\{((y^1)^2 + (y^2)^2)^{1/2} + (c\cos x^1)y^1 + (c\sin x^1)y^2\}$$

is conformally changed to $\tilde{L} = (x^2/k)e^{x^2}L$, which becomes

(4.16) $$\tilde{L} = ((\bar{y}^1)^2 + (\bar{y}^2)^2)^{1/2} + c\bar{y}^2$$

by the coordinate transformation

(4.17) $$\bar{x}^1 = e^{x^2}\cos x^1,\ \bar{x}^2 = e^{x^2}\sin x^1.$$

L defines a global Randers metric on H^2, which is modified from a Riemannian metric of negative constant curvature $-1/k^2$ called the Poincaré metric. The Randers space (H^2, L) is globally conformal to a locally Minkowski space $(H^2, \tilde{L})$, where $\tilde{L}$ is locally expressed as (4.16) in the coordinate system $(\bar{x}^1, \bar{x}^2)$.

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