CONVEX APPROXIMTION TO CONVEX DATA

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Abstract

In the present paper we shall consider a problem of convexity (or concavity) preserving approximation of plane convex (or concave) data by use of simple rational splines of continuity class C^2.

1. Introduction and Description of Method

Parametric cubic splines are of much use for an approximation of plane curves and data which are not represented as graphs of a single-valued functions. However, they do not always preserve desired properties, for example, convexity or concavity of original curves and data. In the present paper we shall consider a problem of convexity (or concavity) preserving approxmation of plane convex (or concave) data by use of a simple rational spline of continuity class C^2 defined on a partition: $-\infty < \tau_0 < \cdots < \tau_n < +\infty$. For $p > -1$, the spline s is defined as follows:

$$\begin{aligned}&\text{(i)}\quad s(\tau) \text{ is a linear combination of } t,\ r,\ t^3/(1+pt) \text{ and } r^3/(1+pr) \text{ on } [\tau_i,\ \tau_{i+1}]\\ &\qquad (0 \leq i \leq n-1) \text{ with } \tau = \tau_i + t\Delta\tau_i\ (\Delta\tau_i = \tau_{i+1} - \tau_i) \text{ and } r = 1-t\\ &\text{(ii)}\quad s \in C^2[\tau_0,\ \tau_n].\end{aligned} \tag{1}$$

For $p=0$, the spline coincides with the well-known cubic spline, and it reduces to a piecewise linear function as $p \to -1+$ under the condition that $|s'(\tau)| \leq K$ $(\tau_0 \leq \tau \leq \tau_n)$ for a positive constant K independent of p.

Now, for given data (x_i, y_i) $(0 \leq i \leq n)$ chosen in order, the splines u and v of the form (1) can be fitted to the points (τ_i, x_i) and (τ_i, y_i), respectively, i. e.,

$$u_i(=u(\tau_i)) = x_i \quad \text{and} \quad v_i(=v(\tau_i)) = y_i \ (0 \leq i \leq n) \tag{2}$$

where $\tau_0=0$ and $\Delta\tau_i = \{(\Delta x_i)^2 + (\Delta y_i)^2\}^{1/2}$ $(0 \leq i \leq n-1)$.

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Since u (or v) is dependent on $n+3$ parameters, there are two additional conditions to (2) required for a unique determination of it. In what follows, $x[\tau_{i+1}, \tau_i]$ and $x[\tau_{i+1}, \tau_i, \tau_{i-1}]$ are the first and second divided differences of data values: x_i, x_{i+1} and x_{i-1}, x_i, x_{i+1}, respectively. Then we take these to be end ones:

(i) for the closed data:

$$u_0^{(k)}=u_n^{(k)},\ v_0^{(k)}=v_n^{(k)}\ (k=1, 2); \tag{3}$$

(ii) for the open data:

$$B_0(u)\ (=(2+p)u'_0+(1+p)u'_1-(3+2p)u[\tau_1, \tau_0])=B_0(v)=0 \tag{4}$$

$$A_{n-1}(u)\ (=(2+p)u'_n+(1+p)u'_{n-1}-(3+2p)u[\tau_n, \tau_{n-1}])=A_{n-1}(v)=0. \tag{5}$$

The resulting curve Γ: $(u(\tau), v(\tau))$ $(\tau_0\leq\tau\leq\tau_n)$ is the parametric rational spline one. Our method is situated between the parametric cubic spline method $(p=0)$ and the piecewise linear one connecting the data poinst by use of piecewise straigh lines $(p\to -1+)$ where the former of continuity class C^2 is not always convex (or concave) even to convex (or concave) data and the latter of continuity class only C is always convex (or concave) to covex (or concave) data. For our method, we have

Theorem. *For each* i $(0\leq i\leq n-1)$, *suppose that*

$$x[\tau_{i+1}, \tau_i]y[\tau_{j+1}, \tau_j, \tau_{j-1}]-x[\tau_{j+1}, \tau_j, \tau_{j-1}]y[\tau_{i+1}, \tau_i] > 0\ (\text{or} <0)\ (j=i, i+1). \tag{6}$$

Then the parametric rational spline curve Γ *is uniquely determined under* (2)–(3) (or (4)–(5) for (3)), *and it is convex (or concave) on* $[\tau_i, \tau_{i+1}]$ *for* p *greater than and sufficiently close to* -1 *where for the closed data,* $(x_j, y_j)=(x_{n+j}, y_{n+j})$ *and* $\Delta\tau_j=\Delta\tau_{n+j}$ $(j=-1, 0, 1)$, *and for the open data, the convexity (or concavity) condition* (6) *is not required in the case when* $(i, j)=(0, 0)$, $(n-1, n)$.

2. Proof of Theorem

Before we proceed with analysis, we shall require the following three lemmas.

Lemma 1 ([4]). *Let* s *be of the form* (1). *Then we have*

$$\frac{s_{i+1}}{\Delta\tau_i}+\frac{(2+p)}{(1+p}\Big)\Big(\frac{1}{\Delta\tau_i}+\frac{1}{\Delta\tau_{i-1}}\Big)s'i+\frac{s_i'{}_{-1}}{\Delta\tau_{i-\mathrm{i}}} =\frac{(3+2p)}{(1+p)}\Big(\frac{s[\tau_{i+1}, \tau_i]}{\Delta\tau_i}+\frac{s[\tau_i, \tau_{i-1}]}{\Delta\tau_{i-1}}\Big) \quad (1\leq i\leq n-1) \tag{7}$$

Lemma 2. *Let* s *be of the form* (1) *satisfy the end conditions* (3) or (4)–(5). *Then we have for* $0\leq i\leq n-1$:

$$A_i(s)\ (=(2+p)s_i'{}_{+1}+(1+p)s_i'-(3+2p)s[\tau_{i+1}, \tau_i])$$

$$=\Delta\tau_i s[\tau_{i+2},\ \tau_{i+1},\ \tau_i]+O(\delta)\ \ (\delta=1+p) \tag{8}$$

$$\begin{aligned}B_i(s)\ (&=(1+p)s'_{i+1}+(2+p)s'_i-(3+2p)s[\tau_{i+1},\ \tau_i])\\ &=-\Delta\tau_i s[\tau_{i+1},\ \tau_i,\ \tau_{i-1}]+O(\delta)\end{aligned} \tag{9}$$

where for (4) − (5), $A_{n-1}(s)=B_0(s)=0$.

Proof. For (4) − (5), from (7) we have

$$|\, s'_i\,| \leq K\ \ (0\leq i\leq n) \tag{10}$$

with a positive constant K independent of p ([1], p. 21). For (3), we also get the same estimation (10) since then the determining equation for s'_i $(0\leq i\leq n)$ is given by (7) $(0\leq i\leq n-1)$ and $s'_0=s'_n$. Letting δ $(=1+p)\to 0$ in (7), by (10) we have

$$\begin{aligned}s'_j=&\left(\frac{s[\tau_{j+1},\ \tau_j]}{\Delta\tau_j}+\frac{s[\tau_j,\ \tau_{j-1}]}{\Delta\tau_{j-1}}\right)/\left(\frac{1}{\Delta\tau_j}+\frac{1}{\Delta\tau_{j-1}}\right)\\ &+O(\delta)\ \ (1\leq j\leq n-1)\end{aligned} \tag{11}$$

from which follow the desired asymptotic expansions (8) and (9).
Let ϕ and θ be given by

$$\phi(t)=(1-2t)(1+\delta t)/(1+pt)+rt/(1+pt)^2$$
$$\theta(t)=2\delta t(3+3pt+p^2t^2)/(1+pt)^3$$

with $\delta=1+p$ and $r=1-t$.
Then we have

Lemma 3. *For p greater than and sufficiently close to* -1, *we get the inequality:*

$$\begin{aligned}&\{\theta(t)+\theta(r)\}+O(\delta)\{\theta(t)\phi(r)+\theta(r)\phi(t)\}\geq 0\\ &(0\leq t\leq 1,\ r=1-t).\end{aligned} \tag{12}$$

Proof. The above inequality is equivalent to

$$\begin{aligned}&\{t(1+pr)^3(3+3pt+p^2p^2)+r(1+pt)^3(3+3pr+p^2r^2)\}\\ &+O(\delta)\,[t(1+pr)(3+3pt+p^2t^2)\{(1+pr)(1-2r)(1+\delta r)+rt\}\\ &+r(1+pt)(3+3pr+p^2r^2)\{(1+pt)(1-2t)(1+\delta t)+rt\}]\geq 0\\ &(0\leq t\leq 1,\ r=1-t).\end{aligned} \tag{13}$$

Since

$$t(1+pr)^3(3+3pt+p^2t^2)+r(1+pt)^3(3+3pr+p^2r^2)$$
$$\geq (3/4)\,\{t(1+pr)^3+r(1+pt)^3\}\geq (3/4)\,(t^4+r^4)\geq 3/32$$
$$(0\leq t\leq 1,\ r=1-t), \tag{14}$$

we have the the desired inequality (12) for p greater than and sufficiently close to -1.

Now we are ready to prove our Theorem. Since $u_j = x_j$ $(j=i,\ i+1)$, we have

$$u(\tau)=x_i r+x_{i+1}t-\frac{(1+p)}{(3+2p)}rt\,\{(1+\frac{t}{1+pt})A_i(u)$$
$$-(1+\frac{r}{1+pr})B_i(u)\}\ \Delta\tau_i \qquad (0\leq t\leq 1,\ r=1-t) \tag{15}$$

with $\tau=\tau_i+t\Delta\tau_i \qquad (0\leq i\leq n-1)$.

By a simple calcuation, on $[\tau_i,\ \tau_{i+1}]$

$$u'(\tau)=x[\tau_{i+1},\ \tau_i]-\frac{(1+p)}{(3+2p)}\{A_i(u)\ \phi\ (t)+B_i(t)\ \phi\ (r)\} \tag{16}$$

$$u''(\tau)=\frac{(1+p)}{(3+2p)}\{A_i(u)\ \theta\ (t)-B_i(u)\ \theta\ (r)\}\,/\,\Delta\tau_i. \tag{17}$$

Hence we have on $[\tau_i,\ \tau_{i+1}]$ (for (3), $0\leq i\leq n-1$ and for (4)$-$(5), $1\leq i\leq n-2$):

$$\Delta\tau_i\,\{u'(\tau)v''(\tau)-u''(\tau)v'(\tau)\}=[x[\tau_{i+1},\ \tau_i]-\frac{\delta}{1+2\delta}\{A_i(u)\times$$
$$\phi\,(t)+B_i(u)\ \phi\,(r)\}]\ \{A_i(v)\ \theta\,(t)-B_i(v)\ \theta\,(r)\}-\{A_i(u)\ \theta\,(t)$$
$$-B_i(u)\ \theta\,(r)\}\ [y[\tau_{i+1},\ \tau_i]-\frac{\delta}{1+2\delta}\{A_i(v)\ \phi\,(t)+B_i(v)\ \phi\,(r)\}]$$
$$=\{A_i(v)x[\tau_{i+1},\ \tau_i]-A_i(u)y[\tau_{i+1},\ \tau_i]\}\ \theta\,(t)+\{B_i(u)y[\tau_{i+1},\ \tau_i]$$
$$-B_i(v)x[\tau_{i+1},\ \tau_i]\}\ \theta\,(r)+O(\delta)\ \{\theta\,(t)\ \phi\,(r)+\theta\,(r)\ \phi\,(t)\}$$
$$(0\leq t\leq 1,\ r=1-t). \tag{18}$$

Since Γ is convex (or concave) on $[\tau_i,\ \tau_{i+1}]$ if and only if

$$u'(\tau)v''(\tau)-u''(\tau)v'(\tau)\geq 0\ (\text{or}\leq 0) \quad \text{on } [\tau_i,\ \tau_{i+1}], \tag{19}$$

by means of Lemmas 2 and 3 we have the desired result from (18) on $[\tau_i,\ \tau_{i+1}]$ (for (3), $0\leq i\leq n-1$ and for (4)$-$(5), $1\leq i\leq n-2$), i.e., for the closed data, on $[\tau_0,\ \tau_n]$ and for the open data, on $[\tau_0,\ \tau_1]$ and $[\tau_{n-1},\ \tau_n]$ we have from (4) and (5)

$$\Delta\tau_0\,\{u'(\tau)v''(\tau)-u''(\tau)v'(\tau)\}=\{A_0(v)x[\tau_1,\ \tau_0]-A_0(u)\,y[\tau_1,\ \tau_0]\}\ \theta\,(t)$$

$$(\tau = \tau_0 + t\Delta\tau_0,\ 0 \leq t \leq 1) \tag{20}$$

and

$$\Delta\tau_{n-1}\{u'(\tau)v''(\tau) - u''(\tau)v'(\tau)\} = \{B_{n-1}(u)y[\tau_n,\ \tau_{n-1}]$$

$$-B_{n-1}(v)x[\tau_n,\ \tau_{n-1}]\}\ \theta(r) \qquad (\tau = \tau_{n-1} + t\Delta\tau_{n-1},\ 0 \leq t \leq 1). \tag{21}$$

Hence, we have the desired result on $[\tau_0,\ \tau_1]$ and $[\tau_{n-1},\ \tau_n]$ by use of the asymptotic expansions (8) and (9).

This completes the proof of our Theorem.

3. Numerical Illustration

If the data (x_i, y_i) $(0 \leq i \leq n)$ is derived from a sufficiently smooth plane curve $(p(\sigma), q(\sigma))$ $(0 \leq \sigma \leq L)$, i.e., $x_i = p(\sigma_i)$ and $y_i = q(\sigma_i)$ $(0 \leq i \leq n)$, then the similar error analysis in [2] gives for $p \neq 0$:

$$|u'_i - p'_i|,\ |v'_i - q'_i| = |p|O(h^2) \quad (h \to 0) \quad (0 \leq i \leq n)$$

where $h = \max\limits_{0 \leq i \leq n-1} |\sigma_{i+1} - \sigma_i|$.

Thus the error would be small for a small value of $|p|$, while the method with $p=0$

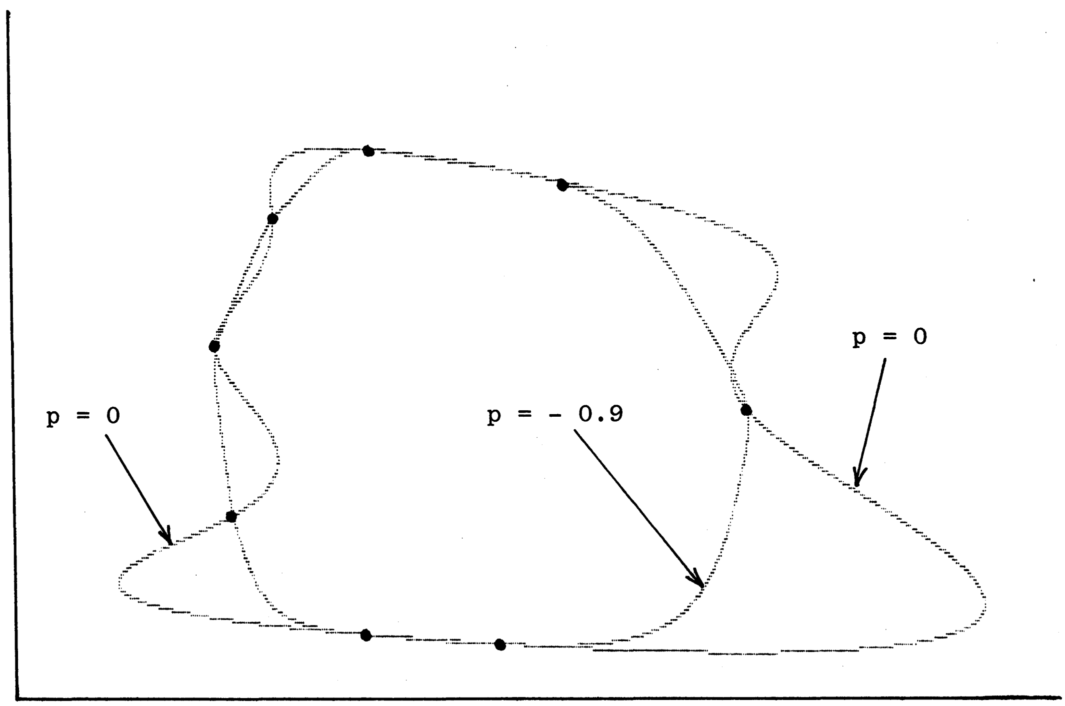

FIG. 1

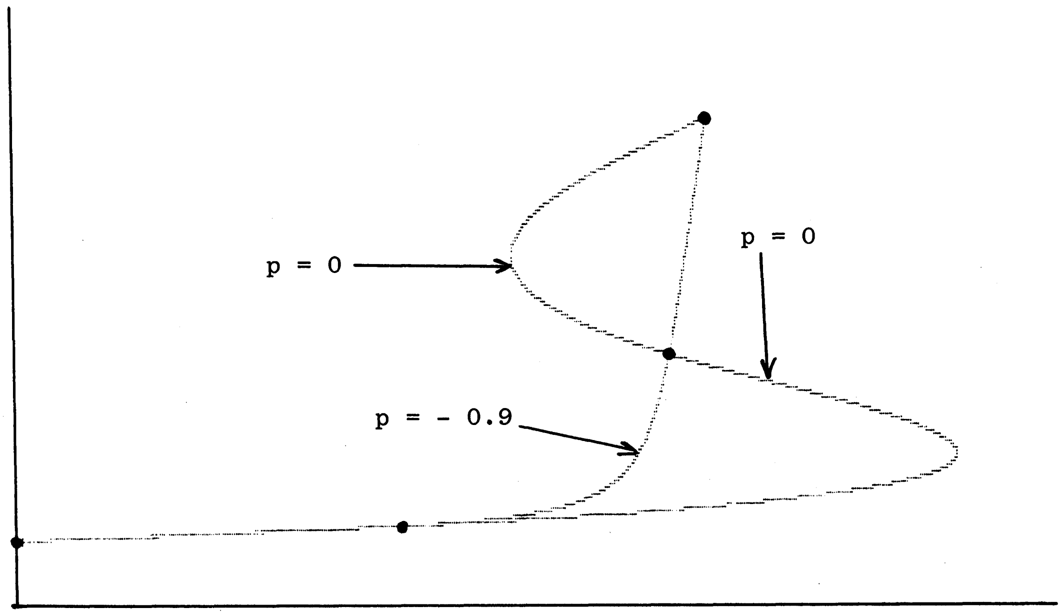

FIG. 2

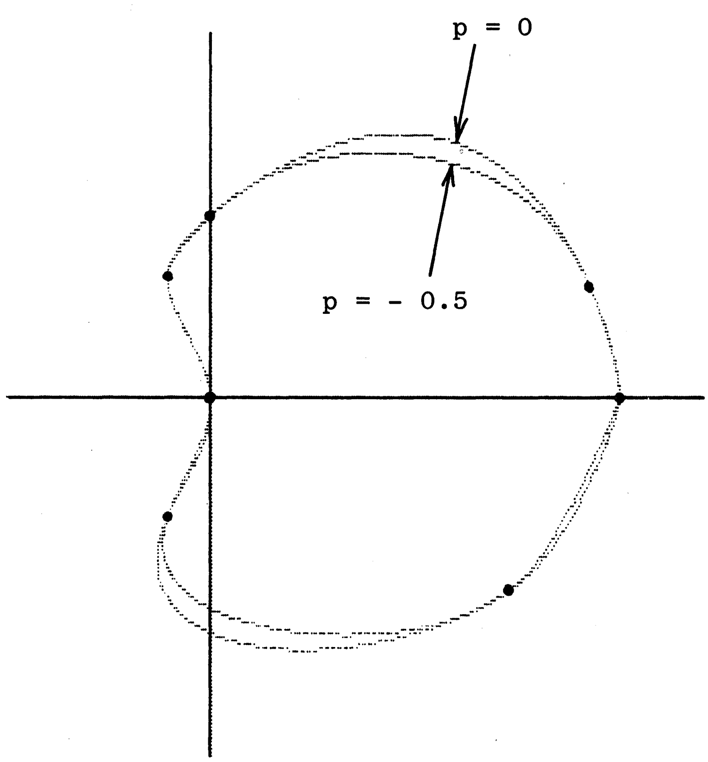

FIG. 3

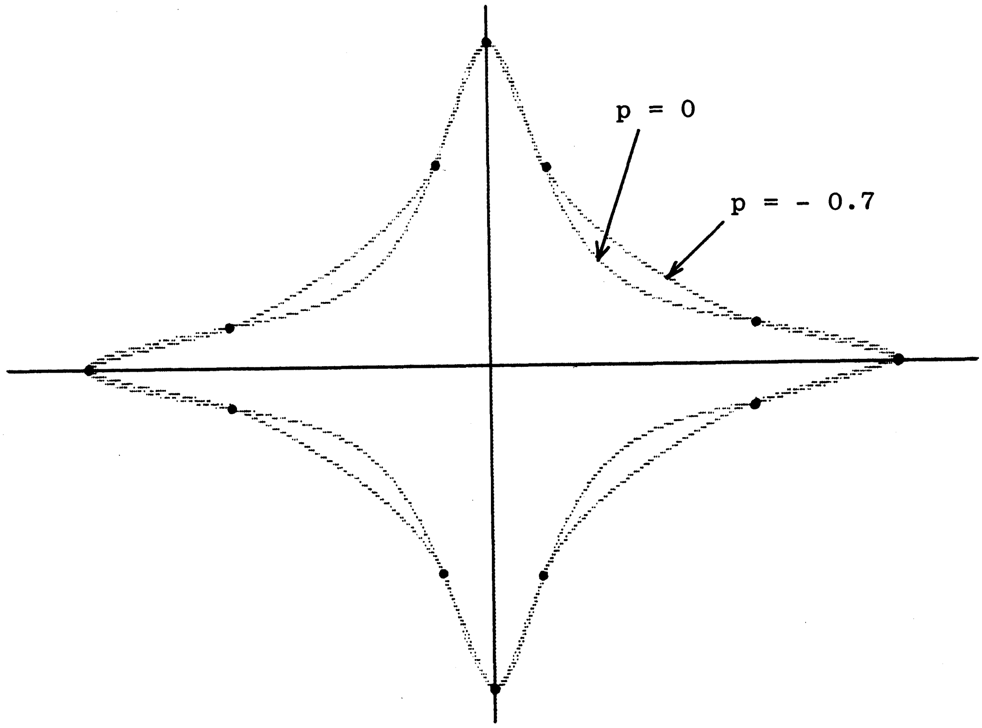

FIG. 4

(the well-known parametric cubic spline one) does not always give convex approximation even to convex data. Therefore, in practical computation, it would be sufficient to decrease the parameter p, starting at zero, until a picture of the spline curve is satisfactory ([3]). In order to illustrate an application of the above stated method, we take four examples where solid circles mean the data points given counterclockwise. The data in FIG. 2 is obtained from $y = 1/(2-x)^2$ (x=0, 1, 1.7, 1.8). Numerical results show that our method gives visually pleasing curves of continuity class C^2.

Beferences

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