# ON COMPLEX FINSLER MANIFOLDS

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# ON COMPLEX FINSLER MANIFOLDS

*Dedicated to Professor Dr. Masao Hashiguchi on the occasion of his 60th birthday*

Tadashi AIKOU*

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**Abstract**

The purpose of the present paper is to investigate complex Finsler manifolds and introduce some special complex Finsler structures to consider the analogy of real Finsler geometry. Furthermore, we give a note on the holomorphic sectional curvature of complex Finsler manifolds.

## Introduction

We know already many papers on complex Finsler geometry (cf. Fukui [1], Ichijyō [4, 6], Kobayashi [7], Rizza [12, 13], Royden [14], Rund [15], etc.). Suggested by Kobayashi [7] and Royden [14], in the present paper, we shall investigate complex Finsler manifolds.

In § 1 we shall introduce the notions of complex Finsler vector bundle and complex Finsler connection, and in § 2, by using the so-called non-linear connection, we show the local expressions of complex Finsler connections.

In § 3 we shall investigate two types of complex Finsler connections which are determined from the given complex Finsler metric. The first one is the Hermitian connection in a complex Finsler vector bundle with a Hermitian metric, that is, the *Finsler-Hermitian connection*, which is essencially the same as the one treated in Kobayashi [7]. The other one is the connection which is treated in Rund [15], that is, the *complex Rund connection.* We shall show that the second connection is uniquely determined by some axioms. Then, with respect to these connections, we shall show the existence of a special complex coordinate system which is used in § 4 and § 7.

In § 4 we shall define the notion of Finsler-Kähler manifold and characterize it by the existence of a special coordinate system which is similar to the case of Kähler manifold.

In § 5 we shall introduce the notion of complex Berwald manifold which is a natural generalization of real case (cf. Matsumoto [11]), and give a necessary condition that a complex Finsler manifold be complex Berwald, which is similar to the result in

* Department of Mathematics, Faculty of Science, Kagoshima University, Kagoshima, Japan.

Ichijyō [5]. Furthermore, we shall investigate complex Finsler manifolds with complex $(\alpha, \beta)$-metics as an example of complex Berwald manifolds.

In §6 we shall introduce the notion of complex locally Minkowski manifold and characterize it in terms of the complex Rund connection.

In the last section, as an application we shall give a note on the holomorphic sectional curvatures treated in Kobayashi [7] and Royden [14].

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## 1. Complex Finsler vector bundles

Let $M$ be a complex manifold of $\dim_C M = n$. Then the complexification $T_C M$ of the real tangent bundle $TM$ is decomposed as $T_C M = T'M \oplus T''M$, where $T'M$ is the holomorphic tangent bundle over $M$ and $T''M$ is the conjugate of $T'M$. As is well-known, $T'M$ is also a complex manifold of $\dim_C T'M = 2n$, and the projection $\pi_T \colon T'M \to M$ is holomorphic.

**Definition 1.1.** A complex vector bundle $\tilde{E}$ over $T'M$ is said to be a *complex Finsler vector bundle*, if $\tilde{E}$ is the pull-back $\pi_T^{-1}(E)$ of a complex vector bundle $E$ over $M$. If $E$ is holomorphic, then $\tilde{E}$ is called a *holomorphic Finsler vector bundle.*

Let $\tilde{E} = \pi_T^{-1}(E)$ be a complex Finsler vector bundle. If we denote by $g_{UV}$ the transition functions of $E$, the ones of $\tilde{E}$ are given by $g_{UV} \circ \pi_T$, where $\{U, V, \cdots\}$ is an open cover of $M$ such that on each $U$ the vector bundle $E$ is trivial.

In the following, we shall denote by $\Gamma(\tilde{E})$ the space of $C^\infty$-sections of $\tilde{E}$. Then a connection in $\tilde{E}$ is a homomorphism $D \colon \Gamma(\tilde{E}) \to \Gamma(\tilde{E} \otimes T_C(T'M)^*)$ satisfying

$$D(fs) = sdf + fDs,$$

where $s \in \Gamma(\tilde{E})$, and $f$ is an arbitrary $C^\infty$-function on $T'M$.

**Definition 1.2.** A connection $D$ in a complex Finsler vector bundle $\tilde{E}$ is called a *complex Finsler connection.*

In the following, we suppose that $\tilde{E}$ is a holomorphic Finsler vector bundle of rank $m$. For a local frame field $s = (s_1, \dots, s_m)$ of $\tilde{E}$ over $\pi_T^{-1}(U)$, we put

$$Ds_\alpha = s_\beta \omega_\alpha{}^\beta.$$

The matrix 1-form $\omega_U = (\omega_\alpha{}^\beta)$ on $\pi_T^{-1}(U)$ is called the connection form of $D$ with respect to $s$. On the intersection $U \cap V$, the connection forms $\omega_U$, $\omega_V$ saitsfy the following relation:

$$(1.1)\qquad \omega_U = \tilde{g}_{UV}{}^{-1}\omega_V \tilde{g}_{UV} + \tilde{g}_{UV}{}^{-1} d\tilde{g}_{UV},$$

where $\tilde{g}_{UV} = g_{UV} \circ \pi_T$ are the transition functions of $\tilde{\boldsymbol{E}}$.

Suppose that a Hermitian metric $g$ is given in $\tilde{\boldsymbol{E}}$. Then a connection $D: \Gamma(\tilde{\boldsymbol{E}}) \to \Gamma(\tilde{\boldsymbol{E}} \otimes T_C(T'M)^*)$ satisfying the following conditions is uniquely determined from $g$, and called the Hermitian connection in $(\tilde{\boldsymbol{E}}, g)$.

(1) $D$ is metrical: $dg(s, t) = g(Ds, t) + g(s, Dt)$ for any $s, t \in \Gamma(\tilde{\boldsymbol{E}})$,

(2) The connection form is (1, 0)-type.

The following proposition is similar to the well-known result (cf. Kobayashi-Wu [9]).

**Proposition 1.1.** *Let $(\tilde{\boldsymbol{E}}, g)$ be a holomorphic Finsler vector bundle with a Hermitian metric $g$. Then the connection form $\omega$ and curvature form $\Omega$ of its Hermitian connection $D$ are given by*

$${}^t\omega = d'g \cdot g^{-1},$$

$$\Omega = d''\omega$$

*respectively.*

## 2. Complex non-linear connections

In the previous section, we introduced a complex Finsler connection $D$: $\Gamma(\tilde{\boldsymbol{E}}) \to \Gamma(\tilde{\boldsymbol{E}} \otimes T_C(T'M)^*)$ in a complex Finsler vector bundle $\tilde{\boldsymbol{E}}$. The complex vector bundle $\tilde{\boldsymbol{E}} \otimes T_C(T'M)^*$, however, is not necessary a complex Finsler vector bundle. In the present section, we state the notion of complex non-linear connection, and show that $\tilde{\boldsymbol{E}} \otimes T_C(T'M)^*$ is also regarded as a complex Finsler vector bundle.

We denote by $\{U, (z^i)\}$ a complex coordinate system on $M$, and let $\{U, X_U\}$ be an open cover of $T'M$ with local holomorphic frame fields $X_U = \{X_1, \dots, X_n\}$. Then we denote by $\{\pi_T{}^{-1}(U), (z^i, \eta^i)\}$ the induced complex coordinate system on $T'M$.

The transition functions of the holomorphic tangent bundle $T'(T'M)$ are given by

$$(2.1)\qquad \begin{pmatrix} \partial z'^j/\partial z^i & 0 \\ (\partial^2 z'^j/\partial z^i \partial z^m)\eta^m & \partial z'^j/\partial z^i \end{pmatrix}.$$

For the differential $\pi_{T*}$ of the natural projection $\pi_T$, we put

$$V(T'M) = \{\xi \in T'(T'M);\ \pi_{T*}(\xi) = 0\}.$$

Then $V(T'M)$ is a vector bundle of rank $n$ over $T'M$. It is clear that $\{\partial/\partial\eta^i\}$ is a local frame field of $V(T'M)$ over $\pi_T{}^{-1}(U)$ and the transition functions of $V(T'M)$ are given by $(\partial z'^j/\partial z^i) \circ \pi_T$. So $V(T'M)$ is a holomorphic Finsler vector bundle of rank $n$.

Now we can take a complex vector sub-bundle $H(T'M)$ of $T'(T'M)$ such that the following $C^{\infty}$-decomposition holds:

$$T'(T'M) = H(T'M) \oplus V(T'M). \tag{2.2}$$

From (2.1), we see that the transition functions of $H(T'M)$ are also given by $(\partial z'^j/\partial z^i)\circ \pi_T$. Hence $H(T'M)$ is also a complex Finsler vector bundle of rank $n$. Then the $C^{\infty}$-decomposition (2.2) is the Whitney sum of complex Finsler vector bundles $H(T'M)$ and $V(T'M)$:

$$T_C(T'M) = T'(T'M) \oplus T''(T'M). \tag{2.3}$$

**Theorem 2.1.** *The holomorphic tangent bundle $T'(TM)$ of $T'M$ is written as a whitney sum* (2.2) *of complex Finsler vector bundles $H(T'M)$ and $V(T'M)$. Then the complex vector bundle $\tilde{E} \otimes T_C(T'M)^*$ is also a complex Finsler vector bundle.*

Let $\tilde{E}$ be a holomorphic Finsler vector bundle with a Finsler connection $D: \Gamma(\tilde{E}) \to \Gamma(\tilde{E} \otimes T_C(T'M)^*)$. Then, from (2.2) and (2.3) we have the following decomposition of $D$:

$$D = D' + D'',$$

where

$$D': \Gamma(\tilde{E}) \longrightarrow \Gamma(\tilde{E} \otimes T'(T'M)^*) \quad \text{and} \quad D'': \Gamma(\tilde{E}) \longrightarrow \Gamma(\tilde{E} \otimes T''(T'M)^*).$$

We have also

$$D' = D'^h + D'^v \quad \text{and} \quad D'' = D''^h + D''^v,$$

where

$$D'^h: \Gamma(\tilde{E}) \longrightarrow \Gamma(\tilde{E} \otimes H(T'M)^*), \qquad D'^v: \Gamma(\tilde{E}) \longrightarrow \Gamma(\tilde{E} \otimes V(T'M)^*)$$

and

$$D''^h: \Gamma(\tilde{E}) \longrightarrow \Gamma(\tilde{E} \otimes \overline{H(T'M)^*}), \quad D''^v: \Gamma(\tilde{E}) \longrightarrow \Gamma(\tilde{E} \otimes \overline{V(T'M)^*}).$$

Then we put

$$D^h = D'^h + D''^h \quad \text{and} \quad D^v = D'^v + D''^v.$$

**Definition 2.1.** The covariant derivation $D^h$ (resp. $D^v$) is called the $h$-(resp. $v$-) *covariant derivation* of the given complex Finsler connection $D$. If the condition $D'' = d''$ is satisfied, then the complex Finsler connection $D$ is said to be (1, 0)-*type*.

For the local expressions of $D^h$ and $D^v$, we consider the following $n$ vector fields $\{\delta/\delta z^i\}$ as a local frame field of $H(T'M)$ over $\pi_T{}^{-1}(U)$:

$$\frac{\delta}{\delta z^i} = \frac{\partial}{\partial z^i} - N^m{}_i \frac{\partial}{\partial \eta^m} \qquad (1 \leqq i \leqq n),$$

where the $n^2$ $C^\infty$-functions $N^i{}_j(z, \eta)$ should satisfy the following transformation law:

$$N'^i{}_l \frac{\partial z'^l}{\partial z^j} = N^l{}_j \frac{\partial z'^i}{\partial z^l} - \frac{\partial z^l}{\partial z'^j} \frac{\partial^2 z'^i}{\partial z^l \partial z^m} \eta^m. \tag{2.4}$$

**Definition 2.2.** The family of functions $N^i{}_j(z, \eta)$ satisfying (2.4) is called a *complex non-linear connection* on $T'M$.

We shall denote by $\{dz^i, \delta\eta^i\}$ the dual frame field of $\{\delta/\delta z^i, \partial/\partial\eta^i\}$, where we put

$$\delta\eta^i = d\eta^i + N^i{}_m(z, \eta)\, dz^m.$$

The 1-forms $\{dz^i\}$ and $\{\delta\eta^i\}$ consist a local frame field of $H(T'M)^*$ and $V(T'M)^*$ over $\pi_T{}^{-1}(U)$ respectively. Then we shall introduce the following notations:

$$\begin{aligned} d'^h f &= \frac{\delta f}{\delta z^i} dz^i = \left(\frac{\partial f}{\partial z^i} - N^m{}_i \frac{\partial f}{\partial \eta^m}\right) dz^i, & d'^v f &= \frac{\partial f}{\partial \eta^i} \delta\eta^i, \\ d''^h f &= \frac{\delta f}{\delta \bar{z}^i} d\bar{z}^i = \left(\frac{\partial f}{\partial \bar{z}^i} - \bar{N}^m{}_i \frac{\partial f}{\partial \bar{\eta}^m}\right) d\bar{z}^i, & d''^v f &= \frac{\partial f}{\partial \bar{\eta}^i} \delta\bar{\eta}^i \end{aligned} \tag{2.5}$$

for any function $f$. Then we put

$$d^h = d'^h + d''^h, \qquad d^v = d'^v + d''^v. \tag{2.6}$$

Obviously, the operators $d^h$ and $d^v$ are real. The following are trivial:

$$d' = d'^h + d'^v, \qquad d'' = d''^h + d''^v. \tag{2.7}$$

Let $D$ be a complex Finsler connection of (1, 0)-type in a holomorphic Finsler connection $\tilde{E}$. Since the connection form $\omega$ is a (1, 0)-form, we may put

$$\omega_\beta{}^\alpha = F_\beta{}^\alpha{}_k dz^k + C_\beta{}^\alpha{}_k \delta\eta^k.$$

Then for an arbitrary section $\xi = \xi^\alpha s_\alpha$, we have

$$D^h \xi = (d^h \xi^\alpha + \xi^\beta F_\beta{}^\alpha{}_k dz^k) s_\alpha,$$

$$D^v \xi = (d^v \xi^\alpha + \xi^\beta C_\beta{}^\alpha{}_k \delta\eta^k) s_\alpha.$$

## 3. Complex Finsler manifolds

Let $M$ be a complex manifold of $\dim_C M = n$, and $T'M$ its holomorphic tangent bundle. We also use the notations in §1 and §2. First we shall define as follows (cf. Kobayashi [7]).

**Definition 3.1.** A function $F(z, \eta)$ on $T'M$ is called a *convex Finsler metric* if the following conditions are satisfied:

(1) $F(z, \eta)$ is smooth on $T'M$-$\{0\}$,

(2) $F(z, \eta) \geqq 0$ and $F(z, \eta) = 0$ if and only if $\eta = 0$,

(3) $F(z, \lambda\eta) = |\lambda|^2 F(z, \eta)$ for all $\lambda \in \boldsymbol{C}$,

(4) The following Hermitian matrix satisfies the convex condition, that is, the Hermitian matrix $(F_{i\bar{j}})$ is positive definite, where

$$F_{i\bar{j}} = \frac{\partial^2 F}{\partial\eta^i \partial\bar{\eta}^j}. \tag{3.1}$$

The complex manifold $M$ with a convex Finsler metric $F(z, \eta)$ is called a *complex Finsler manifold* and denoted by $(M, F)$.

Let $(M, F)$ be a complex Finsler manifold. We shall consider the case of $\boldsymbol{E} = T'M$ in §1:

$$\widetilde{TM} = \pi_T{}^{-1}(T'M)$$

and call it the holomorphic Finsler tangent bundle.

We can consider any local holomorphic frame field $X_U$ of $T'M$ as a local holomorphic frame field of $\widetilde{TM}$. Then a Hermitian metric $g$ in $\widetilde{TM}$ is defined by

$$F_{i\bar{j}} = g(X_i, X_j) \tag{3.2}$$

for the Hermitian matrix $(F_{i\bar{j}})$ of (3.1).

In the following we shall consider typical complex Finsler connections on a complex Finsler manifold $(M, F)$.

### (a) Finsler-Hermitian connections

First we shall consider the Hermitian connection $D: \Gamma(\widetilde{TM}) \to \Gamma(\widetilde{TM} \otimes T_C(T'M)^*)$ in $(\widetilde{TM}, g)$. By Proposition 1.1, the connection form $\omega$ and curvature form $\Omega$ of $D$ are given by

$$\omega_j{}^i = F^{\bar{r}i} d' F_{j\bar{r}}, \tag{3.3}$$

$$\Omega_j{}^i = d'' \omega_j{}^i. \tag{3.4}$$

For the local expressions of $D$, we suppose that a complex non-linear connection $N^i{}_j$ is given on $T'M$. Since $D$ is (1, 0)-type, we get

$$\omega_j{}^i = F_j{}^i{}_k dz^k + C_j{}^i{}_k \delta\eta^k, \tag{3.5}$$

where we put

$$F_{j\ k}^{\ i} = F^{\bar{r}i}\frac{\delta F_{j\bar{r}}}{\delta z^k}, \qquad C_{j\ k}^{\ i} = F^{\bar{r}i}\frac{\partial F_{j\bar{r}}}{\partial \eta^k}. \tag{3.6}$$

To determine $N^i_{\ j}$ from the given metric $F(z, \eta)$, we assume the condition

$$D^h(\eta^m X_m) = 0. \tag{3.7}$$

Because of $D^h = D'^h + d''^h$, the condition (3.7) is equivalent to $N^i_{\ k} = \eta^m F_{m\ k}^{\ i}$. The expression (3.6) and the homogeneity of $F_{i\bar{j}}$ with respect to $\eta$ give

$$N^i_{\ k} = F^{\bar{r}i}\frac{\partial F_{m\bar{r}}}{\partial z^k}\eta^m, \tag{3.8}$$

which is determined by the given convex Finsler metric $F(z, \eta)$. The non-linear connection $N^i_{\ j}$ given by (3.8) is exactly the one appeared in Rund [15] and Royden [14].

**Remark 3.1.** We see that the non-linear connection $N^i_{\ j}$ defined by (3.8) is entirely the one derived from the Euler-Lagrange equation for a $C^\infty$-curve in $(M, F)$ (Royden [14]), that is, a $C^\infty$-curve $z(t)$ is a geodesic in $(M, F)$ if and only if $z(t)$ satisfies the differential equations:

$$\frac{d^2 z^i}{dt^2} + N^i_{\ j}\left(z(t), \frac{dz}{dt}\right)\frac{dz^j}{dt} = \frac{d^2 z^i}{dt^2} + F_{j\ k}^{\ i}\left(z(t), \frac{dz}{dt}\right)\frac{dz^j}{dt}\frac{dz^k}{dt} = 0$$

with the following additional condition

$$F_{i\bar{m}}(F_{j\ k}^{\ i} - F_{k\ j}^{\ i})\frac{dz^j}{dt}\,\overline{\frac{dz^m}{dt}} = 0.$$

**Definition 3.2.** Let $(\widetilde{TM}, g)$ be the holomorphic Finsler tangent bundle over a complex Finsler manifold $(M, F)$. The pair $(D, N)$ of the Hermitian connection $D$ in $(\widetilde{TM}, g)$ and the non-linear connection $N$ given by (3.8) is called the *Finsler-Hermitian connection* on $(M, F)$ and denoted by $H\Gamma$.

**Proposition 3.1.** *Let $(M, F)$ be a complex Finsler manifold. The Finsler-Hermitian connection $H\Gamma$ is uniquely determined by* (3.6) *and* (3.8) *from the given complex Finsler metric $F(z, \eta)$. The curvature form of the Finsler-Hermitian connection $H\Gamma$ is given by* (3.4).

### (b) Complex Rund connections

We shall show the following proposition.

**Proposition 3.2.** *Let $(\widetilde{TM}, g)$ be as the above. Then there exists a unique complex Finsler connection $(D, N)$ satisfying the following conditions:*

(1) *D is* (1, 0)*-type*,
(2) *D is h-metrical*: $d^h g(s, t) = g(D^h s, t) + g(s, D^h t)$ *for any* $s, t \in \Gamma(\widetilde{TM})$,
(3) $D^v = d^v$,
(4) $D^h$ *satisfies* (3.7).

*Proof.* From the conditions (1) and (3), we can put $\omega_j{}^i = F_j{}^i{}_k dz^k$. Then the condition (2) is equivalent to

$$d^h F_{i\bar{j}} = F_{m\bar{j}}\omega_i{}^m + F_{i\bar{m}}\bar{\omega}_j{}^m.$$

Thus, by the condition (1), we have the first equation in (3.6). Furthermore, by the condition (4), we have (3.8). Q.E.D.

**Definition 3.3.** The complex Finsler connection $(D, N)$ given by Proposition 3.3 is called the *complex Rund connection* on $(M, F)$ and denoted by $R\Gamma$.

**Remark 3.2.** The complex Rund connection $R\Gamma$ is first treated by Rund [15]. It is not metrical, but we shall later show an application of this connection.

Now we shall investigate the curvature form $\Omega_j^i$ of $R\Gamma$. Since $R\Gamma$ is not metrical, we calculate $\Omega_j^i = d\omega_j^i + \omega_m^i \wedge \omega_j^m$ directly. Putting

$$R_j{}^i{}_{kl} = \frac{\delta F_j{}^i{}_k}{\delta z^l} - \frac{\delta F_j{}^i{}_l}{\delta z^k} + (F_m{}^i{}_k F_j{}^m{}_l - F_m{}^i{}_l F_j{}^m{}_k), \tag{3.9}$$

$$R^i{}_{kl} = \frac{\delta N^i{}_k}{\delta z^l} - \frac{\delta N^i{}_l}{\delta z^k}, \tag{3.10}$$

Rund [15] shows $R^i{}_{kl} = \eta^m R_m{}^i{}_{kl}$ and the following.

**Proposition 3.3.** (Rund [15]) *The respective quantities* $R_j{}^i{}_{kl}$ *and* $R^i{}_{kl}$ *defined by* (3.9) *and* (3.10) *vanish identically.*

Because of Proposition 3.3, we have

**Proposition 3.4.** *Let* $(M, F)$ *be a complex Finsler manifold. The complex Rund connection* $R\Gamma$ *on* $(M, F)$ *is uniquely determined by* $F_j{}^i{}_k$ *in* (3.6) *and by* $N^i{}_j$ *in* (3.8) *from the given convex Finsler metric* $F(z, \eta)$. *Then the curvature form* $\Omega_j{}^i$ *is given by*

$$\Omega_j{}^i = d''\omega_j{}^i + d''^v\omega_j{}^i. \tag{3.11}$$

With respect to $H\Gamma$ or $R\Gamma$, the following proposition is easily derived.

**Proposition 3.5.** *The coefficients* $N^i{}_j$ *given by* (3.8) *and the coefficients* $F_j{}^i{}_k$ *of* $H\Gamma$ (*or* $R\Gamma$) *given by* (3.6) *satisfy the relation*

$$F_j{}^i{}_k = \frac{\partial N^i{}_k}{\partial \eta^j}.$$

Furthermore, we have easily

**Proposition 3.6.** *Let* $(D, N)$ *be the Finsler-Hermitian (or complex Rund) connection on* $(M, F)$. *For an arbitrary point* $P_0 = (z_0, \eta_0)$ *of* $T'M$, *we can always choose a complex coordinate system* $(z^i, \eta^i)$ *aroud* $P_0$ *satisfying*

$$F_j{}^i{}_k(P_0) + F_k{}^i{}_j(P_0) = 0. \tag{3.12}$$

*Proof.* *For a given complex coordinate system* $\{U, (z^i)\}$ on $M$, we define a new complex coordinate system $\{U, (z'^i)\}$ as

$$z'^i = (z^i - z^i{}_0) - \frac{1}{2}F_j{}^i{}_k(P_0)(z^j - z^j{}_0)(z^k - z^k{}_0).$$

Then, because of (1.1), we get easily the condition (3.12) at $P_0$. Q.E.D.

Suggesting by Proposition 3.6 and Royden [14], we define as follows.

**Definition 3.4.** Let $(M, F)$ be a complex Finsler manifold and $P_0 = (z_0, \eta_0)$ an arbitrary point of $T'M$. Then a coordinate system around $P_0$ is said to be *semi-normal* at $P_0$ if the condition (3.12) is satisfied.

## 4. Finsler-Kähler manifolds

Let $(M, F)$ be a complex Finsler manifold and $(\widetilde{TM}, g)$ its holomorphic Finsler tangent bundle with the Hermitian metric $g$ given by (3.2). Then we define a real (1, 1)-form $\Theta$ on $T'M$ by

$$\Theta = \sqrt{-1}F_{i\bar{j}}dz^i \wedge d\bar{z}^j. \tag{4.1}$$

**Definition 4.1.** The real (1, 1)-form $\Theta$ defined by (4.1) is called the *Finsler-Kähler form* of $(M, F)$.

The following proposition is easily derived.

**Proposition 4.1.** *The Finsler-Kähler form* $\Theta$ *is closed if and only if the metric* $F_{i\bar{j}}$ *is a Kähler metric on* $M$.

Because of Proposition 4.1, the condition $d\Theta = 0$ is too strong to call $(M, F)$ a *Finsler-Kähler manifold.* Now we shall investigate the condition that $\Theta$ be $h$-closed: $d^h\Theta = 0$. As is easily seen, this condition is equivalent to

$$\frac{\delta F_{i\bar{j}}}{\delta z^k} = \frac{\delta F_{k\bar{j}}}{\delta z^i}, \quad \frac{\delta F_{i\bar{j}}}{\delta \bar{z}^k} = \frac{\delta F_{i\bar{k}}}{\delta \bar{z}^j}. \tag{4.2}$$

From (3.6) the condition (4.2) is equivalent to

$$F_j{}^i{}_k = F_k{}^i{}_j. \tag{4.3}$$

The condition (4.2) or (4.3) is similar to the case of Kähler geometry. So we define as follows.

**Definition 4.2.** A complex Finsler manifold $(M, F)$ is called a *Finsler-Kähler manifold* if its Finsler-Kähler form $\Theta$ is $h$-closed: $d^h\Theta = 0$.

As we noted in § 3, for an arbitrary point $P_0$ of $T'M$, we can always choose the semi-normal coordinate system at $P_0$. Thus, if $(M, F)$ is a Finsler-Kähler manifold, we can choose a coordinate system satisfying

$$F_j{}^i{}_k(P_0) = 0.$$

Similarly to the case of Hermitian geometry, a complex coordinate system satisfying

$$(1) \quad F_{i\bar{j}}(P_0) = \delta_{ij},$$

$$(2) \quad d^h F_{i\bar{j}}(P_0) = 0$$

is called the *normal coordinate system* at $P_0$. Hence we have the following theorem, which shows that our definition of Finsler-Kähler manifold is reasonable (cf. Kobayashi-Wu [9]).

**Theorem 4.1.** *A complex Finsler manifold $(M, F)$ is a Finsler-Kähler manifold if and only if for every point $P$ of $T'M$ there exists a complex coordinate system which is normal at $P$.*

## 5. Complex Berwald Manifolds

In the present section, we shall consider the analogy of the notion of Berwald space (or affinely connected space) in real Finsler geometry.

Let $(M, F)$ be a complex Finsler manifold, and $(D, N)$ the Finsler-Hermitian (or complex Rund) connection on $(M, F)$. Then we define as follows (cf. Matsumoto [11]).

**Definition 5.1.** If the coefficients $F_j{}^i{}_k$ of $(D, N)$ given by (3.6) are functions of position $(z^i)$ alone, the Finsler manifold $(M, F)$ is said to be *complex Berwald.*

The following is obvious from the definition.

**Proposition 5.1.** *A complex Finsler manifold $(M, F)$ is complex Berwald if and only if the pull-back $\pi_T{}^*\underline{\omega}$ of a connection $\underline{\omega}$ in $T'M$ gives the complex Rund connection on $(M, F)$.*

As is well-known, a connected real Berwald space is a Finsler space modeled on a Minkowski space (cf. Ichijyō [5]). We shall consider the complex analogy of real case. First we note that each fibre $T'M_P$ of $T'M$ is considered as a complex Minkowski space (cf. Ichijyō [6]). For any $\xi_P \in T'M_P$, the norm $\|\xi_P\|$ is defined by

$$\| \xi_P \|^2 = F(P, \xi_P).$$

We assume that a connected complex Finsler manifold $(M, F)$ is complex Berwald. Then, by Proposition 5.1, the complex Rund connection $R\Gamma$ is given by

$$\omega_j{}^i = \Gamma_j{}^i{}_k(z)\, dz^k, \ \Gamma_j{}^i{}_k(z) = F^{\bar{r}i} \frac{\delta F_{j\bar{r}}}{\delta z^k}, \ N^i{}_j = \eta^m \Gamma_m{}^i{}_j$$

for a connection $\underline{\omega}_j{}^i = \Gamma_j{}^i{}_k(z)\, dz^k$ in $T'M$.

Now let $c: [0, 1] \to M$ be a $C^\infty$-curve in $M$, and $dc/dt$ its tangent vector. Let $\xi(t) = \xi^i(t) \cdot X_i$ be a $C^\infty$-section of $T'M$. Then the covariant derivative $D\xi/dt$ of $\xi$ with respect to $\underline{\omega}$ is given by

$$\frac{D\xi}{dt} = \left( \frac{d\xi^i}{dt} + \underline{\omega}_j{}^i \left( \frac{dc}{dt} \right) \xi^j(t) \right) X_i(c(t)).$$

If the condition $D\xi/dt = 0$ is satisfied, then $\xi(t)$ is said to be *pararell* along $c(t)$. For each $\xi_P \in T'M_P$, there exists a unique $C^\infty$-section $\xi(t)$ satisfying

$$\xi(0) = \xi_P, \qquad \frac{D\xi}{dt} = 0.$$

For this $\xi(t)$, we define a mapping $P_{c(t)}: T'M_P \to T'M_{c(t)}$ by

$$P_{c(t)}(\xi_P) = \xi(t).$$

Obviously, the mapping $P_{c(t)}$ is a complex linear isomorphism. Because of $\| \xi(t) \| = L(c(t))$, we have

$$\begin{aligned} \frac{d \| \xi(t) \|}{dt} &= \left( \frac{\partial L}{\partial z^k} - \Gamma_j{}^i{}_k \frac{\partial L}{\partial \eta^i} \right) \frac{dz^k}{dt} + (\text{conj.}) \\ &= \left( \frac{\delta L}{\delta z^k} \right) (c(t), \xi(t)) + (\text{conj.}), \end{aligned}$$

where we put $F(z, \eta) = L(z, \eta)^2$. Because of $\delta L/\delta z^k = 0$, we see that $d \| \xi(t) \| / dt = 0$. Hence, $\| \xi(t) \|$ is constant along $c(t)$, that is, the complex linear isomorphism $P_{c(t)}$ is isometry. Therefore each complex Minkowski space $T'M_P$ with the norm function $L(P, \ )$ is isometric each other.

**Theorem 5.1.** *Assume that a connected complex Finsler manifold* $(M, F)$ *be complex Berwald. Then* $(M, F)$ *is a manifold modeled on a complex Minkowski space.*

In the following, we shall give an example of complex Berwald manifolds. Let $M$ be a complex manifold with a Hermitian metric $ds^2 = a_{i\bar{j}}(z)\, dz^i \otimes d\bar{z}^j$. Let $b = b_i(z)\, dz^i$

be a (1, 0)-form on $M$. Then we put

$$\alpha(z, \eta) = \{a_{i\bar{j}}(z)\eta^i\bar{\eta}^j\}^{1/2}, \qquad \beta(z, \eta) = b_i(z)\eta^i$$

for each $(z, \eta) \in T'M$. If a function $F(\alpha, \beta)$ of $\alpha$ and $\beta$ satisfies the conditions in Definition 3.1 as a function of $(z, \eta)$, then $F(\alpha, \beta)$ is called a *complex* $(\alpha, \beta)$*-metric.*

Let $F(\alpha, \beta)$ be an $(\alpha, \beta)$-metric on a complex manifold $M$. For the Hermitian connection $\underline{\omega}_j{}^i = \Gamma_j{}^i{}_k(z)\,dz^k$ of $a_{i\bar{j}}$, we put

$$N^i{}_j(z, \eta) = \eta^m \Gamma_m{}^i{}_j(z).$$

Furthermore, we define a complex Finsler connection $D$ in $(\widetilde{TM}, g)$ by the pull-back of $\underline{\omega}$. In the following, we shall investigate the condition that $(D, N)$ gives the complex Rund connection on $(M, F)$. First we have

$$\begin{aligned} D^hF(\alpha, \beta) &= \frac{\partial F}{\partial \alpha} D^h\alpha + \frac{\partial F}{\partial \beta} D^h\beta \\ &= \frac{\partial F}{\partial \alpha} \frac{\delta \alpha}{\delta z^k} + \frac{\partial F}{\partial \beta} \frac{\delta \beta}{\delta z^k} dz^k + (\text{conj.}). \end{aligned}$$

Since we have

$$\frac{\delta \alpha^2}{\delta z^k} = \left(\frac{\partial a_{i\bar{j}}}{\partial z^k} - a_{r\bar{j}}\Gamma_i{}^r{}_k\right)\eta^i\bar{\eta}^j = 0,$$

it follows that $\delta\alpha/\delta z^k = 0$ and $\delta\alpha/\delta\bar{z}^k = 0$. We have also

$$\frac{\delta \beta}{\delta z^k} dz^k + \frac{\delta \beta}{\delta \bar{z}^k} d\bar{z}^k = \left(\frac{\partial b_i}{\partial z^k} - b_r\Gamma_i{}^r{}_k\right)\eta^i dz^k + \left(\frac{\partial b_i}{\partial \bar{z}^k}\right)\eta^i d\bar{z}^k.$$

Thus, if the (1, 0)-form $b_i dz^i$ is holomorphic and pararell with respect to $\underline{\omega}$, then the connection $(D, N)$ satisfies $D^hF = 0$. In this case, the condition $D^hF = 0$ means that the Hermitian metric $g$ in $\widetilde{TM}$ is $h$-metrical. So the connection is the complex Rund connection on $(M, F)$.

**Theorem 5.2.** *Let $M$ be a complex manifold with a complex $(\alpha, \beta)$-metric $F(\alpha, \beta)$. Then, if the* (1, 0)*-form $b = b_i(z)\,dz^i$ is holomorphic and pararell with respect to the Hermitian connection $\Gamma_j{}^i{}_k(z)$, then $(M, F(\alpha, \beta))$ is complex Berwald.*

By (4.3), we have

**Theorem 5.3.** *Let $(M, F(\alpha, \beta))$ be a complex manifold with a complex $(\alpha, \beta)$-metric $F(\alpha, \beta)$ satisfying the assumption in Theorem 5.2. Then, $(M, F(\alpha, \beta))$ is Finsler-Kähler if and only if the Hermnitian metric $a_{i\bar{j}}$ is a Kähler metric on $M$.*

## 6. Complex locally Minkowski manifolds

Let $(M, F)$ be a complex Finsler manifold, and the Hermitian metric in $\widetilde{TM}$ defined by (3.2). Similarly to the case of real Finsler geometry, we define as follows (cf. Matsumoto [11])

**Definition 6.1.** A complex Finsler manifold $(M, F)$ is said to be *complex locally Minkowski*, if there exists an open cover $\{U, X_U\}$ such that on each $\pi_T{}^{-1}(U)$ the function $F$ is a function of the fibre-coordinate only. Such an open cover $\{U, X_U\}$ is said to be *adapted.*

First we show the following:

**Proposition 6.1.** *A complex Finsler manifold* $(M, F)$ *is complex locally Minkowski if and only if there exists an open cover* $\{U, X_U\}$ *such that on each* $\pi_T{}^{-1}(U)$ *the condition* $d^h F_{i\bar{j}} = 0$ *is satisfied.*

*Proof.* If the condition $d^h F_{i\bar{j}} = 0$ is satisfied on each $\pi_T{}^{-1}(U)$, then we have $F_j{}^i{}_k = 0$ from (3.6), and so $N^i{}_j = 0$. Consequently we have $\partial F_{i\bar{j}}/\partial z^k = 0$, that is, $F = F(\eta)$ on each $\pi_T{}^{-1}(U)$.

Conversely, if $F = F(\eta)$ on each $\pi_T{}^{-1}(U)$, then we see that $N^i{}_j = 0$ from (3.8), so we have $d^h F_{i\bar{j}} = 0$. Q.E.D.

From Proposition 6.1, we see that if a complex Finsler manifold $(M, F)$ is complex locally Minkowski, then we have $\omega_j{}^i = 0$ on each $\pi_T{}^{-1}(U)$ with respect to $R\Gamma$ on $(M, F)$, that is, we get

$$DX_U = 0.$$

Hence the complex Rund connection $R\Gamma$ on $(M, F)$ is flat.

Conversely, we assume that the complex Rund connection $R\Gamma$ on $(M, F)$ is flat. Then, from Proposition 3.4, we see that it defines a flat holomorphic connection in $T'M$. So the holomorphic tangent bundle $T'M$ admits a flat structure (cf. Goldberg [2], Kobayashi [8]). If we denote by $\{U, X_U\}$ the flat structure in $T'M$ and denote by $(z^i, \eta^i)$ the local coordinate system on $T'M$ induced from $\{U, X_U\}$, then we have $F_j{}^i{}_k = 0$. Consequently we have $\partial F_{i\bar{j}}/\partial z^k = 0$. Thus $(M, F)$ is complex locally Minkowski.

**Theorem 6.1.** *A complex Finsler manifold* $(M, F)$ *is complex locally Minkowski if and only if the complex Rund connection* $R\Gamma$ *on* $(M, F)$ *is of zero-curvature, that is,* $R\Gamma$ *is flat.*

From (3.11), the curvature form $\Omega_j{}^i$ of $R\Gamma$ is written as

$$\Omega_j{}^i = \frac{\delta F_l{}^i{}_j}{\delta \bar{z}^k} d\bar{z}^k \wedge dz^l + \frac{\partial F_l{}^i{}_j}{\partial \eta^k} \delta\eta^k \wedge dz^l + \frac{\partial F_l{}^i{}_j}{\partial \bar{\eta}^k} \delta\bar{\eta}^k \wedge dz^l.$$

Hence the following relation with complex Berwald manifolds is easily derived from this expression.

**Theorem 6.2.** *A complex Finsler manifold* $(M, F)$ *is complex locally Minkowski if and only if it is complex Berwald and the complex Rund connection* $R\Gamma$ *on* $(M, F)$ *is holomorphic.*

From Theorem 6.1, we have the following result which is similar to the one due to Hashiguchi-Ichijyō [3].

**Theorem 6.3.** *Let* $(M, F(\alpha, \beta))$ *be a complex Finsler manifold which satisfies the assumption in Theorem* 5.2. *If the Hermitian metric* $a_{i\bar{j}}$ *is flat, then* $(M, F)$ *is complex locally Minkowski.*

As we stated in the above, if a complex Finsler manifold $(M, F)$ is complex locally Minkowski, then $T'M$ admits a flat structure. Furthermore, if $M$ is simply connected, then $M$ is a complex parallelizable manifold (cf. Goldberg [2]).

**Corollary 6.1.** *If a simply connected complex manifold* $M$ *admits a complex Finsler metric which is locally Minkowski, then* $M$ *is a complex parallelizable manifold.*

## 7. Holomorphic sectional curvature

In the last section of the present paper, we shall give a note about the holomorphic sectional curvatures defined in Kobayashi [7] and Royden [14].

Let $(M, F)$, $F(z, \eta) = L(z, \eta)^2$, be a complex Finsler manifold of $\dim_{\mathbf{C}} M = n$. Let $(D, N)$ be the Finsler-Hermitian (or complex Rund) connection on $(M, F)$. The curvature form $\Omega$ of $D$ is given by (3.4) (or (3.11)). Then, according to Kobayashi [7], the following Hermitian 2-form $\Psi$ is also called the *curvature form* of $D$:

$$\Psi = \frac{1}{F(z, \eta)}\left(F^{\bar{m}i}\frac{\partial F_{t\bar{m}}}{\partial z^k}\frac{\partial F_{i\bar{s}}}{\partial \bar{z}^l} - \frac{\partial^2 F_{t\bar{s}}}{\partial z^k \partial \bar{z}^l}\right)\eta^t\bar{\eta}^s \cdot dz^k d\bar{z}^l. \tag{7.1}$$

With respect to the non-linear connection $N^i{}_j$ given by (3.8), we have

**Proposition 7.1.** *The curvature form* $\Psi$ *defined by* (7.1) *is expressed as follows*:

$$\Psi = \frac{1}{F(z, \eta)}\left(F_{i\bar{j}}N^i{}_k\bar{N}^j{}_l - \frac{\partial^2 F}{\partial z^k \partial \bar{z}^l}\right)dz^k d\bar{z}^l. \tag{7.2}$$

For an arbitrary $(z, \xi) \in T'M$ satisfying $F(z, \xi) = 1$, the holomorphic sectional curvature $H(z, \xi)$ of $(M, F)$ at $(z, \xi)$ is defined by

$$H(z, \xi) = 2\Psi(\xi, \xi), \tag{7.3}$$

where $\xi = \xi^m \partial/\partial z^m \in \widetilde{TM}_{(z,\xi)}$. Because of (7.1) and (7.2), we have

$$
(7.4)\qquad
\begin{aligned}
H(z, \xi) &= 2\left(F^{\bar{m}i}\frac{\partial F_{t\bar{m}}}{\partial z^k}\frac{\partial F_{i\bar{s}}}{\partial \bar{z}^l} - \frac{\partial^2 F_{t\bar{s}}}{\partial z^k \partial \bar{z}^l}\right)\xi^t \bar{\xi}^s \xi^k \bar{\xi}^l \\
&= 2\left(F_{i\bar{j}} N^i{}_k \bar{N}^j{}_l - \frac{\partial^2 F}{\partial z^k \partial \bar{z}^l}\right)\xi^k \bar{\xi}^l.
\end{aligned}
$$

**Remark 7.1.** From the expression (7.4), we see that if $F_{i\bar{j}}$ is a Hermitian metric $F_{i\bar{j}}(z)$ on $M$, then $H(z, \xi)$ defined by (7.3) is just the holomorphic sectional curvature defined by $\xi \in TM_z$ in the usual sence (cf. Wu [16]).

Next we shall investigate the holomorphic sectional curvature of $(M, F)$ defined in Royden [14]. Let $(z, \xi)$ be an arbitrary point of $T'M$. Then, for a sufficiently small positive $r$, there exists a holomorphic mapping $\varphi$ from the disk $\triangle(r)$ of radius $r$ to $M$ satisfying

$$
(7.5)\qquad \varphi(0) = z, \qquad \varphi_*(0) := \varphi_*\left(\left(\frac{\partial}{\partial \zeta}\right)_0\right) = \xi,
$$

where we denote by $\varphi_*$ the differential of $\varphi$ and by $\zeta$ the coordinate system on $\triangle(r)$. Then, for the given convex Finsler metric $F(z, \eta) = L(z, \eta)^2$, a Hermitian metric $\varphi^*L$ on $\triangle(r)$ is introduced by

$$
\varphi^*L = L(\varphi(\zeta), \varphi_*(\zeta))\,|d\zeta|.
$$

The Gauss curavture $K_{(z,\xi)}(\varphi^*L)$ of $\varphi^*L$ at $(z, \xi)$ is defined by

$$
(7.6)\qquad K_{(z,\xi)}(\varphi^*L) = -\left\{\frac{2}{(\varphi^*L)^2}\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}\log(\varphi^*L)^2\right\}_{\zeta=0}.
$$

Then, according to Royden [14], the holomorphic sectional curvature $K_{(z,\xi)}(L)$ of $(M, F)$ at $(z, \xi)$ is defined by

$$
(7.7)\qquad K_{(z,\xi)}(L) = \sup\{K_{(z,\xi)}(\varphi^*L)\},
$$

where the supremum is taken for all holomorphic mappings $\varphi$ satisfying (7.5). In the following, for the simplicity, we put $(\varphi^*L)^2 = G$.

We may always choose a coordinate system on $\triangle(r)$ satisfying $(\partial G/\partial\zeta)_{\zeta=0} = 0$ and $(\partial G/\partial\bar{\zeta})_{\zeta=0} = 0$. Hence in such a coordinate system on $\triangle(r)$, $K_{(z,\xi)}(L)$ can be written as follows:

$$
K_{(z,\xi)}(L) = -2\inf\left(\frac{\partial^2 G}{\partial\zeta\partial\bar{\zeta}}\right)_{\zeta=0}.
$$

By direct calculations, we have

$$
\left(\frac{\partial^2 G}{\partial\zeta\partial\bar{\zeta}}\right)_{\zeta=0} = \xi^i\bar{\xi}^j\frac{\partial^2 G}{\partial z^i\partial\bar{z}^j} + F_{i\bar{j}}\mu^i\,\overline{(N^j{}_m\xi^m)} + F_{i\bar{j}}(N^i{}_m\xi^m)\bar{\mu}^j + F_{i\bar{j}}\mu^i\bar{\mu}^j,
$$

where we put $\mu^i = (\partial^2\varphi^i/\partial\zeta\partial\bar{\zeta})_{\zeta=0}$ for $\varphi = (\varphi^i,\dots,\varphi^n)$. Because of (7.4) we have

$$H(z, \xi) + 2\left(\frac{\partial^2 G}{\partial\zeta\partial\bar{\zeta}}\right)_{\zeta=0} = 2F_{i\bar{j}}(\mu^i + N^i{}_m\xi^m)\,\overline{(\mu^j + N^j{}_m\xi^m)}\,\cdot$$

Hence we have $H(z, \xi) \geqq -2(\partial^2 G/\partial\zeta\partial\bar{\zeta})_{\zeta=0}$ from the convexity of $(F_{i\bar{j}})$. Furthermore we have

**Proposition 7.2.** *The respective holomorphic sectional curvatures* $H(z, \xi)$ *and* $K_{(z,\xi)}(L)$ *of* (7.3) *and* (7.7) *coincide*:

$$H(z, \xi) = K_{(z,\xi)}(L). \tag{7.8}$$

In fact, when $\varphi$ is a complex line in the semi-normal coordinte at $(z, \xi)$, then $\sup\{K_{(z,\xi)}(\varphi^*L)\}$ attains to the maximum $H(z, \xi)$.

On the other hand, by Theorem 6.2 of Kobayashi [7], if the holomorphic sectional curvature $H(z, \xi)$ is bounded above by a negative constant $-k$ $(k > 0)$, then the manifold $M$ is hyperbolic. By Proposition 7.2, if $H(z, \xi) \leqq -k$, then we also have $K_{(z,\xi)}(L) \leqq -k$, that is, $K(\varphi^*L) \leqq -k$ for an arbitrary holomorphic mapping $\varphi\colon \triangle(r) \to M$. Then, in similarly to Theorem 3.1, IV, §3 in Lang [10], we get

**Proposition 7.3.** *Let* $(M, F)$, $F = L^2$, *be a complex Finsler manifold whose holomorphic sectional curvature* $H(z, \xi)$ *at* $(z, \xi)$ *is bounded above by a negative constant* $-k$. *Then for an arbitrary holomorphic map* $\varphi\colon \triangle(r) \to M$ *satisfying* (7.5), *we have*

$$g_r \geqq \sqrt{k}\,\varphi^*L,$$

*where* $g_r$ *is the Poincarè metric on the disk* $\triangle(r)$:

$$g_r = \frac{2r}{r^2 - |\zeta|^2}|d\zeta|.$$

Thus we have

**Theorem 7.1.** (Kobayashi [7]) *Let* $(M, F)$, $F = L^2$, *be a complex Finsler manifold. Assume that there exists a negative constant* $-k$ *such that* $H(z, \xi) \leqq -k$. *Then* $M$ *is hyperbolic, and we have*

$$F_M(z, \eta) \geqq \sqrt{k}\,L(z, \eta), \tag{7.9}$$

*where* $F_M$ *is the Kobayashi metric on* $M$:

$$F_M(z, \xi) = \inf\left\{\frac{1}{r};\ \varphi\colon \triangle(r) \longrightarrow M \text{ is a holomorphic map satisfying } \varphi(0) = z,\ \varphi_*(0) = \xi\right\}.$$

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