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Shin-ichi NISHIMURA* and Masao HASHIGUCHI**

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Abstract

Let a hypersurface S in an euclidean space R^n be implicitly defined by a differentiable function f in R^n. Then the Gaussian curvature of S is expressed in terms of f itself (cf. [6, Chap. 12]). As an application of this result, in the present paper we discuss the Gaussian curvature of the indicatrix of a Lagrange space $(R^n, \mathscr{L})$.

1. Introduction

In an euclidean xy-plane R^2, let a curve C be implicitly defined by a differentiable function f in R^2 as $f(x, y) = 0$. We put $f_1 = \partial f/\partial x$, $f_2 = \partial f/\partial y$. Around a point $\mathrm{P} \in C$ such that $f_2(\mathrm{P}) \neq 0$ the curve C is graphically expressed by a differentiable function g as $y = g(x)$. Then the curvature κ of C is given by $\kappa = y''/(1 + y'^2)^{3/2}$. If we directly calculate from

$$f_2 y' = -f_1, \qquad f_2^3 y'' = -(f_{11}f_2^2 - 2f_{12}f_1f_2 + f_{22}f_1^2),$$

where $f_{11} = \partial^2 f/\partial x^2$, $f_{12} = f_{21} = \partial^2 f/\partial x \partial y$, $f_{22} = \partial^2 f/\partial y^2$, we have

$$\kappa = \varepsilon \begin{vmatrix} f_{11} & f_{12} & f_1 \\ f_{21} & f_{22} & f_2 \\ f_1 & f_2 & 0 \end{vmatrix} /(f_1^2 + f_2^2)^{3/2} \qquad (\varepsilon = \operatorname{sign} f_2). \tag{1.1}$$

In an euclidean xyz-space R^3, let a surface S be implicitly defined by a differentiable function f in R^3 as $f(x, y, z) = 0$. We put f_i, f_{ij} similarly. Around a point $\mathrm{P} \in S$ such that $f_3(\mathrm{P}) \neq 0$ the surface S is graphically expressed by a differentiable function g as $z = g(x, y)$, and the Gaussian curvature K of S is given by $K = (rt - s^2)/(1 + p^2 + q^2)^2$, where $p = \partial g/\partial x$, $q = \partial g/\partial y$, $r = \partial^2 g/\partial x^2$, $s = \partial^2 g/\partial x \partial y$, $t = \partial^2 g/\partial y^2$. If we directly calculate from

* Kuma Technical High School, Kumamoto, Japan.

** Department of Mathematics, Faculty of Science, Kagoshima University, Kagoshima, Japan.

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$$f_3 p = -f_1,\ f_3 q = -f_2,$$
$$f_3^3 r = -f_{11} f_3^2 + 2 f_{13} f_1 f_3 - f_{33} f_1^2,$$
$$f_3^3 s = -f_{12} f_3^2 + f_{13} f_2 f_3 + f_{23} f_1 f_3 - f_{33} f_1 f_2,$$
$$f_3^3 t = -f_{22} f_3^2 + 2 f_{23} f_2 f_3 - f_{33} f_2^2,$$

we have

$$K = -\begin{vmatrix} f_{11} & f_{12} & f_{13} & f_1 \\ f_{21} & f_{22} & f_{23} & f_2 \\ f_{31} & f_{32} & f_{33} & f_3 \\ f_1 & f_2 & f_3 & 0 \end{vmatrix} \Big/ (f_1^2 + f_2^2 + f_3^2)^2. \tag{1.2}$$

Especially, in the case where a treated function f is a quadratic polynomial of the coordinates:

$$2f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c, \tag{1.3}$$

$$2f(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2px + 2qy + 2rz + d, \tag{1.4}$$

the formulas (1.1) and (1.2) are reduced to

$$\kappa = \varepsilon \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \Big/ (f_1^2 + f_2^2)^{3/2} \qquad (\varepsilon = \text{sign } f_2), \tag{1.5}$$

where $f_1 = ax + hy + g$, $f_2 = hx + by + f$, and

$$K = -\begin{vmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & d \end{vmatrix} \Big/ (f_1^2 + f_2^2 + f_3^2)^2, \tag{1.6}$$

where $f_1 = ax + hy + gz + p$, $f_2 = hx + by + fz + q$, $f_3 = gx + fy + cz + r$, respectively. It is noted that in these formulas the determinants appeared as the numerators are well-known constants independent on rectangular coordinate systems and the values of κ and K depend only on the magnitude of the gradient of f reciprocally.

Generally, in an n-dimensional euclidean space R^n we shall consider a hypersurface S defined by a differentiable function f in R^n as

$$S = \{x \in R^n \mid f(x) = 0,\ (\nabla f)\ (x) \neq 0\}, \tag{1.7}$$

where $x = (x_1, \dots, x_n)$ is a rectangular coordinate system of R^n, and ∇f denotes the gradient of f:

$$\nabla f = {}^t(f_1, \dots, f_n) \qquad (f_i = \partial_i f). \tag{1.8}$$

Throughout the present paper, we put $\partial_i = \partial/\partial x_i$ and denote a vector with components $v_1, \dots, v_n$ by an $n \times 1$ matrix ${}^t(v_1, \dots, v_n)$, but we use also an abridged notation (v_i). A letter ${}^tA$ denotes the transpose of a matrix A. The inner product $\sum_i u_i v_i$ of vectors $\boldsymbol{u} = (u_i)$ and $\boldsymbol{v} = (v_i)$ is denoted by $\boldsymbol{u} \cdot \boldsymbol{v}$, and the length $(\sum_i v_i^2)^{1/2}$ of a vector $\boldsymbol{v} = (v_i)$ by $|\boldsymbol{v}|$. The summation convention is not used.

Now, the notion of Gaussian curvature is generally defined for a hypersurface S in R^n, and in the case where S is implicitly given by (1.7) we can get the same expression as (1.1) and (1.2) (Theorem 2.1). This expression is derived, for example, from Theorem 5 of Thorpe [6, Chap. 12, p 89], but for convenience we shall give a self-contained proof in §2, based on Lemma 2.1 concerning with the determinant of a linear transformation of a hypersubspace of a vector space R^n.

The purpose of the present paper is to apply this result to Finsler geometry. We denote by $y = (y_1, \dots, y_n)$ the canonical coordinate system of the tangent space R^n_x at each point $x \in R^n$, and put $\dot{\partial}_i = \partial/\partial y_i$. Let $(R^n, \mathscr{L})$ be a Lagrange space, where $\mathscr{L}$ is a positive-valued differentiable function in the tangent bundle of R^n and satisfies the regularity condition $\det(\dot{\partial}_i \dot{\partial}_j \mathscr{L}) \neq 0$ (cf. [4, p 11], [1, p 1]).

Each tangent space R^n_x is also regarded as an n-dimensional euclidean space with the rectangular coordinate system y. A hypersurface $I_x = \{y \in R^n_x \mid \mathscr{L}(x, y) = 1\}$ in R^n_x is called the *indicatrix* at x. In §3 we shall express the Gaussian curvature of I_x in terms of $\mathscr{L}$ (Theorem 3.1).

A Lagrange space $(R^n, \mathscr{L})$ becomes a Finsler space (R^n, L) if $\mathscr{L}$ is given by $\mathscr{L} = L^2$, where L is positively homogeneous of degree 1: $L(x, \lambda y) = \lambda L(x, y)$ for $\lambda > 0$. Then Theorem 3.1 is reduced to Thorem 3.2. Given a hypersurface S_x in each tangent space R^n_x a priori, by the well-known method (cf. [3, p 105]) we have a Finsler space whose indicatrix I_x is the given S_x. Thus the Gaussian curvature of S_x is expressed in terms of a Finsler geometry. This fact seems interesting from the standpoint of application.

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2. The Gaussian curvature of a hypersurface

We shall recall here an elementary definition of the Gaussian curvature K of a surface S in an euclidean space R^3. Let S be expressed by parameters u_1, u_2 as $x = x(u_1, u_2)$, where $x = (x_1, x_2, x_3)$ is a rectangular coordinate system of R^3. At each point $\mathrm{P} \in S$, two tangent vector fields $X_\alpha = \partial x/\partial u_\alpha$ $(\alpha = 1, 2)$ constitute a basis of the tangent plane S_P, and the unit vector field $N = (X_1 \times X_2)/|X_1 \times X_2|$ is orthogonal to S_P. Then paying attention to the Weingarten equation

$$N_\beta = -\sum_\alpha h^\alpha_\beta X_\alpha \qquad (N_\beta = \partial N/\partial u_\beta), \tag{2.1}$$

a linear transformation T of S_P is defined by

$$T\colon S_P \longrightarrow S_P \,|\, \boldsymbol{v} = \sum_\beta v_\beta X_\beta \longrightarrow T(\boldsymbol{v}) = -\sum_\beta v_\beta N_\beta. \tag{2.2}$$

Since T is represented by the matrix (h^α_β) with respect to the basis X_1, X_2, the determinant of T gives the Gaussian curvature K of S at P.

It is noted that the vector $\sum_\beta v_\beta N_\beta$ in (2.2) is a derivative $\nabla_{\boldsymbol{v}} N$ of N with respect to $\boldsymbol{v}$. Generally, let Ω be a differentiable geometrical object defined on an open set U of an n-dimensional euclidean space R^n, such as a function and a vector field, and let $\boldsymbol{v} = (v_i)$ be a vector at a point $P \in U$. The *derivative* $\nabla_{\boldsymbol{v}}\Omega$ *of* Ω *with respect to* $\boldsymbol{v}$ is defined by

$$\nabla_{\boldsymbol{v}}\Omega = (\Omega \circ c)'\,(t_0), \tag{2.3}$$

where $x = c(t)$ is any differentiable curve such that $c(t_0) = P$, $c'(t_0) = \boldsymbol{v}$ The derivative $\nabla_{\boldsymbol{v}}\Omega$ is independent on the choice of a curve c, and is expressed by

$$\nabla_{\boldsymbol{v}}\Omega = \sum_i (\partial_i \Omega) v_i. \tag{2.4}$$

Now, let (S, N) be an oriented hypersurface in R^n, where N is a unit vector field orthogonal to S. Let S_P be the tangent space of a point $P \in S$. The notion of a derivative of Ω with respect to $\boldsymbol{v} \in S_P$ is also defined in the case where Ω is defined only on S. Since $\nabla_{\boldsymbol{v}} N \in S_P$ for $\boldsymbol{v} \in S_P$, we have a linear transformation T of S_P defined by

$$T\colon S_P \longrightarrow S_P \,|\, \boldsymbol{v} \longrightarrow T(\boldsymbol{v}) = -\nabla_{\boldsymbol{v}} N. \tag{2.5}$$

This is called the *Weingarten map* of (S, N) at P. The *Gaussian curvature* K of (S, N) at P is defined by the determinant of T.

Remark 2.1. In the case of $n = 3$, this definition of the Gaussian curvature coincides with the elementary definition stated above, independent on the choice of N.

In the case of $n = 2$, the Gaussian curvature K of a parameterized curve C is a cuvature κ of C, if we take N to be the normal vector of C. If N is replaced by $-N$, we have $K = -\kappa$. Since the Weingarten map T is represented by an $(n-1) \times (n-1)$ matrix, if n is odd then K is independent on the choice of N, whereas if n is even then K changes the sign by turning the direction of N.

In the case where a hypersurface S in R^n is given by (1.7), it is noted that the gradient ∇f of a treated function f is orthogonal to S at each point $P \in S$. We have the following expression for the Gaussian curvature K of an oriented hypersurface (S, N).

Theorem 2.1. *Let (S, N) be an oriented hypersurface in R^n, where S is given by (1.7) using a differentiable function f in R^n, and N is a unit vector field orthogonal to S given by*

$$N = \varepsilon \nabla f / |\nabla f| \qquad (\varepsilon = \pm 1). \tag{2.6}$$

Then the Gaussian curvature K of (S, N) is given by

$$K = -\tau \begin{vmatrix} f_{ij} & f_i \\ f_j & 0 \end{vmatrix} / |\nabla f|^{n+1}, \tag{2.7}$$

where $f_i = \partial_i f$, $f_{ij} = \partial_i \partial_j f$, $\nabla f = (f_i)$, and $\tau = (-\varepsilon)^{n+1}$.

Remark 2.2. In the case where n is odd, we have $\tau = 1$. In the case where n is even, we have $\tau = -\varepsilon$. If we choose an orientation N of S by

$$N = -\nabla f / |\nabla f|, \tag{2.8}$$

we have always $\tau = 1$. Even in the case where an orientation N is given a priori, we can take f to be $\tau = 1$, because f and $-f$ give the same S.

For the proof of Theorem 2.1 we shall show that the Weingarten map T of (S, N) at $P \in S$ satisfies the following formula for any $\boldsymbol{u} = (u_i)$, $\boldsymbol{v} = (v_i) \in S_P$:

$$\boldsymbol{u} \cdot T(\boldsymbol{v}) = -(\varepsilon / |\nabla f|) \sum_{i,j} f_{ij} u_i v_j. \tag{2.9}$$

Since $\nabla f = \varepsilon |\nabla f| \, N$ from (2.6), we have for any $\boldsymbol{v} = (v_i) \in S_P$

$$\nabla_{\boldsymbol{v}} \nabla f = \varepsilon \nabla_{\boldsymbol{v}}(|\nabla f|) N + \varepsilon |\nabla f| \nabla_{\boldsymbol{v}} N.$$

Thus from (2.5) we have for any $\boldsymbol{u} = (u_i) \in S_P$

$$\boldsymbol{u} \cdot (\nabla_{\boldsymbol{v}} \nabla f) = -\varepsilon |\nabla f| \boldsymbol{u} \cdot T(\boldsymbol{v}). \tag{2.10}$$

Since the vector field $\nabla f = (f_i)$ is defined on some open set containing S, from (2.4) we have $\nabla_{\boldsymbol{v}} \nabla f = \sum_j (\partial_j \nabla f) v_j$, so we have $\boldsymbol{u} \cdot (\nabla_{\boldsymbol{v}} \nabla f) = \sum_{i,j} f_{ij} u_i v_j$. Thus (2.9) is shown from (2.10).

The proof of Theorem 2.1 is obtained from the following lemma by putting $a_{ij} = -\varepsilon f_{ij} / |\nabla f|$, $n_i = \varepsilon f_i / |\nabla f|$.

Lemma 2.1. *Let V be an n-dimensional real vector space linearly isomorphic to a vector space R^n, and T a linear transformation of an $(n-1)$-dimensional vector subspace*

W of V. We denote any $\boldsymbol{v} \in V$ by $\boldsymbol{v} = (v_i)$ using the corresponding $(v_i) \in R^n$. Let $N = (n_i)$ be a unit vector orthogonal to W. If for any $\boldsymbol{u} = (u_i)$, $\boldsymbol{v} = (v_i) \in W$ the inner product $\boldsymbol{u} \cdot T(\boldsymbol{v})$ is expressed by a matrix $A = (a_{ij})$ as

$$\boldsymbol{u} \cdot T(\boldsymbol{v}) = {}^t\boldsymbol{u} A \boldsymbol{v} \; (= \sum_{i,j} a_{ij} u_i v_j), \tag{2.11}$$

then the determinant K of T is given by

$$K = -\begin{vmatrix} A & N \\ {}^tN & 0 \end{vmatrix} \left(= -\begin{vmatrix} a_{ij} & n_i \\ n_j & 0 \end{vmatrix} \right). \tag{2.12}$$

Proof. In the proof, Greek indices take the values $1, \dots, n-1$. We choose a basis $X_1, \dots, X_{n-1}$ of W, and represent T by an $(n-1) \times (n-1)$ matrix $B = (b_{\alpha\beta})$, where

$$T(X_\beta) = \sum_\alpha b_{\alpha\beta} X_\alpha. \tag{2.13}$$

Then the determinant K of T is obtained as $K = \det B$.

We define $n \times n$ matrices $\tilde{B}$, X, Y by

$$\tilde{B} = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}, \quad X = (X_1, \dots, X_{n-1}, N), \quad Y = (Y_1, \dots, Y_{n-1}, N),$$

where $Y_\beta = T(X_\beta)$, and $(n+1) \times (n+1)$ matrices $\tilde{A}$, $\tilde{X}$ by

$$\tilde{A} = \begin{pmatrix} A & N \\ {}^tN & 0 \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}.$$

Paying attention to (2.13) and $X_\alpha \cdot N = 0$, $N \cdot N = 1$, we have

$${}^tX(X\tilde{B}) = {}^tXY = \begin{pmatrix} X_\alpha \cdot Y_\beta & 0 \\ 0 & 1 \end{pmatrix},$$

from which we have $(\det X)^2 K = \det(X_\alpha \cdot Y_\beta)$.

In the same way, we have

$${}^t\tilde{X}\tilde{A}\tilde{X} = \begin{pmatrix} {}^tX_\alpha A X_\beta & {}^tX_\alpha A N & 0 \\ {}^tN A X_\beta & {}^tN A N & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

from which we have $(\det X)^2 (\det \tilde{A}) = -\det({}^tX_\alpha A X_\beta)$. Since the matrix X is regular, we have $K = -\det \tilde{A}$ from (2.11). Q.E.D.

In the special case where a treated function f is a quadratic polynomial of the coordinates, we have directly from (2.7)

Theorem 2.2. *Let (S, N) be an oriented hypersurface in R^n, where S is a regular quadratic surface defined by*

$$2f(x) = \sum_{i,j} a_{ij}x_i x_j + 2\sum_i b_i x_i + c = 0 \qquad (a_{ij} = a_{ji}), \tag{2.14}$$

and N is a unit vector field orthogonal to S given by (2.6). *Then the Gaussian curvature K of (S, N) is given by*

$$K = -\tau \begin{vmatrix} a_{ij} & b_i \\ b_j & c \end{vmatrix} / (\sum_i f_i^2)^{(n+1)/2}, \tag{2.15}$$

where $f_i = \sum_j a_{ij}x_j + b_i$ and $\tau = (-\varepsilon)^{n+1}$.

Remark 2.3. The expression (2.15) of the Gaussian curvature K of a hypersurface $\lambda f = 0$ is independent on a positive constant λ.

3. The indicatrix of a Lagrange space

Let $(R^n, \mathscr{L})$ be a Lagrange space. At each point $x \in R^n$ the indicatrix I_x is a hypersurface in the tangent space R^n_x, where R^n_x is thought to be an n-dimensional euclidean space with a rectangular coordinate system $y = (y_i)$.

We define a function f by

$$f(x, y) = \mathscr{L}(x, y) - 1, \tag{3.1}$$

and put $\dot{\nabla} f = (\dot{\partial}_i f)$, $\dot{\nabla}\mathscr{L} = (\dot{\partial}_i \mathscr{L})$. Since we have $\dot{\nabla} f = \dot{\nabla}\mathscr{L} \neq 0$ from the regularity of $\mathscr{L}$, the indicatrix I_x is expressed as

$$I_x = \{y \in R^n_x | f(x, y) = 0, (\dot{\nabla} f)\ (x, y) \neq 0\}. \tag{3.2}$$

At each $y \in I_x$ the vector field $\dot{\nabla}\mathscr{L}$ is orthogonal to I_x. Suggested by Remark 2.2, we shall assume that an orientation N_x of I_x is always

$$N = -\dot{\nabla}\mathscr{L} / |\dot{\nabla}\mathscr{L}|. \tag{3.3}$$

Then we have from Theorem 2.1

Theorem 3.1. *Let $(R^n, \mathscr{L})$ be a Lagrange space. At each point $x \in R^n$, the Gaussian curvature K of the indicatrix I_x oriented in the direction opposite to $\dot{\nabla}\mathscr{L} = (\dot{\partial}_i \mathscr{L})$ is given by*

$$K = -\begin{vmatrix} \dot{\partial}_i \dot{\partial}_j \mathscr{L} & \dot{\partial}_i \mathscr{L} \\ \dot{\partial}_j \mathscr{L} & 0 \end{vmatrix} / (\sum_i (\dot{\partial}_i \mathscr{L})^2)^{(n+1)/2}. \tag{3.4}$$

Now, let a Lagrange space $(R^n, \mathscr{L})$ be a Finsler space (R^n, L), where $\mathscr{L} = L^2$. Putting $l_i = \dot{\partial}_i L$, $g_{ij} = (\dot{\partial}_i \dot{\partial}_j L^2)/2$, and $g = \det(g_{ij})$, we have on the indicatrix, where $L(x, y) = 1$,

$$2^{-(n+1)} \begin{vmatrix} \dot{\partial}_i \dot{\partial}_j \mathscr{L} & \dot{\partial}_i \mathscr{L} \\ \dot{\partial}_j \mathscr{L} & 0 \end{vmatrix} = \begin{vmatrix} g_{ij} & l_i \\ l_j & 0 \end{vmatrix} = -g.$$

Thus Theorem 3.1 is reduced to

Theorem 3.2. *Let (R^n, L) be a Finsler space. At each point $x \in R^n$, the Gaussian curvature K of the indicatrix I_x oriented in the direction opposite to $\dot{\nabla} L = (l_i)$ is given by*

$$K = g/(\sum_i l_i^2)^{(n+1)/2}. \tag{3.5}$$

As an example we shall treat a Randers space $(R^n, \alpha + \beta)$, where α and β are a Riemannian metric and a non-vanishing 1-form in R^n respectively. We put

$$\alpha = (\sum_{i,j} a_{ij}(x) y_i y_j)^{1/2}, \quad \beta = \sum_i b_i(x) y_i. \tag{3.6}$$

Each indicatrix I_x of a Riemannian space (R^n, α) is a quadratic hypersurface of the coordinates y_i with the center $y = 0$:

$$2f(x, y) = \sum_{i,j} a_{ij} y_i y_j - 1 = 0, \tag{3.7}$$

whereas the indicatrix I_x of a Randers space is expressed as

$$2f(x, y) = \sum_{i,j} (a_{ij} - b_i b_j) y_i y_j + 2 \sum_i b_i y_i - 1 = 0. \tag{3.8}$$

Under the necessity of using a metric with non-central indicatrix, as the simplest possible asymmetrical modification of a Riemannian metric, Randers introduced a Finsler space with a metric $L = \alpha + \beta$, which is a unique positive-valued Finsler metric such that each indicatrix is a quadratic hypersurface of the coordinates y_i (cf. [5], [2, p 34]).

Now, from (3.8) we have $\dot{\partial}_i f = \sum_j (a_{ij} - b_i b_j) y_j + b_i$, which becomes $\dot{\partial}_i f = \sum_j a_{ij} y_j + \alpha b_i$ on the indicatrix. Since from $L = \alpha + \beta$ we have $l_i = (\sum_j a_{ij} y_j + \alpha b_i)/\alpha$, the vector $\dot{\nabla} f = (\dot{\partial}_i f)$ has the same direction as $\dot{\nabla} L = (l_i)$. Thus the vector field

$$N = -\dot{\nabla} f / |\dot{\nabla} f| \tag{3.9}$$

gives the orientation assumed in (3.3). Since we have

$$\begin{vmatrix} a_{ij} - b_i b_j & b_i \\ b_j & -1 \end{vmatrix} = -\det(a_{ij}),$$

applying Theorem 2.2 to (3.8) we have

Theorem 3.3. *Let (R^n, L) be a Randers space, where $L = \alpha + \beta$. At each point $x \in R^n$, the Gaussian curvature K of the indicatrix I_x is given by*

$$K = \det(a_{ij})/(\sum_i f_i^2)^{(n+1)/2}, \tag{3.10}$$

where $f_i = \sum_j a_{ij} y_j + \alpha b_i$, provided I_x is oriented in the direction opposite to $\dot{\nabla} f = (f_i)$.

Remark 3.1. Since $f_i = \alpha l_i$ in Theorem 3.3, if we compare (3.10) with (3.5), we have $g = \det(a_{ij})/\alpha^{n+1}$ on the indicatrix, from which at any $y \in R^n_x$ we have $g = (L/\alpha)^{n+1} \det(a_{ij})$.

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