

# Linear Estimation of Signal Using Covariance Information in Given Uncertain Observations

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**Abstract** This paper designs recursive least-squares filter and fixed-point smoother, which use the observed value, the probability that the signal exists, and the covariance information relevant to the signal and observation noises, on the estimation problem associated with the uncertain observations in linear continuous systems.

## 1. INTRODUCTION

Nahi (1969) has proposed the celebrated technique in the state estimation of dynamical systems associated with the uncertain observations. By the uncertain observations we mean that the observation set does not necessarily contain the signal in the entire interval of observation and certain observations may contain noise alone. The estimation problem with the uncertain observations has been viewed as an important research in the area of the detection and estimation problems for communication systems (Nahi 1969). In Nahi (1969), the recursive least-squares one-stage-prediction algorithm is devised, provided that the probability for the existence of the signal is available with complete information of a state-space model in linear discrete-time systems.

By the way, the recursive Wiener filter (Kailath 1974) by use of the covariance information of the observed value has been reported. Also, Nakamori (1990) has derived the recursive algorithms for the least-squares estimates from the Wiener-Hopf integral equation for white Gaussian plus colored observation noise in linear continuous systems. In Nakamori (1990), it is assumed that the observed value, the variance of white Gaussian observation noise and the autocovariance functions of the signal and colored noise, in the semi-degenerate kernel form, are known. The estimation technique using the covariance information has been researched in the context of the detection and estimation of the signal (Trees 1968). However, given the uncertain observations, the estimation problem using the covariance information has not been solved.

This paper is concerned with the optimum estimation problems using the covariance information regarding the estimation of the signal in terms of the uncertain observations in linear continuous stochastic systems. The observation equation is given by  $y(t) = z(t) + v(t)$ ,  $z(t) = x_u(t) + v_c(t)$ ,  $x_u(t) = U(t)x(t)$ , where the signal  $x(t)$  might be correlated with colored observation noise  $v_c(t)$ . We use the probability  $p(t)$  at time  $t$  for the existence of the signal (Nahi 1969) as a priori information. We also assume the knowledge of the observed value, the crosscovariance function of  $x_u(t)$  with  $y(s)$ , the autocovariance function of  $z(t)$ , and the variance of white Gaussian observation noise  $v(t)$ . At first,

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the filtering and fixed-point smoothing algorithms of  $x_u(t)$  are proposed in Theorem 1. The estimation algorithms of  $x(t)$  are presented in Theorem 2 by incorporating the innovations process evaluated by the equations of Theorem 1 into the estimation algorithms by Nakamori (1991). In the estimation of  $x(t)$ , the crosscovariance function of  $x(t)$  with  $y(s)$  and the autocovariance function of  $z(t)$  are required particularly for  $p(t)=1$  as an additional information. The numerical considerations involved are illustrated by examples.

## 2. ESTIMATION PROBLEM WITH UNCERTAIN OBSERVATIONS

Let an observation equation be described by

$$y(t) = z(t) + v(t), \quad z(t) = x_u(t) + v_c(t), \quad x_u(t) = U(t)x(t), \quad (1)$$

where  $y(t)$  is an  $n \times 1$  observed value vector,  $x(t)$  is an  $n \times 1$  zero-mean signal vector,  $v_c(t)$  is a zero-mean colored observation noise, and  $v(t)$  is white Gaussian observation noise satisfying

$$E[v(t)] = 0, \quad (2)$$

$$E[v(t)v^T(s)] = R(t)\delta(t-s), \quad 0 \leq t, s < \infty. \quad (3)$$

$U(t)$  is a scalar quantity taking on values of 0 and 1 with

$$p(t) = Pr\{U(t)=1\}, \quad (4)$$

$$1-p(t) = Pr\{U(t)=0\}. \quad (5)$$

Here, we denote  $p(t)$  by the probability that observation at time  $t$  contains the signal. The ideal value of  $U(t)$  might be 1 when the signal  $x(t)$  exists, and  $U(t)=0$  for the case where the observed value contains noises only. Also, the expected values of  $U(t)$ ,  $U(t)U(s)$  and  $U^2(t)$  are written as

$$E[U(t)] = p(t) \quad (6)$$

$$E[U(t)U(s)] = p(t)p(s), \quad t \neq s, \quad (7)$$

$$E[U^2(t)] = p(t) \quad (8)$$

(Nahi, 1969).

We assume that the signal  $x(t)$  and colored noise  $v_c(t)$  are uncorrelated with white Gaussian noise  $v(s)$  as

$$E[x(t) v^T(s)] = 0, \quad E[v_c(t) v^T(s)] = 0, \quad 0 \leq t, \quad s < \infty, \quad (9)$$

and that the signal might be correlated with colored noise.

Let the fixed-point smoothing estimate  $\hat{x}_u(t, T)$  of  $x_u(t)$  be given by

$$\hat{x}_u(t, T) = \int_0^T h(t, s', T) y(s') ds' \quad (10)$$

as a linear integral transform of the observation set  $\{y(s'), 0 \leq s' \leq T\}$ , where  $t$  is the fixed-point and  $h(t, s, T)$  is referred to as an impulse response function.

Let us consider the linear least-squares estimation problem which minimizes a cost function

$$J = E[\|x_u(t) - \hat{x}_u(t, T)\|^2]. \quad (11)$$

Minimizing  $J$ , we obtain the Wiener-Hopf equation

$$E[x_u(t) y^T(s)] = \int_0^T h(t, s', T) E[y(s') y^T(s)] ds' \quad (12)$$

by an orthogonal projection lemma (Nakamori 1990)

$$x_u(t) - \int_0^T h(t, s', T) y(s') ds' \perp y(s), \quad 0 \leq s, \quad t \leq T. \quad (13)$$

Here, " $\perp$ " denotes the notation of the orthogonality.

Substituting (1) into (12), and using (3), we have

$$h(t, s, T) R(s) = K_{uy}(t, s) - \int_0^T h(t, s', T) K_z(s', s) ds', \quad (14)$$

where  $K_{uy}(t, s)$  denotes the crosscovariance function of  $x_u(t)$  with  $y(s)$ , and  $K_z(t, s)$  the autocovariance function of  $z(t)$ .

Following the expression for the covariance function (Nakamori 1990), we let the crosscovariance function  $K_{uy}(t, s)$  be expressed in the semi-degenerate kernel form by

$$\begin{aligned} K_{uy}(t, s) &= E[x_u(t) y^T(s)] \\ &= E[U(t) x(t) y^T(s)] \\ &= \begin{cases} C(t) H^T(s), & 0 \leq s \leq t, \\ M(t) N^T(s), & 0 \leq t \leq s, \end{cases} \end{aligned} \quad (15)$$

where  $C(t)$  and  $H(s)$  are  $n \times n'$  bounded matrices, and  $M(t)$  and  $N(s)$  are  $n \times m'$  bounded matrices. Likewise, the autocovariance function  $K_z(t, s)$  is expressed in the semi-degenerate kernel form by

$$\begin{aligned}
 K_z(t, s) &= E[z(t) z^T(s)] \\
 &= \begin{cases} G(t) L^T(s), & 0 \leq s \leq t, \\ L(t) G^T(s), & 0 \leq t \leq s, \end{cases} \quad (16)
 \end{aligned}$$

where  $G(t)$  and  $L(s)$  are  $n \times 1$  bounded matrices.

Preliminary to the discussion on estimating the signal  $x(t)$ , we consider to estimate  $x_u(t)$ . In Theorem 1, with the assumptions above, we design the recursive algorithms for the linear least-squares filtering and fixed-point smoothing estimates of  $x_u(t)$  by use of the covariance information.

### 3. RECURSIVE FILTERING AND FIXED-POINT SMOOTHING ALGORITHMS OF $x_u(t)$

**Theorem 1.** Let the probability for  $U(t)=1$  be  $p(t)$  in the observation equation (1) for the signal observed with additional white Gaussian plus colored noise. Here, the signal might be correlated with coloured noise. Let the crosscovariance function  $K_{xy}(t, s)$  of  $x(t)$  with  $y(s)$  and the autocovariance function  $K_z(t, s)$  of  $z(t)$  be expressed in the semi-degenerate kernel form. Also, let the observed value and the variance of white Gaussian observation noise be given. Then the recursive algorithms for linear least-squares filtering and fixed-point smoothing estimates of  $x_u(t)$  consist of (17)–(26) in continuous stochastic systems.

Fixed-point smoothing estimate:

$$\partial \hat{x}_u(t, T) / \partial T = h(t, T, T) (y(t) - G(t) e(t)) \quad (17)$$

Filtering estimate:

$$\hat{x}_u(T, T) = C(T) V(T) \quad (18)$$

$$dV(T) / dT = \Phi(T, T) (y(T) - G(T) e(T)), \quad V(0) = 0 \quad (19)$$

$$de(T) / dT = J(T, T) (y(T) - G(T) e(T)), \quad e(0) = 0 \quad (20)$$

$$\Phi(T, T) = (H^T(T) - W(T) G^T(T)) R^{-1}(T) \quad (21)$$

$$J(T, T) = (L^T(T) - r(T) G^T(T)) R^{-1}(T) \quad (22)$$

$$dr(T) / dT = J(T, T) (L(T) - G(T) r(T)), \quad r(0) = 0 \quad (23)$$

$$h(t, T, T) = (M(t) N^T(T) - S(t, T) G^T(T)) R^{-1}(T) \quad (24)$$

$$\partial S(t, T) / \partial T = h(t, T, T) (L(T) - G(T) r(T)), \quad S(t, t) = C(t) W(t) \quad (25)$$

$$dW(T) / dT = \Phi(T, T) (L(T) - G(T) r(T)), \quad W(0) = 0 \quad (26)$$

**(Proof)**

Let us differentiate (14) with respect to  $T$ .

$$\partial h(t, s, T) / \partial TR(s) = -h(t, T, T) K_z(T, s) - \int_0^T \partial h(t, s', T) / \partial TK_z(s', s) ds' \quad (27)$$

If we introduce an auxiliary function  $J(T, s)$  which satisfies

$$J(T, s) R(s) = L^T(s) - \int_0^T J(T, s') K_z(s', s) ds', \quad (28)$$

we have a partial differential equation for  $h(t, s, T)$

$$\partial h(t, s, T) / \partial T = -h(t, T, T) G(T) J(T, s). \quad (29)$$

Similarly, if we differentiate (28) with respect to  $T$ , we have

$$\partial J(T, s) / \partial TR(s) = -J(T, T) K_z(T, s) - \int_0^T \partial J(T, s') / \partial TK_z(s', s) ds'. \quad (30)$$

From (16), (28) and (30), we obtain a partial differential equation for  $J(T, s)$

$$\partial J(T, s) / \partial T = -J(T, T) G(T) J(T, s). \quad (31)$$

Now, from (28), the function  $J(T, T)$  in (31) satisfies

$$J(T, T) R(T) = L^T(T) - \int_0^T J(T, s') K_z(s', T) ds'. \quad (32)$$

Substituting the expression  $K_z(s', T) = L(s') G^T(T)$  for  $0 \leq s' \leq T$  from (16) into (32), we have

$$J(T, T) R(T) = L^T(T) - \int_0^T J(T, s') L(s') ds' G^T(T). \quad (33)$$

If we introduce a function  $r(T)$  defined by

$$r(T) = \int_0^T J(T, s') L(s') ds', \quad (34)$$

we obtain (22) for  $J(T, T)$ .

If we differentiate (34) with respect to  $T$  and substitute (31) into the resultant equation, we have

$$dr(t) / dT = J(T, T) L(T) - J(T, T) G(T) \int_0^T J(T, s') L(s') ds'. \quad (35)$$

From (34), we can rewrite (35) as (23), where the initial condition on the differential equation (23) at  $T=0$  is  $r(0)=0$  from (34).

From (14), the function  $h(t, T, T)$ , which appeared in (29), satisfies

$$h(t, T, T) R(T) = K_{uy}(t, T) - \int_0^T h(t, s', T) K_z(s', T) ds'. \quad (36)$$

If we use the expressions  $K_{uy}(t, T) = M(t) N^T(T)$  for  $0 \leq t \leq T$  and  $K_z(s', T) = L(s') G^T(T)$  for  $0 \leq s' \leq T$  from (15)–(16) in (36), we have

$$h(t, T, T) R(T) = M(t) N^T(T) - \int_0^T h(t, s', T) L(s') ds' G^T(T). \quad (37)$$

If we introduce a function  $S(t, T)$  defined by

$$S(t, T) = \int_0^T h(t, s', T) L(s') ds', \quad (38)$$

we obtain the equation (24) for  $h(t, T, T)$ .

If we differentiate (38) with respect to  $T$  and substitute (29) into the resultant equation, we have

$$\partial S(t, T) / \partial T = h(t, T, T) L(T) - h(t, T, T) G(T) \int_0^T J(T, s') L(s') ds'. \quad (39)$$

From (34), we can rewrite (39) as (25).

If we put  $t=T$  in (14), we have

$$h(T, s, T) R(s) = K_{uy}(T, s) - \int_0^T h(T, s', T) K_z(s', s) ds'. \quad (40)$$

If we substitute  $K_{uy}(T, s) = C(T) H^T(s)$  from (15) into (40), we have

$$h(T, s, T) R(s) = C(T) H^T(s) - \int_0^T h(T, s', T) K_z(s', s) ds'. \quad (41)$$

Let us introduce an auxiliary function  $\Phi(T, s)$  which satisfies

$$\Phi(T, s) R(s) = H^T(s) - \int_0^T \Phi(T, s') K_z(s', s) ds'. \quad (42)$$

From (41) and (42), we obtain

$$h(T, s, T) = C(T) \Phi(T, s). \quad (43)$$

The initial condition on the partial differential equation (25) at  $T=t$  is  $S(t, t)$ . From (38),  $S(t, t)$  is formulated as

$$S(t, t) = \int_0^t h(t, s', t) L(s') ds'. \quad (44)$$

From (43), we can rewrite (44) as

$$S(t, t) = C(t) \int_0^t \Phi(t, s') L(s') ds'. \quad (45)$$

If we introduce a function  $W(t)$  defined by

$$W(T) = \int_0^T \Phi(T, s') L(s') ds', \quad (46)$$

we obtain the initial condition as  $S(t, t) = C(t) W(t)$ .

If we differentiate (42) with respect to  $T$ , we have

$$\partial \Phi(T, s) / \partial T R(s) = -\Phi(T, T) K_z(T, s) - \int_0^T \partial \Phi(T, s') / \partial T K_z(s', s) ds'. \quad (47)$$

If we substitute  $K_z(T, s) = G(T) L^T(s)$  for  $0 \leq s \leq T$  from (16) into (47) and compare the resultant equation with (28), we obtain a partial differential equation for  $\Phi(T, s)$

$$\partial \Phi(T, s) / \partial T = -\Phi(T, T) G(T) J(T, s). \quad (48)$$

From (42), the function  $\Phi(T, T)$  in (48) satisfies

$$\Phi(T, T) R(T) = H^T(T) - \int_0^T \Phi(T, s') K_z(s', T) ds'. \quad (49)$$

Since  $K_z(s', T) = L(s') G^T(T)$  for  $0 \leq s' \leq T$ , we can rewrite (49) as

$$\Phi(T, T) R(T) = H^T(T) - \int_0^T \Phi(T, s') L(s') ds' G^T(T). \quad (50)$$

Also, by use of (46), (50) becomes (21).

If we differentiate (46) with respect to  $T$ , we have

$$dW(T) / dT = \Phi(T, T) L(T) + \int_0^T \partial \Phi(T, s') / \partial T L(s') ds'. \quad (51)$$

If we substitute (48) into (51) and use (34), we obtain (26). The initial condition on the differential equation for  $W(T)$  at  $T=0$  is  $W(0)=0$  from (46).

If we differentiate (10) with respect to  $T$ , we have

$$\partial \hat{x}_u(t, T) / \partial T = h(t, T, T) y(T) + \int_0^T \partial h(t, s', T) / \partial T y(s') ds'. \quad (52)$$

If we substitute (29) into (52) and introduce a function

$$e(T) = \int_0^T J(T, s') y(s') ds', \quad (53)$$

we obtain the partial differential equation (17) for the fixed-point smoothing estimate  $\hat{x}_u(t, T)$ .

If we differentiate (53) with respect to  $T$ , we have

$$de(T)/dT = J(T, T) y(T) + \int_0^T \partial J(T, s') / \partial T y(s') ds'. \quad (54)$$

If we substitute (31) into (54) and use (53), we obtain (20). The initial condition on the differential equation (20) at  $T=0$  is  $e(0)=0$  from (53).

The filtering estimate  $\hat{x}_u(T, T)$  of  $x_u(T) (= U(T)x(T))$  is formulated as

$$\hat{x}_u(T, T) = \int_0^T h(T, s', T) y(s') ds' \quad (55)$$

by putting  $t=T$  in (10). If we substitute (43) into (55), and introduce a function  $V(T)$  defined by

$$V(T) = \int_0^T \Phi(T, s') y(s') ds', \quad (56)$$

we obtain (18).

If we differentiate (56) with respect to  $T$ , we have

$$dV(T)/dT = \Phi(T, T) y(T) + \int_0^T \partial \Phi(T, s') / \partial T y(s') ds'. \quad (57)$$

If we substitute (48) into (57) and use (53), we obtain (19). The initial condition on the differential equation for  $V(T)$  at  $T=0$  is  $V(0)=0$  from (56).  $\square$

In Theorem 1, we proposed the algorithms for the filtering and fixed-point smoothing estimates of  $x_u(t) (= U(t)x(t))$ . It should be noted that the filtering estimate of  $x_u(t) + v_c(t)$  is calculated by  $G(t)e(t)$ . Letting  $\hat{z}(t, t) = G(t)e(t)$ , the innovations process for  $x_u(t) + v_c(t)$  becomes  $y(t) - \hat{z}(t, t)$ . We focus our attention to estimating the signal  $x(t)$  by use of the covariance information under the assumption that the uncertain observations are given. Here, we examine to apply the linear least-squares filtering and fixed-point smoothing algorithms (Nakamori 1991), in the estimation problem of the signal with certain observations, i. e., for  $p(t)=1$ , based on the innovations theory to the present estimation problem of  $x(t)$ . In Theorem 2, we propose the algorithms for the filtering and fixed-point smoothing estimates of  $x(t)$  by incorporating the estimation algorithms developed by Nakamori (1991) with the equations that calculate the innovations process  $y(t) - G(t)e(t)$ .

#### 4. RECURSIVE FILTERING AND FIXED-POINT SMOOTHING ALGORITHMS OF $x(t)$

**Theorem 2.** Let the observation equation be expressed by (1). Let the crosscovariance function  $K_{xy}(t, s)$  of  $x(t)$  with  $y(s)$  and the autocovariance function  $K_c(t, s)$  of  $z(t)$  be expressed in the semi-degenerate kernel form as follows particularly for  $p(t)=1$ .



$$K_{xy}(t, s) = \begin{cases} \alpha(t) \beta^T(s), & 0 \leq s \leq t, \\ \varepsilon(t) \zeta^T(s), & 0 \leq t \leq s \end{cases} \quad (58)$$

Here,  $\alpha(t)$  and  $\beta(s)$  are  $n \times n'$  bounded matrices, and  $\varepsilon(t)$  and  $\zeta(s)$  are  $n \times m'$  bounded matrices.

$$K_c(t, s) = \begin{cases} A(t) B^T(s), & 0 \leq s \leq t, \\ B(t) A^T(s), & 0 \leq t \leq s \end{cases} \quad (59)$$

Here,  $A(t)$  and  $B(s)$  are  $n \times 1$  bounded matrices. Let  $\hat{z}(t, t)$  denote the filtering estimate of  $x_u(t) + v_c(t)$ . Then the sequential algorithms for the linear least-squares filtering and the fixed-point smoothing estimates of  $x(t)$  consist of (60)–(72).

$$\begin{aligned} \hat{x}(t, T) &: \text{fixed-point smoothing estimate of } x(t). \\ \partial \hat{x}(t, T) / \partial T &= \varepsilon(t) (\zeta^T(T) - E^T(T, t) A^T(T)) R^{-1}(T) (y(T) - \hat{z}(T, T)) \\ &\quad - \alpha(t) D^T(T, t) A^T(T) R^{-1}(T) (y(T) - \hat{z}(T, T)) \end{aligned} \quad (60)$$

$$\partial D(T, t) / \partial T = -J(T, T) A(T) D(T, t) \quad (61)$$

$$\partial E(T, t) / \partial T = J(T, T) (\zeta(T) - A(T) E(T, t)), \text{ initial condition } E(t, t) = 0 \quad (62)$$

$$\begin{aligned} \hat{x}(t, t) &: \text{filtering estimate of } x(t). \\ \hat{x}(t, t) &= \alpha(t) Q(t) \end{aligned} \quad (63)$$

$$\begin{aligned} \hat{z}(t, t) &: \text{filtering estimate of } x_u(t) + v_c(t). \\ \hat{z}(t, t) &= G(t) e(t) \end{aligned} \quad (64)$$

$$dD(t, t) / dt = f(t, t) (\beta(t) - A(t) D(t, t)), \text{ initial condition } D(0, 0) = 0 \quad (65)$$

$$dQ(t) / dt = (\beta^T(t) - q^T(t) A^T(t)) R^{-1}(t) (y(t) - \hat{z}(t, t)), \text{ initial condition } Q(0) = 0 \quad (66)$$

$$dq(t) / dt = f(t, t) (\beta(t) - A(t) q(t)), \text{ initial condition } q(0) = 0 \quad (67)$$

$$f(t, t) = (B^T(t) - g(t) A^T(t)) R^{-1}(t) \quad (68)$$

$$dg(t) / dt = f(t, t) (B(t) - A(t) g(t)), \text{ initial condition } g(0) = 0 \quad (69)$$

$$de(T) / dT = J(T, T) (y(T) - G(T) e(T)), e(0) = 0 \quad (70)$$

$$J(T, T) = (L^T(T) - r(T) G^T(T)) R^{-1}(T) \quad (71)$$

$$dr(T)/dT = J(T, T)(L(T) - G(T)r(T)), r(0) = 0 \quad (72)$$

Here,  $y(t) - \hat{z}(t, t)$  is the innovations process in the estimation problem concerned with the uncertain observations.

(Proof)

(60)–(63) and (65)–(69) have been derived by Nakamori (1991) in the context of linear least-squares estimation problem with the certain observations, i. e., for  $p(t) = 1$ , given the covariance information. These filtering and fixed-point smoothing algorithms have been obtained based on the innovations approach. We note that the quantity  $y(T) - \hat{z}(T, T)$  in (60) represents the innovations process at time  $T$ . In the current estimation problem using the uncertain observations,  $y(T)$  is the uncertain observed value and  $\hat{z}(T, T)$  is the filtering estimate of  $x_u(T) + v_c(T)$ . It follows from Theorem 1 that the filtering estimate  $\hat{z}(t, t)$  is calculated by (64) and (70)–(72) sequentially.  $\square$

## 5. A NUMERICAL SIMULATION EXAMPLE

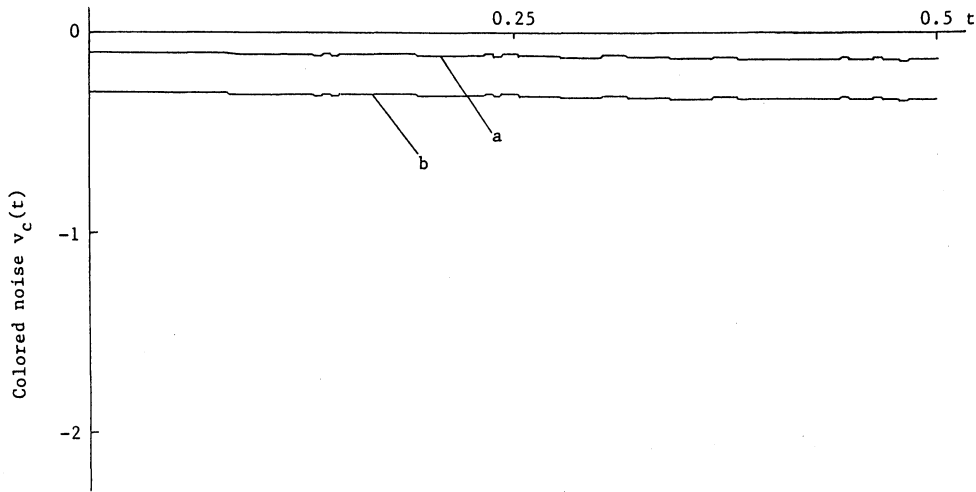
Let the observation equation be given by (1) for a scalar signal which is observed with additive white Gaussian and colored noises. Let the signal  $x(t)$  be generated by

$$dx(t)/dt = -5x(t) + u(t), \quad E[u(t)u(s)] = 100\delta(t-s), \quad E[x^2(0)] = 10, \quad (73)$$

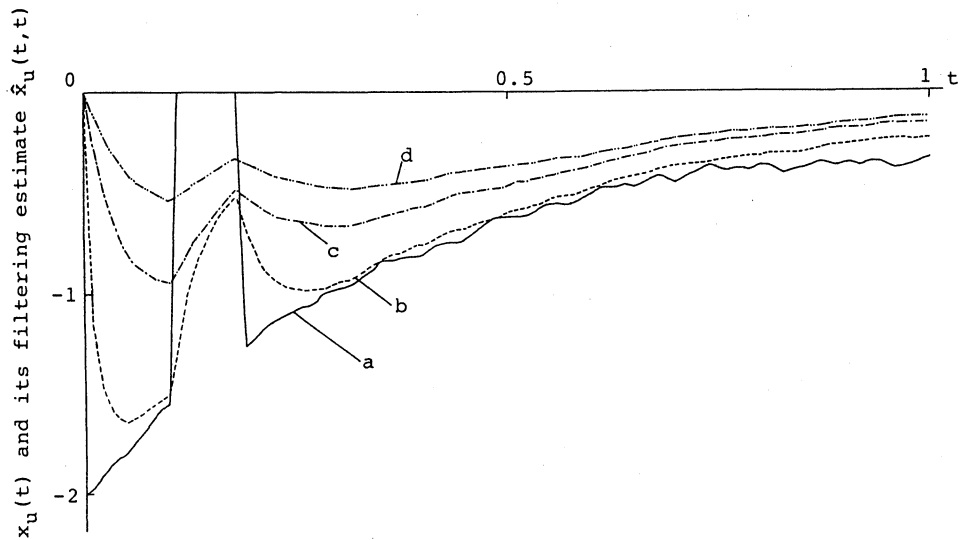
where the autocovariance function  $K_x(t, s)$  of  $x(t)$  is expressed by  $K_x(t, s) = 10e^{-5|t-s|}$  (Baggeroer, 1970). Also, let the process of colored noise  $v_c(t)$  be generated by

$$dv_c(t)/dt = w(t), \quad E[w(t)w(s)] = 10\delta(t-s), \quad E[v_c^2(0)] = 0, \quad (74)$$

where the autocovariance function  $K_c(t, s)$  of  $v_c(t)$  is given by  $K_c(t, s) = 10 \min(t, s)$  (Baggeroer 1970). The crosscovariance function  $K_{uy}(t, s)$  of  $x_u(t)$  with  $y(s)$  is expressed by (15). Since  $x(t)$  is uncorrelated with  $v_c(s)$ , we obtain  $C(t) = 10p(t)e^{-5t}$  ( $= N(t)$ ) and  $H(s) = p(s)e^{5s}$  ( $= M(s)$ ) from  $K_x(t, s) = 10e^{-5|t-s|}$ . Also, the autocovariance function  $K_z(t, s)$  of  $z(t)$  ( $= x_u(t) + v_c(t)$ ) is given by (16). The functions  $G(t)$  and  $L(s)$  become  $G(t) = [p(t) 10e^{-5t} 10]$  and  $L(s) = [p(s) e^{5s} s]$ . If we substitute the functions  $C(T)$ ,  $H(T)$ ,  $N(T)$ ,  $M(t)$ ,  $G(T)$  and  $L(T)$  into Theorem 1, we can calculate the filtering estimate  $\hat{x}_u(t, t)$  and the fixed-point smoothing estimate  $\hat{x}_u(t, T)$ . Graphs (a) and (b) in Fig. 1 illustrate the colored noise process  $v_c(t)$  generated by (74) vs.  $t$ , starting with initial conditions  $v_c(0) = -0.1$  and  $v_c(0) = -0.3$  respectively. Fig. 2 shows the process  $x_u(t)$  (graph (a)) and its filtering estimate  $\hat{x}_u(t, t)$  vs.  $t$  when the initial value of the colored noise is  $v_c(0) = -0.1$  and the probability  $p(t)$  is  $p = 0.5$ , where we introduced the notation  $p$  for  $p(t)$ . Graphs (b), (c) and (d) illustrate  $\hat{x}_u(t, t)$  for white Gaussian observation noises  $N(0, 0.1^2)$ ,  $N(0, 0.3^2)$  and  $N(0, 0.5^2)$  respectively. Fig. 3 shows the mean-square values of the filtering and fixed-point smoothing errors vs.  $p$  when  $v_c(0) = -0.1$ . Graphs (a), (b) and (c) illustrate the mean-square value (M. S. V.) of the filtering error  $x_u(t) - \hat{x}_u(t, t)$  for



**Fig. 1** The colored noise process  $v_c(t)$  vs.  $t$ .  
 (a)  $v_c(t)$  for the initial condition  $v_c(0) = -0.1$ .  
 (b)  $v_c(t)$  for the initial condition  $v_c(0) = -0.3$ .



**Fig. 2** The process  $x_u(t)$  and its filtering estimate  $\hat{x}_u(t, t)$  vs.  $t$  for  $N(0, 0.1^2)$ ,  $N(0, 0.3^2)$  and  $N(0, 0.5^2)$  when  $v_c(0) = -0.1$  and  $p = 0.5$ .  
 (a)  $x_u(t)$ .  
 (b)  $\hat{x}_u(t, t)$  for  $N(0, 0.1^2)$ .  
 (c)  $\hat{x}_u(t, t)$  for  $N(0, 0.3^2)$ .  
 (d)  $\hat{x}_u(t, t)$  for  $N(0, 0.5^2)$ .

$N(0, 0.1^2)$ ,  $N(0, 0.3^2)$  and  $N(0, 0.5^2)$ . Graphs (d), (e) and (f) illustrate the M. S. V. of the fixed-point smoothing error  $x_u(t) - \hat{x}_u(t, T)$  for  $N(0, 0.1^2)$ ,  $N(0, 0.3^2)$  and  $N(0, 0.5^2)$ . Here, the M. S. V. of the filtering error is calculated by  $\sum_{i=1}^{1000} (x_u(i\Delta) - \hat{x}_u(i\Delta, i\Delta))^2 / 1000$ ,  $\Delta = 0.001$ . Also, the M. S. V. of the fixed-point smoothing error is calculated by  $\sum_{i=1}^{1000} \sum_{j=1}^{100} (x_u(i\Delta) - \hat{x}_u(i\Delta, i\Delta + j\Delta))^2 / 100000$ . Fig. 4 illustrates the mean-square values of the filtering and fixed-point smoothing errors vs.  $p$  when

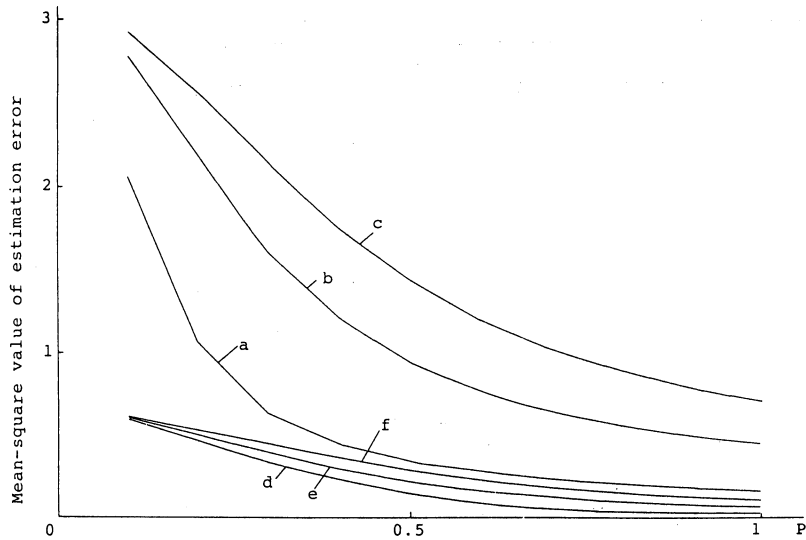


Fig. 3 The mean-square values of the filtering error  $x_u(t) - \hat{x}_u(t, t)$  and the fixed-point smoothing error  $x_u(t) - \hat{x}_u(t, T)$  vs.  $p$  for  $N(0, 0.1^2)$ ,  $N(0, 0.3^2)$  and  $N(0, 0.5^2)$  when  $v_c(0) = -0.1$ .

- (a) The M. S. V. of  $x_u(t) - \hat{x}_u(t, t)$  for  $N(0, 0.1^2)$ . (d) The M. S. V. of  $x_u(t) - \hat{x}_u(t, T)$  for  $N(0, 0.1^2)$ .  
 (b) The M. S. V. of  $x_u(t) - \hat{x}_u(t, t)$  for  $N(0, 0.3^2)$ . (e) The M. S. V. of  $x_u(t) - \hat{x}_u(t, T)$  for  $N(0, 0.3^2)$ .  
 (c) The M. S. V. of  $x_u(t) - \hat{x}_u(t, t)$  for  $N(0, 0.5^2)$ . (f) The M. S. V. of  $x_u(t) - \hat{x}_u(t, T)$  for  $N(0, 0.5^2)$ .

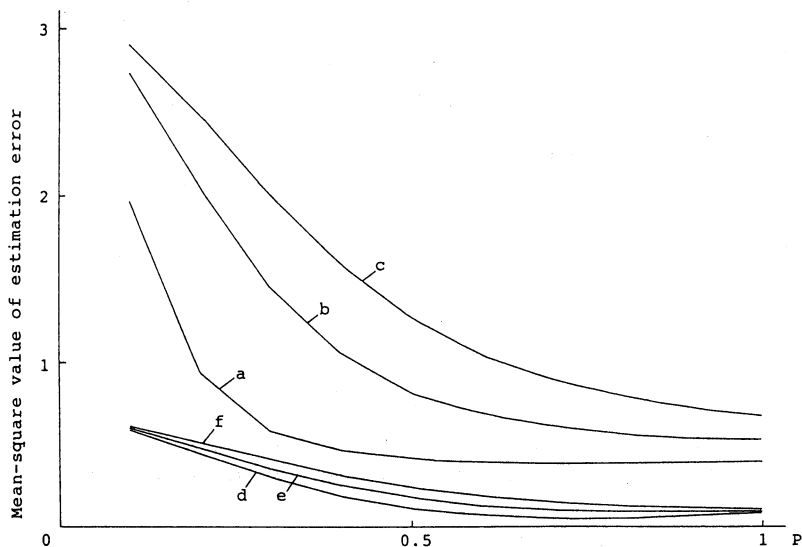


Fig. 4 The mean-square values of the filtering error  $x_u(t) - \hat{x}_u(t, t)$  and the fixed-point smoothing error  $x_u(t) - \hat{x}_u(t, T)$  vs.  $p$  for  $N(0, 0.1^2)$ ,  $N(0, 0.3^2)$  and  $N(0, 0.5^2)$  when  $v_c(0) = -0.3$ .

- (a) The M. S. V. of  $x_u(t) - \hat{x}_u(t, t)$  for  $N(0, 0.1^2)$ . (d) The M. S. V. of  $x_u(t) - \hat{x}_u(t, T)$  for  $N(0, 0.1^2)$ .  
 (b) The M. S. V. of  $x_u(t) - \hat{x}_u(t, t)$  for  $N(0, 0.3^2)$ . (e) The M. S. V. of  $x_u(t) - \hat{x}_u(t, T)$  for  $N(0, 0.3^2)$ .  
 (c) The M. S. V. of  $x_u(t) - \hat{x}_u(t, t)$  for  $N(0, 0.5^2)$ . (f) The M. S. V. of  $x_u(t) - \hat{x}_u(t, T)$  for  $N(0, 0.5^2)$ .

$v_c(0) = -0.3$ . Graphs (a), (b) and (c) illustrate the M. S. V. of the filtering error  $x_u(t) - \hat{x}_u(t, t)$  for  $N(0, 0.1^2)$ ,  $N(0, 0.3^2)$  and  $N(0, 0.5^2)$ . Graphs (d), (e) and (f) illustrate the M. S. V. of the fixed-point smoothing error  $x_u(t) - \hat{x}_u(t, T)$  for  $N(0, 0.1^2)$ ,  $N(0, 0.3^2)$  and  $N(0, 0.5^2)$ . From Fig. 3 and Fig. 4, we

