

Design of Linear Stationary Stochastic Estimators in Relation to H_∞ Estimation Technique

Seiichi NAKAMORI*

(Received 30 September, 1998)

Abstract

This paper designs the fixed-point smoother and filter suitable for estimating the wide-sense stationary stochastic signal in relation to the H_∞ estimation approach in continuous-time systems. Performance measure for the design of the estimators is newly introduced by referring to that of the infinite-horizon H_∞ estimation problem in the Krein spaces [1],[2]. At first, we propose the estimation algorithms using the covariance information in linear continuous wide-sense stationary stochastic systems. Secondly, to improve the estimation accuracy of the recursive least-squares (RLS) estimators [3] using the covariance information, the suboptimal fixed-point smoother and filter using the covariance information are proposed. Finally, the recursive H_∞ like fixed-point smoother and filter using the state-space parameters are derived from those using the covariance information in a unified manner in linear continuous wide-sense stationary stochastic systems.

1. Introduction

Recently, by use of the state-space parameters, the H_∞ and its related estimation techniques [1],[2],[4]-[8] have attracted great attention in the deterministic and stochastic estimation methods of signal. Incidentally, as an alternative approach to the least-squares estimation problem based on the state-space model, the recursive Wiener fixed-point smoother and filter using the covariance information of the signal and the observation noise are developed in linear continuous stochastic systems [3].

The performance criterion concerned with the H_∞ estimation problem is formulated in the deterministic manner by nature. The problem formulation in the finite-horizon and infinite-horizon H_∞ estimation problems is limited within deterministic representations and would not fit to the estimation in linear stationary stochastic systems. In [1],[2] the deterministic H_∞ technique in the Krein spaces is developed. In the H_∞ estimation problem [1],[2], the perfor-

* Department of Technology, Faculty of Education, Kagoshima University, 1-20-6, Kohrimoto, Kagoshima 890-0065, Japan

mance criterion is represented as Eq.(4) in section 2. In the H_2 estimation problem, the value of γ_∞^2 (see Eq.(4)) is set to ∞ [1],[2],[4]-[8]. Consequently, the estimation accuracy of the H_∞ estimator is superior to the H_2 estimator. Taking into account of these aspects, we introduce the performance criterion newly. We examine to design the estimators, which correspond to the H_∞ estimator, to improve the estimation accuracy in comparison with the RLS Wiener estimators [3], which correspond to the H_2 estimator in the H_∞ estimation problem, in linear wide-sense stationary stochastic systems [9]. At first, in section 2, the stochastic signal estimation problem is introduced for the estimation of the wide-sense stationary signal. Based on the current performance criterion (see Eq.(5)), as in the deterministic H_∞ estimation technique [1],[2], we obtain the observation equation in the wide-sense stationary stochastic systems. Assuming that the observation equation is given, we consider the linear least-squares estimation problem using the covariance information in wide-sense stationary stochastic systems.

In the observed values, an artificial observed value $\tilde{z}(t)$ (see Eq.(9)) is included. In [Theorem 1], by use of the covariance information of the signal and the observation noise, recursive algorithms for the fixed-point smoothing and filtering estimates of the signal $z(t)$ (see Eq.(8)) are proposed. In [Theorem 2], based on the algorithms of [Theorem 1], recursive Wiener fixed-point smoother and filter using the covariance information are proposed. Recursive Wiener fixed-point smoother and filter for the signal $z(t)$ use the system matrix F , the observation matrix H (see Eq.(8)), the crosscovariance function $K_{xy}(t, T)$ of the state variable $x(t)$ with observed value $y(T)$, the crossvariance function $K_{xy}(T, T)$, the variance Ξ of the observation noise (see Eq.(10)) and the observed value $y(T)$ (see Eq.(9)). From the estimation equations in [Theorem 2], [Theorem 3] formulates the algorithms using the covariance information for the fixed-point smoothing estimates $\hat{z}_1(t, T)$ and $\hat{z}_2(t, T)$ at the fixed point t and the filtering estimates $\hat{z}_1(T, T)$ and $\hat{z}_2(T, T)$ for the components $z_1(t)$ and $z_2(t)$ of the signal vector $z(t)$. According to the derivation of the H_∞ suboptimal estimators [2], we propose in [Theorem 4] the recursive suboptimal fixed-point smoother and filter using the covariance information by setting $\tilde{z}(T) = \hat{z}_2(T, T)$ in the estimation algorithms of [Theorem 3]. Here, the estimation algorithms of [Theorem 4] necessitate the information of the observation matrices C and L (see Eq.(8)), the system matrix F , the autocovariance function $K_x(t, T)$ of the state variable $x(t)$, the autovariance function $K_x(T, T)$, the crosscovariance function $K_{xy_1}(t, T)$ of the state variable $x(t)$ with the observed value $y_1(T)$, the crossvariance function $K_{xy_1}(T, T)$, γ (see Eq.(5)) and the observed value $y_1(T)$ (see Eq.(2)). For $\gamma^2 = \infty$, the fixed-point smoothing and filtering algorithms in [Theorem 4] are reduced to the RLS Wiener algorithms for the fixed-point

smoothing and filtering estimates in [3]. [Theorem 5] shows the optimal fixed-point smoothing and filtering algorithms using the state-space parameters. They are derived from the estimation algorithms of [Theorem 3] using the covariance information. The algorithms of [Theorem 5] uses the system matrix F , the observation matrices C and L , the input matrix B (see Eq.(1)), γ , the observed values $y_1(T)$ and $\bar{z}(T)$. In [Theorem 6], assuming that the observed value $\bar{z}(T)$ of $z_2(T)$ introduced artificially is equal to $\hat{z}_2(T, T)$ in the estimation equations of [Theorem 5], we propose the suboptimal algorithms using the state-space parameters for the fixed-point smoothing and filtering estimates of $z_1(t)$ and $z_2(t)$. The suboptimal filtering algorithm using the state-space parameters in [Theorem 6] is same as that based on the game theory approach [5] in linear continuous stochastic systems. For $\gamma^2 = \infty$, the suboptimal filter is reduced to the Kalman filter. The suboptimal fixed-point smoothing algorithm in [Theorem 6] is proposed for the first time in this context.

2. Problem Formulation

Let the linear time-invariant state-space model for the state variable $x(t)$ be given by

$$\begin{aligned} \frac{dx(t)}{dt} &= Fx(t) + Bu(t), \quad x(0) = x_0, \\ E[u(t)u^T(s)] &= \Pi_0 \delta(t - s). \end{aligned} \tag{1}$$

Here, $x(t) \in R_n, u(t) \in R_l, F$ represents the system matrix and B the input matrix for $u(t)$. Let $z_1(t)$ represent a signal expressed by $z_1(t) = Cx(t)$. We assume that the signal $z_1(t)$ is observed with additive white Gaussian noise $v_1(t)$.

$$y_1(t) = z_1(t) + v_1(t), \quad z_1(t) = Cx(t), \quad E[v_1(t)v_1^T(s)] = R\delta(t - s) \tag{2}$$

Here, $y_1(t)$ represents the observed value and C m by n observation matrix. We assume that $v_1(t)$ and $u(t)$ are uncorrelated. We also assume that (F, C) is observable and x_0 is a random variable with the mean zero and the variance Q_0 .

Let a signal $z_2(t)$ be represented by

$$z_2(t) = Lx(t), \tag{3}$$

where L is r by n vector.

Let $L_2[O, T]$ denote the usual Hilbert space of square integrable functions. In the finite-horizon H_∞ estimation problem, the estimators are designed so as to achieve the following performance measure [2]:

$$\sup_{x_0, u(t) \in L_2, x(t), v_1(t) \in L_2} \frac{\int_0^T (\tilde{z}(t) - Lx(t))^T (\tilde{z}(t) - Lx(t)) dt}{x_0^T \hat{Q}_0^{-1} x_0 + \int_0^T u^T(t) u(t) dt + \int_0^T v_1^T(t) v_1(t) dt} < \gamma^2 \quad (4)$$

Here, $\tilde{z}(t)$ is the artificial observed value for $z_2(t)$ and \hat{Q}_0 is a definite weighting matrix. \hat{Q}_0 reflects a priori knowledge as to how close $x(0)$ is to its initial guess. We assume that initial guess of $x(0)$ is zero without loss of generality. For the case of $T=\infty$, the performance criterion (4) is concerned with the infinite-horizon H_∞ estimation problem. In the H_∞ estimation problem, we make no assumption of the nature of the noises disturbances $u(t)$ and $v_1(t)$ (e.g. normally distributed, uncorrelated, etc.) and consider to estimate $z_2(t)=Lx(t)$. It is clear that the problem formulation for the H_∞ estimation problem would not fit to treat the estimation problem which assumes the priori statistics for $u(t)$ and $v_1(t)$ in (1) and (2) in wide-sense stationary stochastic systems [9].

In this paper, instead of the H_∞ estimation problem, we consider to estimate the signal $z_2(t)=Lx(t)$ in the wide-sense stationary stochastic systems by taking into account of the statistical properties for $u(t)$ and $v_1(t)$ in (1) and (2). Along with this idea, we newly introduce the performance criterion represented by

$$\sup \frac{E[(\tilde{z}(t) - Lx(t))^T (\tilde{z}(t) - Lx(t))]}{E[x_0^T Q_0^{-1} x_0] + E[u^T(t) \Pi_0^{-1} u(t)] + E[(y_1(t) - Cx(t))^T R^{-1} (y_1(t) - Cx(t))]} < \gamma^2 \quad (5)$$

in the wide-sense stationary stochastic systems. Under the criterion of (5), we can now treat the estimation problem for the wide-sense stationary stochastic signal process in which the ergodic process is included. If we introduce the stochastic quantity of the form

$$J_f = E[x_0^T Q_0^{-1} x_0] + E[u^T(t) \Pi_0^{-1} u(t)] + E[(y_1(t) - Cx(t))^T R^{-1} (y_1(t) - Cx(t))] - \gamma^{-2} E[(\tilde{z}(t) - Lx(t))^T (\tilde{z}(t) - Lx(t))], \quad (6)$$

we find that the performance criterion (5) in the current stochastic estimation problem is transformed into the relationship satisfying $J_f > 0$. Henceforth,

$$J_f = E[x_0^T(0) Q_0^{-1} x_0] + E[u^T(t) \Pi_0^{-1} u(t)] + E\left[\left(\begin{bmatrix} y_1(t) \\ \tilde{z}(t) \end{bmatrix} - \begin{bmatrix} C \\ L \end{bmatrix} x(t)\right)^T \begin{bmatrix} R & 0 \\ 0 & -\gamma^2 I \end{bmatrix}^{-1} \left(\begin{bmatrix} y_1(t) \\ \tilde{z}(t) \end{bmatrix} - \begin{bmatrix} C \\ L \end{bmatrix} x(t)\right)\right] > 0 \quad (7)$$

is obtained in relation to the H_∞ estimation problem [2]. Let us introduce the vector $z(t)$ which consists of the signals $z_1(t)$ and $z_2(t)$.

$$\begin{aligned} z(t) &= Hx(t) \\ &= \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}, \quad H = \begin{bmatrix} C \\ L \end{bmatrix}, \quad z_1(t) = Cx(t), \quad z_2(t) = Lx(t) \end{aligned} \quad (8)$$

Let us also introduce the observation vector $y(t)$

$$y(t) = \begin{bmatrix} y_1(t) \\ \tilde{z}(t) \end{bmatrix} \quad (9)$$

which consists of $y_1(t)$ and the artificial observation $\tilde{z}(t)$ for $z_2(t)$. As in the observation equation in the Krein spaces [2], by checking the condition for a minimum of $J_f > 0$, the observation equation in the stationary stochastic continuous-time systems might be written as

$$\begin{aligned} y(t) &= Hx(t) + v(t), \quad z(t) = Hx(t), \quad v(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}, \\ E[v(t)v^T(s)] &= \Xi \delta(t-s), \quad \Xi = \begin{bmatrix} R & 0 \\ 0 & -r^2 \end{bmatrix}. \end{aligned} \quad (10)$$

The observation equation (10) is analogous to that in the Krein spaces [2] of the linear deterministic H_∞ estimation problem. Provided that the observation equation (10) is given, we consider the stochastic estimation problem in the sense of linear least-squares estimation problem for the fixed-point smoothing and filtering estimates of the signal $z(t)$ and the filtering estimate of the state variable $x(t)$.

It should be noted that the performance criterion even in the infinite-horizon H_∞ estimation problem is distinct from the current one given by (5). In the H_∞ estimation problem, we do not assume a priori knowledge on the variances of the noises $u(\bullet)$ and $v_1(\bullet)$ with their uncorrelation property.

Let the fixed-point smoothing estimate $\hat{x}(t, T)$ of the state variable $x(t)$ be expressed as

$$\hat{x}(t, T) = \int_0^T h(t, s, T) y(s) ds \quad (11)$$

as an integral transformation of the observed data set $\{y(s), 0 \leq s \leq T\}$. Here, $h(t, s, T)$ represents the impulse response function. "t" is referred to as the fixed point. The fixed-point smoothing estimate of the signal $z(t)$ is expressed as $\hat{z}(t, T) = H\hat{x}(t, T)$. We consider the linear least-squares smoothing problem which minimizes the cost function

$$J = E\|x(t) - \hat{x}(t, T)\|^2 \quad (12)$$

in linear continuous-time stochastic systems, given the observation equation (10). Let $K_{xy}(t, s)$ represent the crosscovariance function of the state variable $x(t)$ with the observed value $y(s)$. The optimal impulse response function, which minimizes (12), satisfies the Wiener-Hopf integral equation [3]

$$K_{xy}(t, s) = \int_0^T h(t, s', T) E[y(s') y^T(s)] ds' \quad (13)$$

If we substitute (10) into (13), we obtain

$$h(t, s, T) \Xi = K_{xy}(t, s) - \int_0^T h(t, s', T) H K_{xy}(s', s) ds', \quad (14)$$

since the variance of $v(t)$ is Ξ from (10).

In sections 3 and 4, by use of the covariance information of the signal $z(t)$ and the observation noise $v(t)$, we present the recursive algorithms for the fixed-point smoothing estimate $\hat{z}(t, T)$ of the signal $z(t)$ and the filtering estimates $\hat{z}(T, T)$ of $z(T)$ and $\hat{x}(T, T)$ of $x(T)$.

3. Recursive Wiener Smoother and Filter

Let $\Phi(T, 0)$ represent the state transition matrix of the system matrix F . $\Phi(t, s)$, $0 \leq s \leq t$, satisfies $\frac{\partial \Phi(t, s)}{\partial t} = F \Phi(t, s)$. Let $K_z(t, s)$ represent the autocovariance function of the signal $z(t)$. $K_z(t, s)$ is expressed as

$$K_z(t, s) = H \Phi(t, s) K_{xy}(s, s) l(t-s) + K_{xy}^T(t, t) \Phi^T(s, t) H^T l(s-t), \quad (15)$$

where $l(t-s)$ represents the unit step function. [Theorem 1] presents the fixed-point smoothing and filtering algorithms of $z(t)$ using the crosscovariance function $K_{xy}(t, T)$ of the state variable $x(t)$ with the observed value $y(T)$, the crossvariance function $K_{xy}(T, T)$, the state transition matrix $\Phi(T, 0)$, the observation matrix H , the variance Ξ of the observation noise $v(t)$ and the observed value $y(T)$.

[Theorem 1]

Let the variance of the initial value x_0 of $x(t)$ at $t = 0$ be $Q_0 > 0$. Let the observation equation be given by (10). Then the recursive algorithms for the fixed-point smoothing estimate $\hat{z}(t, T)$ at the fixed point t and the filtering estimates $\hat{z}(T, T)$ of $z(T)$ and $\hat{x}(T, T)$ of $x(T)$, which achieve the criterion of (5), consist of (16) ~ (24). Here, the estimators use the crosscovariance function $K_{xy}(t, T)$ of the state variable $x(t)$ with the observed value $y(T)$, the crossvariance function $K_{xy}(T, T)$, the state-transition matrix $\Phi(T, 0)$ for the system matrix F ,

the observation matrix H , the variance Ξ of the observation noise $v(t)$ and the observed value $y(T)$.

$\hat{z}(t, T)$: Fixed-point smoothing estimate of the signal $z(t)$ at the fixed point t .

$$\frac{\partial \hat{z}(t, T)}{\partial T} = Hh(t, T, T)(y(T) - \hat{z}(T, T)) \quad (16)$$

$$h(t, T, T) = (K_{xy}(t, T) - U(t, T)\Phi^T(T, 0)H^T)\Xi^{-1} \quad (17)$$

$$\frac{\partial U(t, T)}{\partial T} = h(t, T, T)(K_{xy}^T(T, T)\Phi^T(0, T) - H\Phi(T, 0)W(T)) \quad (18)$$

$$U(T, T) = \Phi(T, 0)W(T) \quad (19)$$

$$J(T, T) = (\Phi(0, T)K_{xy}(T, T) - W(T)\Phi^T(T, 0)H^T)\Xi^{-1} \quad (20)$$

$$\begin{aligned} \frac{dW(T)}{dT} &= J(T, T)(K_{xy}^T(T, T)\Phi^T(0, T) - H\Phi(T, 0)W(T)), \\ W(0) &= 0 \end{aligned} \quad (21)$$

$\hat{z}(T, T)$: Filtering estimate of the signal $z(T)$.

$$\hat{z}(T, T) = H\hat{x}(T, T) \quad (22)$$

$\hat{x}(T, T)$: Filtering estimate of the state variable $x(T)$.

$$\hat{x}(T, T) = \Phi(T, 0)e(T) \quad (23)$$

$$\frac{de(T)}{dT} = J(T, T)(y(T) - H\Phi(T, 0)e(T)), \quad e(0) = 0 \quad (24)$$

Proof: Let us differentiate (14) with respect to T .

$$\frac{\partial h(t, s, T)}{\partial T} \Xi = -h(t, T, T)HK_{xy}(T, s) - \int_0^T \frac{\partial h(t, s', T)}{\partial T} HK_{xy}(s', s) ds' \quad (25)$$

If we introduce an auxiliary function $q(T, s)$ which satisfies

$$q(T, s)\Xi = HK_{xy}(T, s) - \int_0^T q(T, s')HK_{xy}(s', s) ds', \quad (26)$$

we obtain the differential equation

$$\frac{\partial h(t, s, T)}{\partial T} = -h(t, T, T)q(T, s) \quad (27)$$

for $h(t, s, T)$ by comparing (25) with (26). (26) is written as

$$q(T, s)\Xi = H\Phi(T, 0)\Phi(0, s)K_{xy}(s, s) - \int_0^T q(T, s')HK_{xy}(s', s)ds' \quad (28)$$

by using the property of the transition matrix $\Phi(T, s)$. If we introduce an auxiliary function $J(T, s)$ which satisfies

$$J(T, s)\Xi = \Phi(0, s)K_{xy}(s, s) - \int_0^T J(T, s')HK_{xy}(s', s)ds', \quad (29)$$

we obtain

$$q(T, s) = H\Phi(T, 0)J(T, s). \quad (30)$$

If we differentiate (29) with respect to T , we have

$$\frac{\partial J(T, s)}{\partial T}\Xi = -J(T, T)HK_{xy}(T, s) - \int_0^T \frac{\partial J(T, s')}{\partial T}HK_{xy}(s', s)ds'. \quad (31)$$

From (28), (30) and (31), we obtain the differential equation

$$\begin{aligned} \frac{\partial J(T, s)}{\partial T} &= -J(T, T)q(T, s) \\ &= -J(T, T)H\Phi(T, 0)J(T, s) \end{aligned} \quad (32)$$

for the function $J(T, s)$.

Now, the function $h(t, T, T)$ in (27) can be formulated as follows. If we put $s=T$ in (14) and use the expression of $K_{xy}(s', T)$ for $0 \leq s' \leq T$, i.e., $HK_{xy}(s', T) = K_{xy}^T(s', s')\Phi^T(T, s')H^T$, we have

$$\begin{aligned} h(t, T, T)\Xi &= K_{xy}(t, T) - \int_0^T h(t, s', T)HK_{xy}(s', T)ds' \\ &= K_{xy}(t, T) - \int_0^T h(t, s', T)K_{xy}^T(s', s')\Phi^T(T, s')H^T ds'. \end{aligned} \quad (33)$$

If we introduce a function

$$U(t, T) = \int_0^T h(t, s', T)K_{xy}^T(s', s')\Phi^T(0, s')ds', \quad (34)$$

we obtain

$$h(t, T, T) = (K_{xy}(t, T) - U(t, T)\Phi^T(T, 0)H^T)\Xi^{-1}. \quad (35)$$

If we differentiate (34) with respect to T , and use (27) and (30), we have

$$\frac{\partial U(t, T)}{\partial T} = h(t, T, T)K_{xy}^T(T, T)\Phi^T(0, T) - h(t, T, T)H\Phi(T, 0)\int_0^T J(T, s')K_{xy}^T(s', s')\Phi^T(0, s')ds'. \tag{36}$$

If we introduce a function

$$W(T) = \int_0^T J(T, s')K_{xy}^T(s', s')\Phi^T(0, s')ds', \tag{37}$$

we obtain the differential equation

$$\frac{\partial U(t, T)}{\partial T} = h(t, T, T)(K_{xy}^T(T, T)\Phi^T(0, T) - H\Phi(T, 0)W(T)) \tag{38}$$

for the function $U(t, T)$.

If we differentiate (37) with respect to T , we have

$$\frac{dW(T)}{dT} = J(T, T)K_{xy}^T(T, T)\Phi^T(0, T) + \int_0^T \frac{\partial J(T, s')}{\partial T}K_{xy}^T(s', s')\Phi^T(0, s')ds'. \tag{39}$$

If we substitute (32) into (39) and use (37), we obtain the differential equation

$$\frac{dW(T)}{dT} = J(T, T)(K_{xy}^T(T, T)\Phi^T(0, T) - H\Phi(T, 0)W(T)) \tag{40}$$

for the function $W(T)$. The initial condition on the differential equation (40) at $T=0$ is $W(0)=0$ from (37).

The function $J(T, T)$ in (40) is formulated as follows. If we put $s=T$ in (29) and substitute the expression for $K_{xy}(s', T)$ into the resultant equation, we have

$$\begin{aligned} J(T, T)\Xi &= \Phi(0, T)K_{xy}(T, T) - \int_0^T J(T, s')HK_{xy}(s', T)ds' \\ &= \Phi(0, T)K_{xy}(T, T) - \int_0^T J(T, s')K_{xy}^T(s', s')\Phi^T(T, s')H^T ds'. \end{aligned} \tag{41}$$

From (37) and (41), we obtain

$$J(T, T) = (\Phi(0, T)K_{xy}(T, T) - W(T)\Phi^T(T, 0)H^T)\Xi^{-1}. \tag{42}$$

If we differentiate (11) for the fixed-point smoothing estimate $\hat{x}(t, T)$ with respect to T , and use (27) and (30), we have

$$\frac{\partial \hat{x}(t, T)}{\partial T} = h(t, T, T)y(T) - h(t, T, T)H\Phi(T, 0)\int_0^T J(T, s)y(s)ds. \tag{43}$$

If we introduce a function

$$e(T) = \int_0^T J(T, s)y(s)ds, \quad (44)$$

we obtain the differential equation

$$\frac{\partial \hat{x}(t, T)}{\partial T} = h(t, T, T)(y(T) - H\Phi(T, 0)e(T)) \quad (45)$$

for $\hat{x}(t, T)$.

If we differentiate (44) with respect to T , and use (32) and (44), we obtain

$$\frac{de(T)}{dT} = J(T, T)(y(T) - H\Phi(T, 0)e(T)). \quad (46)$$

The initial condition on the differential equation (46) for $e(T)$ at $T=0$ is $e(0)=0$ from (44).

From (11), the filtering estimate $\hat{x}(T, T)$ is written as

$$\hat{x}(T, T) = \int_0^T h(T, s, T)y(s)ds. \quad (47)$$

Let us derive the equation for $h(T, s, T)$. From (14), we have

$$h(T, s, T)\Xi = K_{xy}(T, s) - \int_0^T h(T, s', T)HK_{xy}(s', s)ds'. \quad (48)$$

If we compare (48) with (29), we obtain

$$h(T, s, T) = \Phi(T, 0)J(T, s). \quad (49)$$

If we substitute (49) into (47), we obtain

$$\hat{x}(T, T) = \Phi(T, 0)e(T) \quad (50)$$

from (44).

In the calculation of (38), the initial condition of $U(t, T)$ at $T=t$ is necessary. If we put $T=t$ in (34), we have

$$U(t, t) = \int_0^t h(t, s', t)K_{xy}^T(s', s')\Phi^T(0, s')ds'. \quad (51)$$

From (37), (49) and (51), we obtain

$$U(t, t) = \Phi(t, 0)W(t). \quad (52)$$

□

Now, in [Theorem 2], starting with the stochastic estimation algorithms of [Theorem 1], we propose the recursive Wiener fixed-point smoother and filter. The recursive Wiener smoother

