Design of Linear Stationary Stochastic Estimators in Relation to H_∞ Estimation Technique

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Abstract

This paper designs the fixed-point smoother and filter suitable for estimating the wide-sense stationary stochastic signal in relation to the H_∞ estimation approach in continuous-time systems. Performance measure for the design of the estimators is newly introduced by referring to that of the infinite-horizon H_∞ estimation problem in the Krein spaces [1],[2]. At first, we propose the estimation algorithms using the covariance information in linear continuous wide-sense stationary stochastic systems. Secondly, to improve the estimation accuracy of the recursive least-squares (RLS) estimators [3] using the covariance information, the suboptimal fixed-point smoother and filter using the covariance information are proposed. Finally, the recursive H_∞ like fixed-point smoother and filter using the state-space parameters are derived from those using the covariance information in a unified manner in linear continuous wide-sense stationary stochastic systems.

1. Introduction

Recently, by use of the state-space parameters, the H_∞ and its related estimation techniques [1],[2],[4]-[8] have attracted great attention in the deterministic and stochastic estimation methods of signal. Incidentally, as an alternative approach to the least-squares estimation problem based on the state-space model, the recursive Wiener fixed-point smoother and filter using the covariance information of the signal and the observation noise are developed in linear continuous stochastic systems [3].

The performance criterion concerned with the H_∞ estimation problem is formulated in the deterministic manner by nature. The problem formulation in the finite-horizon and infinite-horizon H_∞ estimation problems is limited within deterministic representations and would not fit to the estimation in linear stationary stochastic systems. In [1],[2] the deterministic H_∞ technique in the Krein spaces is developed. In the H_∞ estimation problem [1],[2], the perfor-

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mance criterion is represented as Eq.(4) in section 2. In the H_2 estimation problem, the value of γ_∞^2(see Eq.(4)) is set to ∞[1],[2],[4]-[8]. Consequently, the estimation accuracy of the H_∞ estimator is superior to the H_2 estimator. Taking into account of these aspects, we introduce the performance criterion newly. We examine to design the estimators, which correspond to the H_∞ estimator, to improve the estimation accuracy in comparison with the RLS Wiener estimators [3], which correspond to the H_2 estimator in the H_∞ estimation problem, in linear wide-sense stationary stochastic systems [9]. At first, in section 2, the stochastic signal estimation problem is introduced for the estimation of the wide-sense stationary signal. Based on the current performance criterion (see Eq.(5)), as in the deterministic H_∞ estimation technique [1],[2], we obtain the observation equation in the wide-sense stationary stochastic systems. Assuming that the observation equation is given, we consider the linear least-squares estimation problem using the covariance information in wide-sense stationary stochastic systems.

In the observed values, an artificial observed value $\breve{z}(t)$ (see Eq.(9)) is included. In [Theorem 1], by use of the covariance information of the signal and the observation noise, recursive algorithms for the fixed-point smoothing and filtering estimates of the signal $z(t)$ (see Eq.(8)) are proposed. In [Theorem 2], based on the algorithms of [Theorem 1], recursive Wiener fixed-point smoother and filter using the covariance information are proposed. Recursive Wiener fixed-point smoother and filter for the signal $z(t)$ use the system matrix F, the observation matrix H (see Eq.(8)), the crosscovariance function $K_{xy}(t,T)$ of the state variable $x(t)$ with observed value $y(T)$, the crossvariance function $K_{xy}(T,T)$, the variance Ξ of the observation noise (see Eq.(10)) and the observed value $y(T)$ (see Eq.(9)). From the estimation equations in [Theorem 2], [Theorem 3] formulates the algorithms using the covariance information for the fixed-point smoothing estimates $\hat{z}_1(t,T)$ and $\hat{z}_2(t,T)$ at the fixed point t and the filtering estimates $\hat{z}_1(T,T)$ and $\hat{z}_1(T,T)$ for the components $z_1(t)$ and $z_2(t)$ of the signal vector $z(t)$. According to the derivation of the H_∞ suboptimal estimators [2], we propose in [Theorem 4] the recursive suboptimal fixed-point smoother and filter using the covariance information by setting $\breve{z}(T)=\hat{z}_2(T,T)$ in the estimation algorithms of [Theorem 3]. Here, the estimation algorithms of [Theorem 4] necessitate the information of the observation matrices C and L (see Eq.(8)), the system matrix F, the autocovariance function $K_x(t,T)$ of the state variable $x(t)$, the autovariance function $K_x(T,T)$, the crosscovariance function $K_{xy_1}(t,T)$ of the state variable $x(t)$ with the observed value $y_1(T)$, the crossvariance function $K_{xy_1}(T,T)$, γ (see Eq.(5)) and the observed value $y_1(T)$(see Eq.(2)). For $\gamma^2=\infty$, the fixed-point smoothing and filtering algorithms in [Theorem 4] are reduced to the RLS Wiener algorithms for the fixed-point smoothing and filtering estimates in [3]. [Theorem 5] shows the optimal fixed-point smoothing and filtering algorithms using the state-space parameters. They are derived from the estimation algorithms of [Theorem 3] using the covariance information. The algorithms of [Theorem 5] uses the system matrix F, the observation matrices C and L, the input matrix B(see Eq.(1)), γ, the observed values $y_1(T)$ and $\check{z}(T)$. In [Theorem 6], assuming that the observed value $\check{z}(T)$ of $z_2(T)$ introduced artificially is equal to $\hat{z}_2(T,T)$ in the estimation equations of [Theorem 5], we propose the suboptimal algorithms using the state-space parameters for the fixed-point smoothing and filtering estimates of $z_1(t)$ and $z_2(t)$. The suboptimal filtering algorithm using the state-space parameters in [Theorem 6] is same as that based on the game theory approach [5] in linear continuous stochastic systems. For $\gamma^2=\infty$, the suboptimal filter is reduced to the Kalman filter. The suboptimal fixed-point smoothing algorithm in [Theorem 6] is proposed for the first time in this context.

2. Problem Formulation

Let the linear time-invariant state-space model for the state variable $x(t)$ be given by

$$\frac{dx(t)}{dt} = Fx(t) + Bu(t), x(0) = x_0,$$

$$E[u(t)u^T(s)] = \Pi_0\, \delta(t-s). \tag{1}$$

Here, $x(t) \in R_n$, $u(t) \in R_l$. F represents the system matrix and B the input matrix for $u(t)$. Let $z_1(t)$ represent a signal expressed by $z_1(t)=Cx(t)$. We assume that the signal $z_1(t)$ is observed with additive white Gaussian noise $v_1(t)$.

$$y_1(t)=z_1(t)+v_1(t), \qquad z_1(t)=Cx(t), \qquad E[v_1(t)v_1^T(s)] = R\delta(t-s) \tag{2}$$

Here, $y_1(t)$ represents the observed value and C m by n observation matrix. We assume that $v_1(t)$ and $u(t)$ are uncorrelated. We also assume that (F,C) is observable and x_0 is a random variable with the mean zero and the variance Q_0.

Let a signal $z_2(t)$ be represented by

$$z_2(t)=Lx(t), \tag{3}$$

where L is r by n vector.

Let $L_2[O,T]$ denote the usual Hilbert space of square integrable functions. In the finite-horizon H_∞ estimation problem, the estimators are designed so as to achieve the following performance measure [2]:

$$\sup_{x_0,u(t)\in L_2,x(t),v_1(t)\in L_2} \frac{\int_0^T (\breve{z}(t)-Lx(t))^T(\breve{z}(t)-Lx(t))dt}{x_0^T\hat{Q}_0^{-1}x_0+\int_0^T u^T(t)u(t)dt+\int_0^T v_1^T(t)v_1(t)dt} < \gamma_\infty^2 \tag{4}$$

Here, $\breve{z}(t)$ is the artificial observed value for $z_2(t)$ and $\hat{Q}_0$ is a definite weighting matrix. $\hat{Q}_0$ reflects a priori knowledge as to how close $x(0)$ is to its initial guess. We assume that initial guess of $x(0)$ is zero without loss of generality. For the case of $T=\infty$, the performance criterion (4) is concerned with the infinite-horizon H_∞ estimation problem. In the H_∞ estimation problem, we make no assumption of the nature of the noises disturbances $u(t)$ and $v_1(t)$ (e.g. normally distributed, uncorrelated, etc.) and consider to estimate $z_2(t)=Lx(t)$. It is clear that the problem formulation for the H_∞ estimation problem would not fit to treat the estimation problem which assumes the priori statistics for $u(t)$ and $v_1(t)$ in (1) and (2) in wide-sense stationary stochastic systems [9].

In this paper, instead of the H_∞ estimation problem, we consider to estimate the signal $z_2(t)=Lx(t)$ in the wide-sense stationary stochastic systems by taking into account of the statistical properties for $u(t)$ and $v_1(t)$ in (1) and (2). Along with this idea, we newly introduce the performance criterion represented by

$$\sup \frac{E[(\breve{z}(t)-Lx(t))^T(\breve{z}(t)-Lx(t))]}{E[x_0^TQ_0^{-1}x_0]+E[u^T(t)\Pi_0^{-1}u(t)]+E[(y_1(t)-Cx(t))^TR^{-1}(y_1(t)-Cx(t))]} < \gamma^2 \tag{5}$$

in the wide-sense stationary stochastic systems. Under the criterion of (5), we can now treat the estimation problem for the wide-sense stationary stochastic signal process in which the ergodic process is included. If we introduce the stochastic quantity of the form

$$\begin{aligned} J_f &= E[x_0^TQ_0^{-1}x_0]+E[u^T(t)\Pi_0^{-1}u(t)]+E[(y_1(t)-Cx(t))^TR^{-1}(y_1(t)-Cx(t))] \\ &\quad -\gamma^{-2}E[(\breve{z}(t)-Lx(t))^T(\breve{z}(t)-Lx(t))], \end{aligned} \tag{6}$$

we find that the performance criterion (5) in the current stochastic estimation problem is transformed into the relationship satisfying $J_f > 0$. Henceforth,

$$\begin{aligned} J_f &= E[x_0^T(0)Q_0^{-1}x_0]+E[u^T(t)\Pi_0^{-1}u(t)]+ \\ &E\left[\left(\begin{bmatrix} y_1(t) \\ \breve{z}(t) \end{bmatrix}-\begin{bmatrix} C \\ L \end{bmatrix}x(t)\right)^T\begin{bmatrix} R & 0 \\ 0 & -\gamma^2 I \end{bmatrix}^{-1}\left(\begin{bmatrix} y_1(t) \\ \breve{z}(t) \end{bmatrix}-\begin{bmatrix} C \\ L \end{bmatrix}x(t)\right)\right] > 0 \end{aligned} \tag{7}$$

is obtained in relation to the H_∞ estimation problem [2]. Let us introduce the vector $z(t)$ which consists of the signals $z_1(t)$ and $z_2(t)$.

$$\begin{aligned} z(t) &= Hx(t) \\ &= \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}, \quad H = \begin{bmatrix} C \\ L \end{bmatrix}, \quad z_1(t) = Cx(t), \quad z_2(t) = Lx(t) \end{aligned} \tag{8}$$

Let us also introduce the observation vector $y(t)$

$$y(t) = \begin{bmatrix} y_1(t) \\ \breve{z}(t) \end{bmatrix} \tag{9}$$

which consists of $y_1(t)$ and the artificial observation $\breve{z}(t)$ for $z_2(t)$. As in the observation equation in the Krein spaces [2], by checking the condition for a minimum of $J_f > 0$, the observation equation in the stationary stochastic continuous-time systems might be written as

$$\begin{aligned} &y(t) = Hx(t) + v(t), \quad z(t) = Hx(t), \quad v(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}, \\ &E\left[v(t)v^T(s)\right] = \Xi\delta(t-s), \quad \Xi = \begin{bmatrix} R & 0 \\ 0 & -r^2 \end{bmatrix}. \end{aligned} \tag{10}$$

The observation equation (10) is analogous to that in the Krein spaces [2] of the linear deterministic H_∞ estimation problem. Provided that the observation equation (10) is given, we consider the stochastic estimation problem in the sense of linear least-squares estimation problem for the fixed-point smoothing and filtering estimates of the signal $z(t)$ and the filtering estimate of the state variable $x(t)$.

It should be noted that the performance criterion even in the infinite-horizon H_∞ estimation problem is distinct from the current one given by (5). In the H_∞ estimation problem, we do not assume a priori knowledge on the variances of the noises $u(\bullet)$ and $v_1(\bullet)$ with their uncorrelation property.

Let the fixed-point smoothing estimate $\hat{x}(t,T)$ of the state variable $x(t)$ be expressed as

$$\hat{x}(t,T) = \int_0^T h(t,s,T)y(s)ds \tag{11}$$

as an integral transformation of the observed data set $\{y(s), 0 \leqq s \leqq T\}$. Here, $h(t,s,T)$ represents the impulse response function. "t" is referred to as the fixed point. The fixed-point smoothing estimate of the signal $z(t)$ is expressed as $\hat{z}(t,T)=H\hat{x}(t,T)$. We consider the linear least-squares smoothing problem which minimizes the cost function

$$J = E\|x(t) - \hat{x}(t,T)\|^2 \tag{12}$$

in linear continuous-time stochastic systems, given the observation equation (10). Let $K_{xy}(t,s)$ represent the crosscovariance function of the state variable $x(t)$ with the observed value $y(s)$. The optimal impulse response function, which minimizes (12), satisfies the Wiener-Hopf integral equation [3]

$$K_{xy}(t,s) = \int_0^T h(t,s',T)E[y(s')y^T(s)]ds'. \tag{13}$$

If we substitute (10) into (13), we obtain

$$h(t,s,T)\Xi = K_{xy}(t,s) - \int_0^T h(t,s',T)HK_{xy}(s',s)ds', \tag{14}$$

since the variance of $v(t)$ is Ξ from (10).

In sections 3 and 4, by use of the covariance information of the signal $z(t)$ and the observation noise $v(t)$, we present the recursive algorithms for the fixed-point smoothing estimate $\hat{z}(t,T)$ of the signal $z(t)$ and the filtering estimates $\hat{z}(T,T)$ of $z(T)$ and $\hat{x}(T,T)$ of $x(T)$.

3. Recursive Wiener Smoother and Filter

Let $\Phi(T,0)$ represent the state transition matrix of the system matrix F. $\Phi(t,s)$, $0 \le s \le t$, satisfies $\dfrac{\partial\Phi(t,s)}{\partial t} = F\Phi(t,s)$. Let $K_z(t,s)$ represent the autocovariance function of the signal $z(t)$. $K_z(t,s)$ is expressed as

$$K_z(t,s) = H\Phi(t,s)K_{xy}(s,s)1(t-s) + K_{xy}^T(t,t)\Phi^T(s,t)H^T 1(s-t), \tag{15}$$

where $1(t-s)$ represents the unit step function. [Theorem 1] presents the fixed-point smoothing and filtering algorithms of $z(t)$ using the crosscovariance function $K_{xy}(t,T)$ of the state variable $x(t)$ with the observed value $y(T)$, the crossvariance function $K_{xy}(T,T)$, the state transition matrix $\Phi(T,0)$, the observation matrix H, the variance Ξ of the observation noise $v(t)$ and the observed value $y(T)$.

[Theorem 1]

Let the variance of the initial value x_0 of $x(t)$ at $t = 0$ be $Q_0 > 0$. Let the observation equation be given by (10). Then the recursive algorithms for the fixed-point smoothing estimate $\hat{z}(t,T)$ at the fixed point t and the filtering estimates $\hat{z}(T,T)$ of $z(T)$ and $\hat{x}(T,T)$ of $x(T)$, which achieve the criterion of (5), consist of (16) ~ (24). Here, the estimators use the crosscovariance function $K_{xy}(t,T)$ of the state variable $x(t)$ with the observed value $y(T)$, the crossvariance function $K_{xy}(T,T)$, the state-transition matrix $\Phi(T,0)$ for the system matrix F, the observation matrix H, the variance Ξ of the observation noise $v(t)$ and the observed value $y(T)$.

$\hat{z}(t,T)$: Fixed-point smoothing estimate of the signal $z(t)$ at the fixed point t.

$$\frac{\partial \hat{z}(t,T)}{\partial T} = Hh(t,T,T)(y(T) - \hat{z}(T,T)) \tag{16}$$

$$h(t,T,T) = (K_{xy}(t,T) - U(t,T)\Phi^T(T,0)H^T)\Xi^{-1} \tag{17}$$

$$\frac{\partial U(t,T)}{\partial T} = h(t,T,T)(K_{xy}^T(T,T)\Phi^T(0,T) - H\Phi(T,0)W(T)) \tag{18}$$

$$U(T,T) = \Phi(T,0)W(T) \tag{19}$$

$$J(T,T) = (\Phi(0,T)K_{xy}(T,T) - W(T)\Phi^T(T,0)H^T)\Xi^{-1} \tag{20}$$

$$\frac{dW(T)}{dT} = J(T,T)(K_{xy}^T(T,T)\Phi^T(0,T) - H\Phi(T,0)W(T)),$$
$$W(0)=0 \tag{21}$$

$\hat{z}(T,T)$: Filtering estimate of the signal $z(T)$.

$$\hat{z}(T,T)=H\hat{x}(T,T) \tag{22}$$

$\hat{x}(T,T)$: Filtering estimate of the state variable $x(T)$.

$$\hat{x}(T,T)=\Phi(T,0)e(T) \tag{23}$$

$$\frac{de(T)}{dT} = J(T,T)(y(T) - H\Phi(T,0)e(T)), \quad e(0) = 0 \tag{24}$$

Proof: Let us differentiate (14) with respect to T.

$$\frac{\partial h(t,s,T)}{\partial T}\Xi = -h(t,T,T)HK_{xy}(T,s) - \int_0^T \frac{\partial h(t,s',T)}{\partial T}HK_{xy}(s',s)ds' \tag{25}$$

If we introduce an auxiliary function $q(T,s)$ which satisfies

$$q(T,s)\Xi = HK_{xy}(T,s) - \int_0^T q(T,s')HK_{xy}(s',s)ds', \tag{26}$$

we obtain the differential equation

$$\frac{\partial h(t,s,T)}{\partial T} = -h(t,T,T)q(T,s) \tag{27}$$

for $h(t,s,T)$ by comparing (25) with (26). (26) is written as

$$q(T,s)\Xi = H\Phi(T,0)\Phi(0,s)K_{xy}(s,s) - \int_0^T q(T,s')HK_{xy}(s',s)ds' \tag{28}$$

by using the property of the transition matrix $\Phi(T,s)$. If we introduce an auxiliary function $J(T,s)$ which satisfies

$$J(T,s)\Xi = \Phi(0,s)K_{xy}(s,s) - \int_0^T J(T,s')HK_{xy}(s',s)ds', \tag{29}$$

we obtain

$$q(T,s) = H\Phi(T,0)J(T,s). \tag{30}$$

If we differentiate (29) with respect to T, we have

$$\frac{\partial J(T,s)}{\partial T}\Xi = -J(T,T)HK_{xy}(T,s) - \int_0^T \frac{\partial J(T,s')}{\partial T}HK_{xy}(s',s)ds'. \tag{31}$$

From (28), (30) and (31), we obtain the differential equation

$$\begin{aligned}\frac{\partial J(T,s)}{\partial T} &= -J(T,T)q(T,s)\\ &= -J(T,T)H\Phi(T,0)J(T,s)\end{aligned} \tag{32}$$

for the function $J(T,s)$.

Now, the function $h(t,T,T)$ in (27) can be formulated as follows. If we put $s=T$ in (14) and use the expression of $K_{xy}(s',T)$ for $0 \le s' \le T$, i.e., $HK_{xy}(s',T) = K_{xy}^T(s',s')\Phi^T(T,s')H^T$, we have

$$\begin{aligned}h(t,T,T)\Xi &= K_{xy}(t,T) - \int_0^T h(t,s',T)HK_{xy}(s',T)ds'\\ &= K_{xy}(t,T) - \int_0^T h(t,s',T)K_{xy}^T(s',s')\Phi^T(T,s')H^T ds'.\end{aligned} \tag{33}$$

If we introduce a function

$$U(t,T) = \int_0^T h(t,s',T)K_{xy}^T(s',s')\Phi^T(0,s')ds', \tag{34}$$

we obtain

$$h(t,T,T) = (K_{xy}(t,T) - U(t,T)\Phi^T(T,0)H^T)\Xi^{-1}. \tag{35}$$

If we differentiate (34) with respect to T, and use (27) and (30), we have

$$\frac{\partial U(t,T)}{\partial T} = h(t,T,T)K_{xy}^T(T,T)\Phi^T(0,T) - h(t,T,T)H\Phi(T,0)\int_0^T J(T,s')K_{xy}^T(s',s')\Phi^T(0,s')ds'. \tag{36}$$

If we introduce a function

$$W(T) = \int_0^T J(T,s')K_{xy}^T(s',s')\Phi^T(0,s')ds', \tag{37}$$

we obtain the differential equation

$$\frac{\partial U(t,T)}{\partial T} = h(t,T,T)(K_{xy}^T(T,T)\Phi^T(0,T) - H\Phi(T,0)W)(T)) \tag{38}$$

for the function *U(t,T)*.

If we differentiate (37) with respect to *T*, we have

$$\frac{dW(T)}{dT} = J(T,T)K_{xy}^T(T,T)\Phi^T(0,T) + \int_0^T \frac{\partial J(T,s')}{\partial T}K_{xy}^T(s',s')\Phi^T(0,s')ds'. \tag{39}$$

If we substitute (32) into (39) and use (37), we obtain the differential equation

$$\frac{dW(T)}{dT} = J(T,T)(K_{xy}^T(T,T)\Phi^T(0,T) - H\Phi(T,0)W(T)) \tag{40}$$

for the function *W(T)*. The initial condition on the differential equation (40) at *T=0* is *W(0)=0* from (37).

The function *J(T,T)* in (40) is formulated as follows. If we put *s=T* in (29) and substitute the expression for $K_{xy}(s',T)$ into the resultant equation, we have

$$\begin{aligned} J(T,T)\Xi &= \Phi(0,T)K_{xy}(T,T) - \int_0^T J(T,s')HK_{xy}(s',T)ds' \\ &= \Phi(0,T)K_{xy}(T,T) - \int_0^T J(T,s')K_{xy}^T(s',s')\Phi^T(T,s')H^T ds'. \end{aligned} \tag{41}$$

From (37) and (41), we obtain

$$J(T,T) = (\Phi(0,T)K_{xy}(T,T) - W(T)\Phi^T(T,0)H^T)\Xi^{-1}. \tag{42}$$

If we differentiate (11) for the fixed-point smoothing estimate $\hat{x}(t,T)$ with respect to *T*, and use (27) and (30), we have

$$\frac{\partial \hat{x}(t,T)}{\partial T} = h(t,T,T)y(T) - h(t,T,T)H\Phi(T,0)\int_0^T J(T,s)y(s)ds. \tag{43}$$

If we introduce a function

$$e(T)=\int_0^T J(T,s)y(s)ds, \tag{44}$$

we obtain the differential equation

$$\frac{\partial\hat{x}(t,T)}{\partial T}=h(t,T,T)(y(T)-H\Phi(T,0)e(T)) \tag{45}$$

for $\hat{x}(t,T)$.

If we differentiate (44) with respect to T, and use (32) and (44), we obtain

$$\frac{de(T)}{dT}=J(T,T)(y(T)-H\Phi(T,0)e(T)). \tag{46}$$

The initial condition on the differential equation (46) for $e(T)$ at $T=0$ is $e(0)=0$ from (44).

From (11), the filtering estimate $\hat{x}(T,T)$ is written as

$$\hat{x}(T,T)=\int_0^T h(T,s,T)y(s)ds. \tag{47}$$

Let us derive the equation for $h(T,s,T)$. From (14), we have

$$h(T,s,T)\Xi=K_{xy}(T,s)-\int_0^T h(T,s',T)HK_{xy}(s',s)ds'. \tag{48}$$

If we compare (48) with (29), we obtain

$$h(T,s,T)=\Phi(T,0)J(T,s). \tag{49}$$

If we substitute (49) into (47), we obtain

$$\hat{x}(T,T)=\Phi(T,0)e(T) \tag{50}$$

from (44).

In the calculation of (38), the initial condition of $U(t,T)$ at $T=t$ is necessary. If we put $T=t$ in (34), we have

$$U(t,t)=\int_0^t h(t,s',t)K_{xy}^T(s',s')\Phi^T(0,s')ds'. \tag{51}$$

From (37), (49) and (51), we obtain

$$U(t,t)=\Phi(t,0)W(t). \tag{52}$$

□

Now, in [Theorem 2], starting with the stochastic estimation algorithms of [Theorem 1], we propose the recursive Wiener fixed-point smoother and filter. The recursive Wiener smoother and filter use the system matrix F in (1), the observation matrix H in (10), the crosscovariance function $K_{xy}(t,T)$ of the state variable $x(t)$ with the observed value $y(T)$, the crossvariance function $K_{xy}(T,T)$, the variance Ξ of the observation noise $v(t)$, and the observed value $y(T)$.

[Theorem 2]

Let the autocovariance function $K_z(t,s)$ of the signal $z(t)$ be expressed as (15), and let the information of the system matrix F, the observation matrix H, the crosscovariance function $K_{xy}(t,T)$ of the state variable $x(t)$ with the observed value $y(T)$, the crossvariance function $K_{xy}(T,T)$, the variance Ξ of the observation noise $v(t)$ and the observed value $y(T)$ be given. Then the recursive Wiener algorithms for the fixed-point smoothing estimate $\hat{z}(t,T)$ at the fixed point t and the filtering estimates $\hat{z}(T,T)$ of $z(T)$ and $\hat{x}(T,T)$ of $x(T)$ consist of (53)~(59).

$\hat{z}(t,T)$: Fixed-point smoothing estimate of the signal $z(t)$ at the fixed point t.

$$\frac{\partial \hat{z}(t,T)}{\partial T} = Hh(t,T,T)(y(T) - \hat{z}(T,T)) \tag{53}$$

$$h(t,T,T) = (K_{xy}(t,T) - p(t,T)H^T)\Xi^{-1} \tag{54}$$

$p(t,T)$: Autovariance function of the fixed-point smoothing estimate $\hat{x}(t,T)$.

$$\frac{\partial p(t,T)}{\partial T} = h(t,T,T)(K_{xy}^T(T,T) - Hp(T,T)) + p(t,T)F^T \tag{55}$$

$p(T,T)$: Autovariance function of the filtering estimate $\hat{x}(T,T)$.

$$\frac{dp(T,T)}{dT} = Fp(T,T) + p(T,T)F^T + h(t,T,T)(K_{xy}^T(T,T) - Hp(T,T)),$$

$$p(0,0)=0 \tag{56}$$

$\hat{z}(T,T)$: Filtering estimate of the signal $z(T)$.

$$\hat{z}(T,T)=H\hat{x}(T,T) \tag{57}$$

$\hat{x}(T,T)$: Filtering estimate of the state variable $x(T)$.

$$\frac{d\hat{x}(T,T)}{dT} = F\hat{x}(T,T) + h(T,T,T)(y(T) - H\hat{x}(T,T)),$$

$$\hat{x}(0,0)=0 \tag{58}$$

$$h(T,T,T)=(K_{xy}(T,T)-p(T,T)H^T)\Xi^{-1} \tag{59}$$

Proof: From (45) and (50), we obtain

$$\frac{\partial\hat{x}(t,T)}{\partial T}=h(t,T,T)(y(T)-H\hat{x}(T,T)). \tag{60}$$

If we put

$$p(t,T)=U(t,T)\Phi^T(T,0), \tag{61}$$

we can rewrite (17) as

$$h(t,T,T)=(K_{xy}(t,T)-p(t,T)H^T)\Xi^{-1}. \tag{62}$$

If we differentiate (61) with respect to T, we have

$$\frac{\partial p(t,T)}{\partial T}=[h(t,T,T)(K_{xy}^T(T,T)\Phi^T(0,T)-H\Phi(T,0)W(T))]\Phi^T(T,0)+p(t,T)F^T \tag{63}$$

from (38) and the relationship $\frac{\partial\Phi(T,0)}{\partial T}=F\Phi(T,0)$. If we take the relationship

$$\begin{aligned}p(T,T)&=U(T,T)\Phi^T(T,0)\\&=\Phi(T,0)W(T)\Phi^T(T,0),\end{aligned} \tag{64}$$

from (19) and (61), into consideration, we obtain the differential equation

$$\frac{\partial p(t,T)}{\partial T}=h(t,T,T)(K_{xy}^T(T,T)-Hp(T,T))+p(t,T)F^T \tag{65}$$

for the function $p(t,T)$.

Let us differentiate (64) with respect to T.

$$\frac{dp(T,T)}{dT}=\frac{\partial\Phi(T,0)}{\partial T}W(T)\Phi^T(T,0)+\Phi(T,0)W(T)\frac{\partial\Phi^T(T,0)}{\partial T}+\Phi(T,0)\frac{dW(T)}{dT}\Phi^T(T,0) \tag{66}$$

If we use the relationship $\frac{\partial\Phi(T,0)}{\partial T}=F\Phi(T,0)$, we can rewrite (66) as

$$\begin{aligned}\frac{dp(T,T)}{dT}&=Fp(T,T)+p(T,T)F^T+\Phi(T,0)J(T,T)(K_{xy}^T(T,T)\Phi^T(0,T)-H\Phi(T,0)W(T))\Phi^T(T,0)\\&=Fp(T,T)+p(T,T)F^T+h(T,T,T)(K_{xy}^T(T,T)-Hp(T,T))\end{aligned} \tag{67}$$

from (21), (49) and (64). The initial condition on the differential equation (67) for $p(T,T)$ at $T=0$ is $p(0,0)=0$ from (37) and (64).

If we differentiate (50) with respect to T, and use (24), (49) and (50), we obtain the differential equation

$$\begin{aligned}\frac{d\hat{x}(T,T)}{dT} &= F\hat{x}(T,T) + \Phi(T,0)J(T,T)(y(T) - H\Phi(T,0)e(T)) \\ &= F\hat{x}(T,T) + h(T,T,T)(y(T) - H\hat{x}(T,T))\end{aligned} \tag{68}$$

for the filtering estimate $\hat{x}(T,T)$.

The filter gain $h(T,T,T)$ in (68) can be formulated as follows. We obtain

$$\begin{aligned}h(T,T,T) &= \Phi(T,0)J(T,T) \\ &= \Phi(T,0)(\Phi(0,T)K_{xy}(T,T) - W(T)\Phi^T(T,0)H^T)\Xi^{-1} \\ &= (K_{xy}(T,T) - p(T,T)H^T)\Xi^{-1}\end{aligned} \tag{69}$$

in terms of (20), (49) and (64).

From linear estimation theory [9], we find that $P(t,T)$ in (62) and $P(T,T)$ in (69) represent the autovariance functions of the fixed-point smoothing estimate $\hat{x}(t,T)$ and the filtering estimate $\hat{x}(T,T)$ respectively. □

The recursive Wiener fixed-point smoother and filter have been derived in [Theorem 2] based on the invariant imbedding method [3] in a unified manner. In [Theorem 2], the stochastic estimation algorithms for the fixed-point smoothing estimates $\hat{z}_1(t,T)$ and $\hat{z}_2(t,T)$ and the filtering estimates $\hat{z}_1(T,T)$ and $\hat{z}_2(T,T)$ of $z_1(T)$ and $z_2(T)$ are not given explicitly against those in [1],[2],[4]-[8] for the H_∞ estimation problem. Hence, in [Theorem 3], we formulate the estimation algorithms for the fixed-point smoothing estimates $\hat{z}_1(t,T)$ and $\hat{z}_2(t,T)$ and the filtering estimates $\hat{z}_1(T,T)$ and $\hat{z}_2(T,T)$ by expanding the signal vector $z(T)$ and the function vector $h(t,T,T)$ in the algorithms of [Theorem 2] into their vector components as $z(T)=[z_1(T)\ z_2(T)]^T$ and $h(t,T,T)=[h_1(t,T,T)\ \ h_2(t,T,T)]$.

[Theorem 3]

Let $\hat{z}_1(t,T)$ and $\hat{z}_2(t,T)$ represent the smoothing estimates of $z_1(t)$ and $z_2(t)$ at the fixed point t respectively. Let $\hat{z}_1(T,T)$ and $\hat{z}_2(T,T)$ represent the filtering estimates of $z_1(T)$ and $z_2(T)$ respectively. Let $y_1(T)(=Cx(T)+v_1(T))$ and $\breve{z}(T)(=Lx(T)+v_2(T))$ be the observed values defined in the observation equation (10). Let the information of the system matrix F, the observation matrices C and L, the crosscovariance function $K_{xy_1}(t,T)$ of the state variable $x(t)$ with the observed value $y_1(T)$, the crossvariance function $K_{xy_1}(T,T)$, the autocovariance function $K_x(t,T)$ of the state variable $x(t)$, the autovariance function $K_x(T,T)$ of the state variable $x(T)$, γ, the observed value $y_1(T)$ and the artificial observed value $\breve{z}(T)$ be given. Then the recursive Wiener algorithms for the fixed-point smoothing estimates $\hat{z}_1(t,T)$ of the signal $z_1(t)$ and $\hat{z}_2(t,T)$ of the signal $z_2(t)$ at the fixed point t and the filtering estimates $\hat{z}_1(T,T)$, $\hat{z}_2(T,T)$ and $\hat{x}(T,T)$ consist of (70) ～ (78).

$\hat{z}_1(t,T)$: Fixed-point smoothing estimate of the signal $z_1(t)$ at the fixed point t.

$$\frac{\partial \hat{z}_1(t,T)}{\partial T} = C\{[K_{xy_1}(t,T) - p(t,T)C^T]R^{-1}[y_1(T) - \hat{z}_1(T,T)] - \gamma^{-2}[K_x(t,T)L^T - p(t,T)L^T][\breve{z}(T) - \hat{z}_2(T,T)]\} \tag{70}$$

$\hat{z}_2(t,T)$: Fixed-point smoothing estimate of the signal $z_2(t)$ at the fixed point t.

$$\frac{\partial \hat{z}_2(t,T)}{\partial T} = L\{[K_{xy_1}(t,T) - p(t,T)C^T]R^{-1}[y_1(T) - \hat{z}_1(T,T)] - \gamma^{-2}[K_x(t,T)L^T - p(t,T)L^T][\breve{z}(T) - \hat{z}_2(T,T)]\} \tag{71}$$

$$h_1(t,T,T) = [K_{xy_1}(t,T) - p(t,T)C^T]R^{-1},\quad K_{xy_1}(t,T) = K_x(t,T)C^T \tag{72}$$

$$h_2(t,T,T) = -\gamma^{-2}[K_x(t,T) - p(t,T)]L^T \tag{73}$$

$p(t,T)$: Autovariance function of the fixed-point smoothing estimate $\hat{x}(t,T)$.

$$\frac{\partial p(t,T)}{\partial T} = h_1(t,T,T)[K_{xy_1}^T(T,T) - Cp(T,T)] + h_2(t,T,T)[LK_x(T,T) - Lp(T,T)] + p(t,T)F^T \tag{74}$$

$p(T,T)$: Autovariance function of the filtering estimate $\hat{x}(T,T)$.

$$\frac{dp(T,T)}{dT} = Fp(T,T) + p(T,T)F^T + [K_{xy_1}(T,T) - p(T,T)C^T]R^{-1}[K_{xy_1}^T(T,T) - Cp(T,T)] - \gamma^{-2}[K_x(T,T)L^T - p(T,T)L^T][LK_x(T,T) - Lp(T,T)],\quad p(0,0)=0 \tag{75}$$

$\hat{z}_1(T,T)$: Filtering estimate of the signal $z_1(T)$.

$$\hat{z}_1(T,T) = C\hat{x}(T,T) \tag{76}$$

$\hat{z}_2(T,T)$: Filtering estimate of the signal $z_2(T)$.

$$\hat{z}_2(T,T)=L\hat{x}(T,T) \tag{77}$$

$\hat{x}(T,T)$: Filtering estimate of the state variable $x(T)$.

$$\frac{d\hat{x}(T,T)}{dT}=F\hat{x}(T,T)+(K_{xy_1}(T,T)-p(T,T)C^T)R^{-1}(y_1(T)-C\hat{x}(T,T))-$$
$$\gamma^{-2}(K_x(T,T)L^T-p(T,T)L^T)(\breve{z}(T)-L\hat{x}(T,T)),$$
$$\hat{x}(0,0)=0 \tag{78}$$

Here, $K_x(t,T)$ and $K_{xy_1}(t,T)$ are calculated by

$$K_x(t,T)=K_x(T,T)\Phi^T(t,T), \qquad \Phi(t,T)=\Phi^{-1}(T,t),$$
$$\frac{\partial\Phi(T,t)}{\partial T}=F\Phi(T,t), \quad K_{xy_1}(t,T)=K_x(t,T)C^T,$$

in terms of $K_x(T,T)$ and F.

Proof: Let $h(t,T,T)$ be expressed by

$$h(t,T,T)=[h_1(t,T,T) \quad h_2(t,T,T)]. \tag{79}$$

If we substitute (8), (9), (10) and (79) into the estimation equations of [Theorem 2], we readily obtain the recursive Wiener algorithms for the fixed-point smoothing and filtering estimates in [Theorem 3]. □

If we put $\breve{z}(T)=\hat{z}_2(T,T)(=L\hat{x}(T,T))$ in [Theorem 3], in accordance with the derivation of the H_∞ suboptimal filter in [2], we obtain the recursive suboptimal estimation algorithms of [Theorem 4] by use of the covariance information.

[Theorem 4]

Let $\hat{z}_1(t,T)$ and $\hat{z}_2(t,T)$ represent the smoothing estimates of $\hat{z}_1(t)$ and $\hat{z}_2(t)$ respectively. Let $\hat{z}_1(T,T)$ and $\hat{z}_2(T,T)$ represent the filtering estimates of $z_1(T)$ and $z_2(T)$ respectively. Let $y_1(T)(=Cx(T)+v_1(T))$ be the observed values defined in the observation equation (10). Let the information of the system matrix F, the observation matrices C and L, the crosscovariance function $K_{xy_1}(t,T)$ of the state variable $x(t)$ with the observed value $y_1(T)$, the crossvariance function $K_{xy_1}(T,T)$, the autocovariance function $K_x(t,T)$ of the state variable $x(t)$, the autovariance function $K_x(T,T)$ of the state variable $x(T)$, γ, the observed value $y_1(T)$ and the artificial observed value $\breve{z}(T)$ be given. Then the recursive suboptimal Wiener algorithms to those of [Theorem 3] for the fixed-point smoothing estimates $\hat{z}_1(t,T)$ of the signal $z_1(t)$ and $\hat{z}_2(t,T)$ of the signal $z_2(t)$ at the fixed point t and the filtering estimates $\hat{z}_1(T,T)$, $\hat{z}_2(T,T)$ and $\hat{x}(T,T)$ consist of (80) ~ (88).

$\hat{z}_1(t,T)$: Fixed-point smoothing estimate of the signal $z_1(t)$ at the fixed point t.

$$\frac{\partial z_1(t,T)}{\partial T} = C[K_{xy_1}(t,T) - p(t,T)C^T][y_1(T) - \hat{z}_1(T,T)] \tag{80}$$

$\hat{z}_2(t,T)$: Fixed-point smoothing estimate of the signal $z_2(t)$ at the fixed point t.

$$\frac{\partial \hat{z}_2(t,T)}{\partial T} = L[K_{xy_1}(t,T) - p(t,T)C^T][y_1(T) - \hat{z}_1(T,T)] \tag{81}$$

$$h_1(t,T,T) = [K_{xy_1}(t,T) - p(t,T)C^T]R^{-1}$$
$$K_{xy_1}(t,T) = K_x(t,T)C^T \tag{82}$$

$$h_2(t,T,T) = -\gamma^{-2}[K_x(t,T)L^T - p(t,T)L^T] \tag{83}$$

$p(t,T)$: Autovariance function of the fixed-point smoothing estimate $\hat{x}(t,T)$.

$$\frac{\partial p(t,T)}{\partial T} = h_1(t,T,T)[K_{xy_1}^T(T,T) - Cp(T,T)] + h_2(t,T,T)[LK_x(T,T) - Lp(T,T)] + p(t,T)F^T \tag{84}$$

$p(T,T)$: Autovariance function of the filtering estimate $\hat{x}(T,T)$.

$$\frac{dp(T,T)}{dT} = Fp(T,T) + p(T,T)F^T + [K_{xy_1}(T,T) - p(T,T)C^T]R^{-1}[K_{xy_1}^T(T,T) - Cp(T,T)] -$$
$$\gamma^{-2}[K_x(T,T)L^T - p(T,T)L^T][LK_x(T,T) - Lp(T,T)],$$
$$p(0,0)=0 \tag{85}$$

$\hat{z}_1(T,T)$: Filtering estimate of the signal $z_1(T)$.

$$\hat{z}_1(T,T)=C\hat{x}(T,T) \tag{86}$$

$\hat{z}_2(T,T)$: Filtering estimate of the signal $z_2(T)$.

$$\hat{z}_2(T,T)=L\hat{x}(T,T) \tag{87}$$

$\hat{x}(T,T)$: Filtering estimate of the state variable $x(T)$.

$$\frac{d\hat{x}(T,T)}{dT} = F\hat{x}(T,T) + (K_{xy_1}(T,T) - p(T,T)C^T)R^{-1}(y_1(T) - C\hat{x}(T,T)),$$
$$\hat{x}(0,0) = 0 \tag{88}$$

Here, $K_x(t,T)$ and $K_{xy_1}(t,T)$ are calculated by

$$K_x(t,T) = K_x(T,T)\Phi^T(t,T), \quad \Phi(t,T) = \Phi^{-1}(T,t),$$
$$\frac{\partial\Phi(T,t)}{\partial T} = F\Phi(T,t), \quad K_{xy_1}(t,T) = K_x(t,T)C^T,$$

in terms of $K_x(T,T)$ and F.

The autocovariance function $K_x(T,T)$ of the state variable $x(T)$ is calculated by

$$\frac{dK_x(T,T)}{dT} = FK_x(T,T) + K_x(T,T)F^T + B\Pi_0 B^T \tag{89}$$

For $\gamma_\infty^2=\infty$, the H_∞ filter is reduced to the H_2 filter and the structure of the H_∞ smoother is same as the H_2 smoother [4]. Similar relationship would fit to the suboptimal estimation algorithms in the wide-sense stationary stochastic systems. For $\gamma^2=\infty$, the fixed-point and filtering algorithms in [Theorem 4] are reduced to the RLS Wiener algorithms for the fixed-point smoothing and filtering estimates in [3].

Next, from [Theorem 6] in section 4, we show that the filtering equations using the state-space parameters are same as those based on the game theory approach [5].

4. Derivation of Estimators Using the State-Space Parameters

In [Theorem 5], we show the recursive fixed-point smoothing and filtering algorithms using the state-space parameters. The algorithms are derived from those of [Theorem 4] using the covariance information.

[Theorem 5]

Let $\hat{z}_1(t,T)$ and $\hat{z}_2(t,T)$ represent the fixed-point smoothing estimates of $z_1(t)$ and $z_2(t)$ respectively. $\hat{z}_1(T,T)$ and $\hat{z}_2(T,T)$ represent the filtering estimates of $z_1(T)$ and $z_2(T)$ respectively. Let $y_1(T)(=Cx(T)+v_1(T))$ and $\breve{z}(T)(=Lx(T)+v_2(T))$ be the observed values defined in the observation equation (10). Let the information of the system matrix F, the observation matrices C and L, the input matrix B in the state-space model (1), γ, the observed value $y_1(T)$ and the artificial observed value $\breve{z}(T)$ in (9) be given. Then the recursive optimal algorithms for the fixed-point smoothing estimates $\hat{z}_1(t,T)$ of the signal $z_1(t)$ and $\hat{z}_2(t,T)$ of the signal $z_2(t)$ at the fixed point t and the filtering estimates $\hat{z}_1(T,T)$, $\hat{z}_2(T,T)$ and $\hat{x}(T,T)$ consist of (90)～(98).

$\hat{z}_1(t,T)$: Fixed-point smoothing estimate of the signal $z_1(t)(=Cx(t))$ at the fixed point t.

$$\frac{\partial \hat{z}_1(t,T)}{\partial T} = C\{h_1(t,T,T)[y_1(T)-\hat{z}_1(T,T)] + h_2(t,T,T)[\breve{z}(T)-\hat{z}_2(T,T)]\} \tag{90}$$

$\hat{z}_2(t,T)$: Fixed-point smoothing estimate of the signal $z_2(t)(=Lx(t))$ at the fixed point t.

$$\frac{\partial \hat{z}_2(t,T)}{\partial T} = L\{h_1(t,T,T)[y\ (T)-\hat{z}_1(T,T)] + h_2(t,T,T)[\ \breve{z}(T)-\hat{z}_2(T,T)]\} \tag{91}$$

$$h_1(t,T,T) = Q(t,T)C^T R^{-1}$$
$$(= (K_x(t,T)-p(t,T))C^T R^{-1}) \tag{92}$$

$$h_2(t,T,T) = -\gamma^{-2} Q(t,T)L^T$$
$$(= -\gamma^{-2}(K_x(t,T)-p(t,T))L^T) \tag{93}$$

$Q(t,T)$: Autovariance function of the fixed-point smoothing error $x(t)$-$\hat{x}(t,T)$.

$$\frac{\partial Q(t,T)}{\partial T} = Q(t,T)F^T - h_1(t,T,T)CQ(T,T) - h_2(t,T,T)LQ(T,T) \tag{94}$$

$Q(T,T)$: Autovariance function of the filtering error $x(T)$-$\hat{x}(T,T)$.

$$\frac{dQ(T,T)}{dT} = FQ(T,T) + Q(T,T)F^T + Q(T,T)[C^T R^{-1} C - \gamma^{-2} L^T L]Q(T,T) + B\Pi_0 B^T,$$
$$Q(0,0)=Q_0 \tag{95}$$

$\hat{z}_1(T,T)$: Filtering estimate of the signal $z_1(T)(=Cx(T))$.

$$\hat{z}_1(T,T)=C\hat{x}(T,T)) \tag{96}$$

$\hat{z}_2(T,T)$: Filtering estimate of the signal $z_2(T)(=Lx(T))$.

$$\hat{z}_2(T,T)=\mathrm{L}\hat{x}(T,T) \tag{97}$$

$\hat{x}$(T,T): Filtering estimate of the state variable $x(T)$.

$$\frac{d\hat{x}(T,T)}{dt} = F\hat{x}(T,T) + Q(T,T)C^T R^{-1}[y_1(T) - C\hat{x}(T,T)] - \gamma^{-2}Q(T,T)L^T[\breve{z}(T) - L\hat{x}(T,T)],$$
$$\hat{x}(0,0) = 0 \tag{98}$$

Proof: [Theorem 5] is derived along with [Theorem 3]. $Q(t,T)$ represent the autocovariance function of the smoothing error $x(t)$-$\hat{x}(t,T)$. $Q(t,T)$ is calculated by subtracting the autovariance function $p(t,T)$ of the fixed-point smoothing estimate $\hat{x}(t,T)$ from the autocovariance function $K_x(t,T)$ of $x(t)$, i.e., $Q(t,T)=K_x(t,T)-p(t,T)$. If we use the equation for $Q(t,T)$ and $Q(T,T)(=K_x(T,T)-p(T,T))$, we obtain (90) ~ (94) and (98). If we differentiate $Q(T,T)$ and use (75) and (89), we obtain (95). The initial condition on the differential equation (95) for the variance $Q(T,T)$ of the filtering error $x(T)$-$\hat{x}(T,T)$ at $T = 0$ is obtained by substituting $K_x(0,0)=Q_0$ and $p(0,0)=0$ into the equation $Q(0,0)= K_x(0,0)-p(0,0)$.

In [Theorem 6], we design the suboptimal recursive algorithms using the state-space parameters for the fixed-point smoothing and filtering estimates from the optimal recursive algorithms of [Theorem 5]. The suboptimal algorithms are derived by putting $\breve{z}(T) = L\hat{x}(T,T)(= \hat{z}_2(T,T))$ in the estimation equations of [Theorem 5] based on the state-space model.

[Theorem 6]

Let $\hat{z}_1(t,T)$ and $\hat{z}_1(t,T)$ represent the fixed-point smoothing estimates of $z_1(t)$ and $z_2(t)$ respectively. $\hat{z}_1(T,T)$ and $\hat{z}_2(T,T)$ represent the filtering estimates of $z_1(T)$ and $z_2(T)$ respectively. $y_1(T)(= Cx(T) + v_1(T))$ and $\breve{z}(T)(= Lx(T) + v_2(T))$ be the observed values defined in the observation equation (10). Let the information of the system matrix F, the observation matrices C and L, the input matrix B in the state-space model (1), γ, the observed value $y_1(T)$ and the artificial observed value $\breve{z}_1(T)$ in (9) be given. Then the recursive suboptimal algorithms for the fixed-point smoothing estimates $\hat{z}_1(t,T)$ of the signal $z_1(t)$ and $\hat{z}_2(t,T)$ of the signal $z_2(t)$ at the fixed point t and the filtering estimates $\hat{z}_1(T,T)$, $\hat{z}_2(T,T)$ and $\hat{x}(T,T)$ consist of (99)~(107).

$\hat{z}_1(t,T)$: Fixed-point smoothing estimate of the signal $z_1(t)(=Cx(t))$ at the fixed point t.

$$\frac{\partial \hat{z}_1(t,T)}{\partial T} = Ch_1(t,T,T)[y_1(T) - \hat{z}_1(T,T)] \tag{99}$$

$\hat{z}_2(t,T)$: Fixed-point smoothing estimate of the signal $\hat{z}_2(t)(=Lx(t))$ at the fixed point t.

$$\frac{\partial \hat{z}_2(t,T)}{\partial T} = Lh_1(t,T,T)[y_1(T) - \hat{z}_1(T,T)] \tag{100}$$

$$h_1(t,T,T) = Q(t,T)C^T R^{-1}$$
$$(= (K_x(t,T) - p(t,T))C^T R^{-1}) \tag{101}$$

$$h_2(t,T,T) = -\gamma^{-2} Q(t,T)L^T$$
$$(= -\gamma^{-2}(K_x(t,T) - p(t,T))L^T) \tag{102}$$

$Q(t,T)$: Autovariance function of the fixed-point smoothing error $x(t)\text{-}\hat{x}(t,T)$.

$$\frac{\partial Q(t,T)}{\partial T} = Q(t,T)F^T - h_1(t,T,T)CQ(T,T) - h_2(t,T,T)LQ(T,T) \tag{103}$$

$Q(T,T)$: Autovariance function of the filtering error $x(T)\text{-}\hat{x}(T,T)$.

$$\frac{dQ(T,T)}{dT} = FQ(T,T) + Q(T,T)F^T + Q(T,T)[C^T R^{-1} C - \gamma^{-2} L^T L]Q(T,T) + B\Pi_0 B^T,$$
$$Q(0,0) = Q_0 \tag{104}$$

$\hat{z}_1(T,T)$: Filtering estimate of the signal $z_1(T)(=Cx(T))$.

$$\hat{z}_1(T,T) = C\hat{x}(T,T) \tag{105}$$

$\hat{z}_2(T,T)$: Filtering estimate of the signal $z_2(T)(=Lx(T))$.

$$\hat{z}_2(T,T) = L\hat{x}(T,T) \tag{106}$$

$\hat{x}(T,T)$: Filtering estimate of the state variable $x(T)$.

$$\frac{d\hat{x}(T,T)}{dt} = F\hat{x}(T,T) + Q(T,T)C^T R^{-1}[y_1(T) - C\hat{x}(T,T)],$$
$$\hat{x}(0,0) = 0 \tag{107}$$

It should be noted that the suboptimal filtering algorithm for $\hat{z}_2(T,T)$ in [Theorem 6] is same as that of [5] based on the game theory approach in linear continuous stochastic systems. For $\gamma^2=\infty$, the suboptimal filtering equations for $\hat{x}(T,T)$ are reduced to those of the Kalman filter. The present estimation technique is useful in the wide-sense stationary stochastic systems. The recursive suboptimal fixed-point smoother in [Theorem 6] is proposed for the first time in this context.

There exists a bounded symmetric matrix function $Q(T,T)>0$ for t $\in[0,\infty)$ that satisfies (104), and such that the unforced linear time-invariant system

$\frac{dQ(T,T)}{dT}=[F+Q(T,T)(C^TR^{-1}C-\gamma^{-2}L^TL)]Q(T,T)$ is exponentially stable. For $T=\infty$, the solution of the Riccati differential equation is equal to that of the steady state Riccati equation.

5. A Numerical Simulation Example

Let the observed value $y_1(t)$ be generated by the following state-space model.

$$\begin{aligned}
y_1(t) &= z_1(t)+v_1(t)\\
&=[1\quad 0]\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+v_1(t),\quad z_1(t)=x_1(t),\quad C=[1\quad 0],
\end{aligned}$$

$$\begin{bmatrix}\frac{dx_1(t)}{dt}\\ \frac{dx_2(t)}{dt}\end{bmatrix}=\begin{bmatrix}0 & 1\\ -3 & -4\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}1\\ -2\end{bmatrix}w(t),$$

$$E[w(t)w(s)]=\delta(t-s). \tag{108}$$

Let $K_{z_1}(t,s)$ represent the autocovariance function of $z_1(t)$. The autocovariance function $K_{z_1}(t,s)$ of $z_1(t)$ is expressed as

$$K_{z_1}(t,s)=\frac{3}{16}e^{-(t-s)}+\frac{5}{48}e^{-3(t-s)} \tag{109}$$

for $0\le s\le t$.

Let $K_{xz_1}(t,s)(=K_{xy_1}(t,s))$ represent the crosscovariance function of the state variable $x(t)$ with the signal $z_1(s)$ and $K_{xz_2}(t,s)$ the crosscovariance function of $x(t)$ with $z_2(s)$. We assume that the scalar quantities $z_1(t)$ and $z_2(t)$ are related by $z_2(t)=az_1(t)$ for a given parameter $a(=0.95)$. Hence, in the recursive suboptimal estimation algorithms of [Theorem 4] for the filtering estimates $\hat{z}_1(T,T)$ and $\hat{z}_2(T,T)$ and the fixed-point smoothing estimates $\hat{z}_1(t,T)$ and $\hat{z}_2(t,T)$, we need the information of the system matrix F, the observation vectors C and L, the crossvariance function $K_{xy_1}(T,T)(=K_{xz_1}(T,T))$, the variance R of the observation noise $v_1(t)$, γ, the observed value $y_1(T)$ and the parameter a. Under this assumption, the autocovariance functions $K_{z_1}(t,s)$ of $z_1(t)$ and $K_{z_2}(t,s)$ of $z_2(t)$ are expressed as

$$K_{z_1}(t,s)=C\Phi(t,s)K_{xy_1}(s,s)1(t-s)+K_{xy_1}{}^T(t,t)\Phi^T(s,t)C^T1(s-t), \tag{110}$$

$$K_{z_2}(t,s)=a^2C\Phi(t,s)K_{xy_1}(s,s)1(t-s)+a^2K_{xy_1}{}^T(t,t)\Phi^T(s,t)C^T1(s-t). \tag{111}$$

For the autocovariance function $K_{z_1}(t,s)$ expressed as (110), the system matrix F, the crossvariance function $K_{xy_1}(t,t)$ of the state variable $x(t)$ with the observed value $y_1(t)$ and the observation vector C are obtained as follows [3],[10].

$$\Gamma_{ij}(t,s)=\begin{bmatrix} K_{0,0}(t,s) & K_{0,1}(t,s) & K_{0,2}(t,s)\cdots & K_{0,j-1}(t,s) \\ K_{1,0}(t,s) & K_{1,1}(t,s) & K_{1,2}(t,s)\cdots & K_{1,j-1}(t,s) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ K_{i-1,0}(t,s) & K_{i-1,1}(t,s) & K_{i-1,2}(t,s)\cdots & K_{i-1,j-1}(t,s) \end{bmatrix}, \tag{112}$$

where $K_{ij}(t,s)=\dfrac{\partial^i\partial^j K_{z_1}(t,s)}{\partial t^i \partial s^j}, 0\le s\le t$. For the matrix $\Gamma_{nn}(t,t)$ with the rank n, we find that

$$F=\frac{\partial\Gamma_{nn}(t,s)}{\partial t}\Big|_{t=s}\Gamma_{nn}^{-1}(t,t), \tag{113}$$

$$K_{xy_1}(t,t)=\begin{bmatrix} K_{0,0}(t,s)|_{s=t} \\ K_{1,0}(t,s)|_{s=t} \\ \cdots\cdots \\ K_{n-1,0}(t,s)|_{s=t} \end{bmatrix}, \tag{114}$$

$$C=[1 \quad 0 \quad \cdots\cdots \quad 0]. \tag{115}$$

Here, F is a time-invariant square matrix of order n and $K_{xy_1}(t,t)$ is an $n\times 1$ vector.

Hence, we can calculate the estimates by use of the system matrix F, the observation vectors C and $L(=aC)$, the crossvariance function $K_{xy_1}(T,T)$, the variance R of the observation noise, and the observed value $y_1(T)$, where F, C and $K_{xy_1}(T,T)$ are calculated from the autocovariance function $K_{z_1}(t,s)$ of the signal $z_1(t)$. We consider to estimate $z_2(t)(=0.95z_1(t))$ for a=0.95. From (112), $\Gamma_{22}(t,s)$ is calculated as

$$\Gamma_{22}(t,s)=\begin{bmatrix} \frac{3}{16}e^{-(t-s)}+\frac{5}{48}e^{-3(t-s)} & \frac{3}{16}e^{-(t-s)}+\frac{5}{16}e^{-3(t-s)} \\ -\frac{3}{16}e^{-(t-s)}-\frac{5}{16}e^{-3(t-s)} & -\frac{3}{16}e^{-(t-s)}-\frac{15}{16}e^{-3(t-s)} \end{bmatrix}.$$

From (113), F is evaluated as $F=\begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$ by $F=\dfrac{\partial\Gamma_{22}(t,s)}{\partial t}\Big|_{s=t}\Gamma_{22}^{-1}(t,t)$ in terms of the invertible matrix $\Gamma_{22}(t,t)$ with rank 2. Also, from (114), $K_{xy_1}(t,t)$ is evaluated as $K_{xy_1}(t,t)=[\frac{7}{24} \quad -\frac{1}{2}]^T$. Here, the observation vector is given by $C=[1 \quad 0]$ from (115).

If we substitute the quantities $a(=0.95)$, F, $K_{xy_1}(t,t)$, C, L, γ and R into the estimation algorithms of [Theorem 4], the fixed-point smoothing and filtering estimates of $z_1(t)$ and $z_2(t)$ are calculated. Fig.1 illustrates the stochastic processes of $z_2(t)$ (graph (a)) and its filtering estimate $\hat{z}_2(t,t)$ vs. t for $\gamma^2=0.5^2$. Graphs (b) and (c) depict $\hat{z}_2(t,t)$ for white Gaussian observation noises $N(0,0.1^2)$ and $N(0,0.3^2)$. Fig.2 illustrates the stochastic processes of $z_1(t)$ (graph

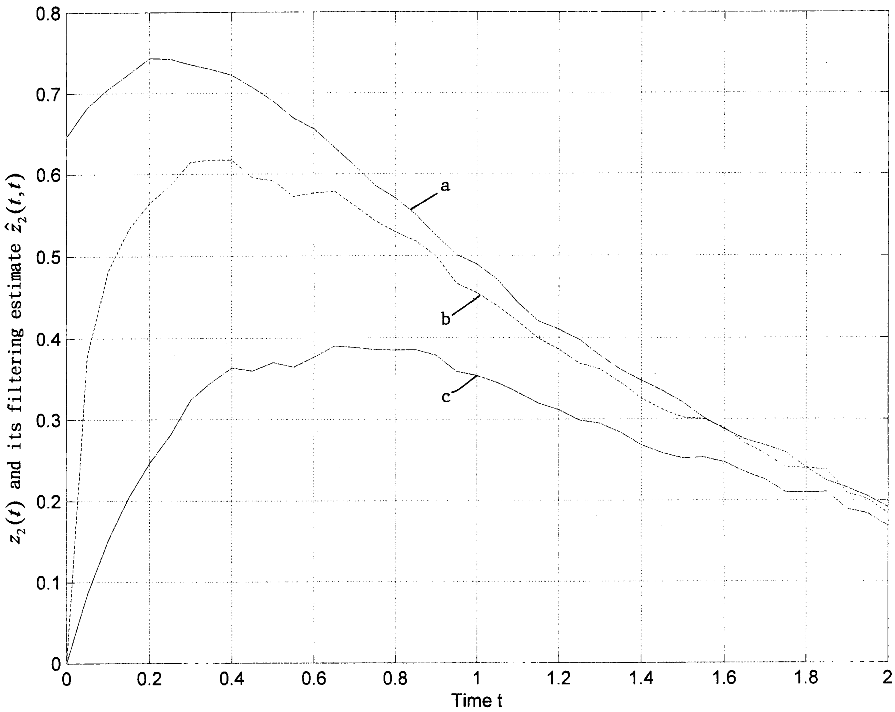

Fig.1 The sequence of $z_2(t)$ and its filtering estimate $\hat{z}_2(t,t)$ vs. t for $\gamma^2=0.5^2$.

(a)⋯⋯ $z_2(t)$

(b)⋯⋯ Filtering estimate $\hat{z}_2(t,t)$ for white Gaussian observation noise $N(0,0.1^2)$.

(c)⋯⋯ Filtering estimate $\hat{z}_2(t,t)$ for white Gaussian observation noise $N(0,0.3^2)$.

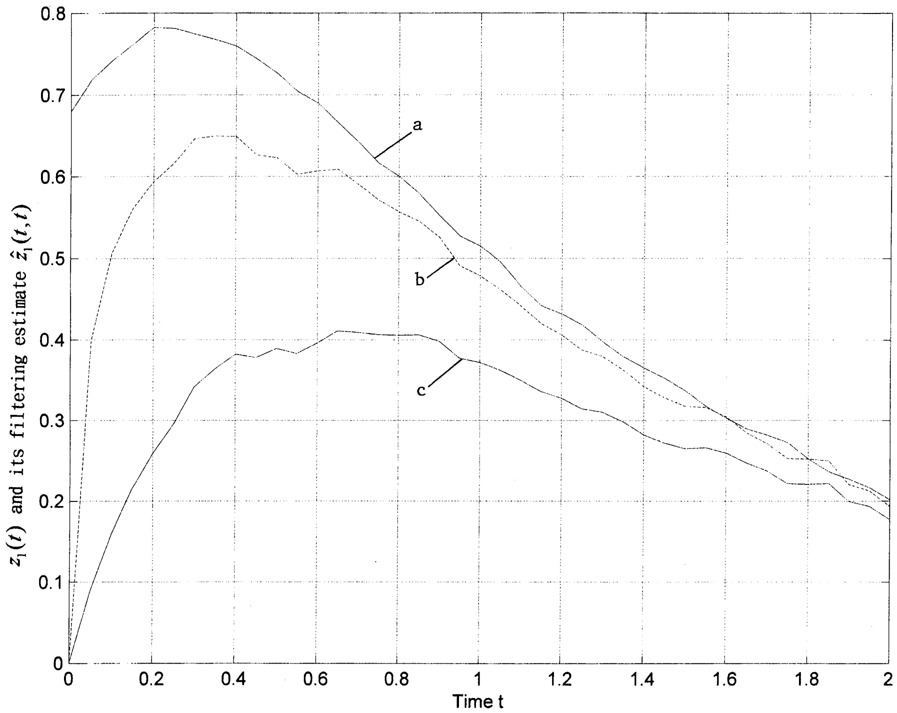

Fig.2 The sequence of $z_l(t)$ and its filtering estimate $\hat{z}_1(t,t)$ vs. t for $\gamma^2=0.5^2$.

(a)⋯⋯ $z_l(t)$

(b)⋯⋯ Filtering estimate $\hat{z}_1(t,t)$ for white Gaussian observation noise $N(0,0.1^2)$.

(c)⋯⋯ Filtering estimate $\hat{z}_1(t,t)$ for white Gaussian observation noise $N(0,0.3^2)$.

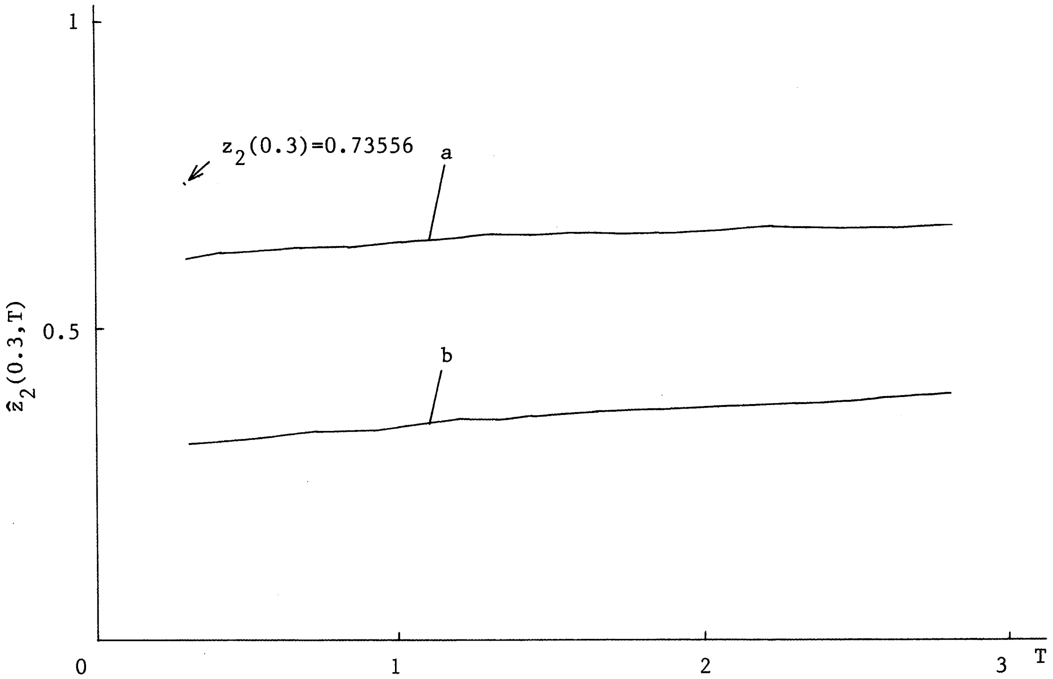

Fig.3 The fixed-point smoothing estimate $\hat{z}_2(0.3,T)$ vs. T for $\gamma^2=0.5^2$.
(a)…… $\hat{z}_2(0.3,T)$ for white Gaussian observation noise $N(0,0.1^2)$.
(b)…… $\hat{z}_2(0.3,T)$ for white Gaussian observation noise $N(0,0.3^2)$.

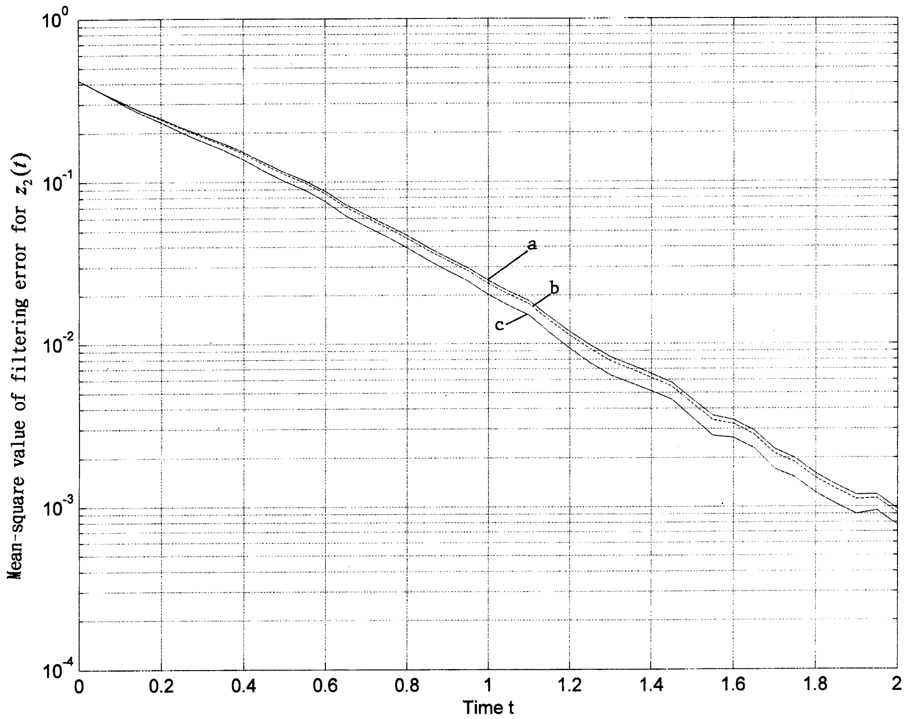

Fig.4 M.S.V. of the filtering error $z_2(t)$- $\hat{z}_2(t,t)$ vs. t for the observation noise $N(0,0.3^2)$.
(a)……M.S.V. of $z_2(t)$- $\hat{z}_2(t,t)$ for $\gamma^2=\infty$.
(b)……M.S.V. of $z_2(t)$- $\hat{z}_2(t,t)$ for $\gamma^2=1$.
(c)……M.S.V. of $z_2(t)$- $\hat{z}_2(t,t)$ for $\gamma^2=0.5^2$.

(a)) and its filtering estimate $\hat{z}_1(t,t)$ vs. t for $\gamma^2=0.5^2$. Graphs (b) and (c) depict $\hat{z}_1(t,t)$ for white Gaussian observation noises $N(0,0.1^2)$ and $N(0,0.3^2)$. Fig.3 illustrates the fixed-point smoothing estimate $\hat{z}_2(0.3,T)$ vs. T for $\gamma^2=0.5^2$, where the value of $z_2(t)$ at the fixed point $t=0.3$ is 0.73556. Graphs (a) and (b) show $\hat{z}_2(0.3,T)$ for the observation noises $N(0,0.1^2)$ and $N(0,0.3^2)$. Fig.4 illustrates the mean-square value (M.S.V.) of the filtering error $z_2(t)-\hat{z}_2(t,t)$ vs. t for the observation noise $N(0,0.3^2)$. Graph (a) shows the M.S.V. of the filtering error for $\gamma^2=\infty$. Graphs (b) and (c) show the M.S.V. of the filtering error for $\gamma^2=1$ and $\gamma^2=0.5^2$ respectively. Fig.5 illustrates the M.S.V. of the filtering error $z_1(t)-\hat{z}_1(t,t)$ vs. t for the observation noise $N(0,0.3^2)$. Graph (a) shows the M.S.V. of the filtering error for $\gamma^2=\infty$. Graphs (b) and (c) show the M.S.V. of the filtering error for $\gamma^2=1$ and $\gamma^2=0.5^2$ respectively. Fig.6 illustrates the M.S.V. of the filtering error $z_2(t)-\hat{z}_2(t,t)$ vs. t for $\gamma^2=0.5^2$. Graphs (a), (b) and (c) show the M.S.V. of $z_2(t)-\hat{z}_2(t,t)$ for the observation noises $N(0,0.5^2)$, $N(0,0.3^2)$ and $N(0,0.1^2)$ respectively. Fig.7 illustrates the M.S.V. of the filtering error $z_1(t)-\hat{z}_1(t,t)$ vs. t for $\gamma^2=0.5^2$. Graphs (a), (b) and (c) show the M.S.V. of $z_1(t)-\hat{z}_1(t,t)$ for the observation noises$N(0,0.5^2)$, $N(0,0.3^2)$ and $N(0,0.1^2)$ respectively. In Figs. 4~7, the M.S.V. is evaluated in terms of the average of 20 trials for the square value of the filtering error. Table 1 compares the mean-square values of the estimation errors with those of the RLS estimation errors for both the filtering estimate $\hat{z}_2(t,t)$ and the fixed-point smoothing estimate $\hat{z}_2(t,T)$. The mean-square values are shown for the sequences of the white Gaussian

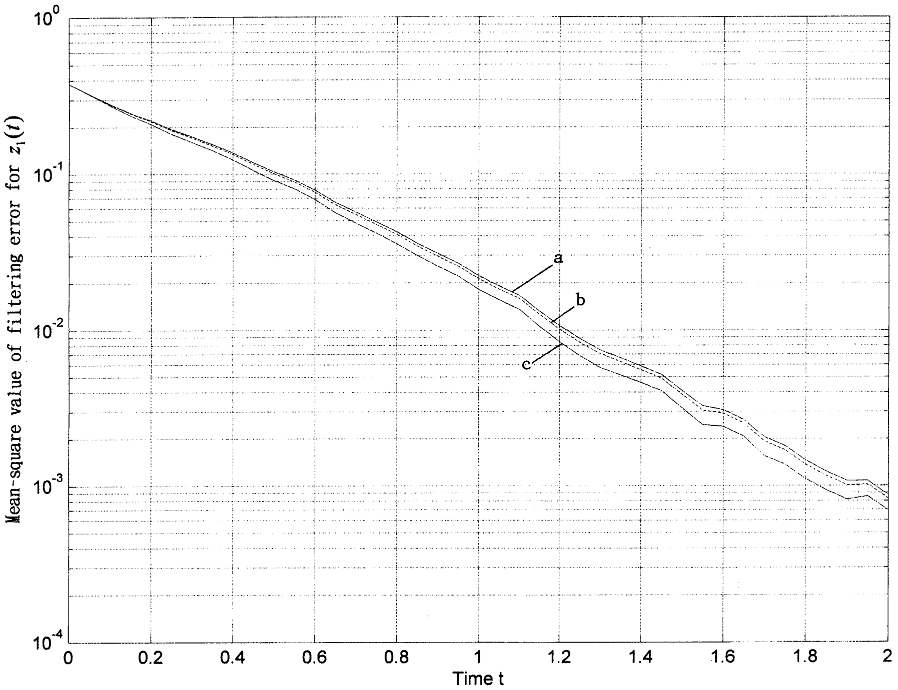

Fig.5 M.S.V. of the filtering error $z_1(t)$- $\hat{z}_1(t,t)$ vs. t for the observation noise $N(0,0.3^2)$.
(a) ······ M.S.V. of $z_1(t)$- $\hat{z}_1(t,t)$ for $\gamma^2 = \infty$.
(b) ······ M.S.V. of $z_1(t)$- $\hat{z}_1(t,t)$ for $\gamma^2 = 1$.
(c) ······ M.S.V. of $z_1(t)$- $\hat{z}_1(t,t)$ for $\gamma^2 = 0.5^2$.

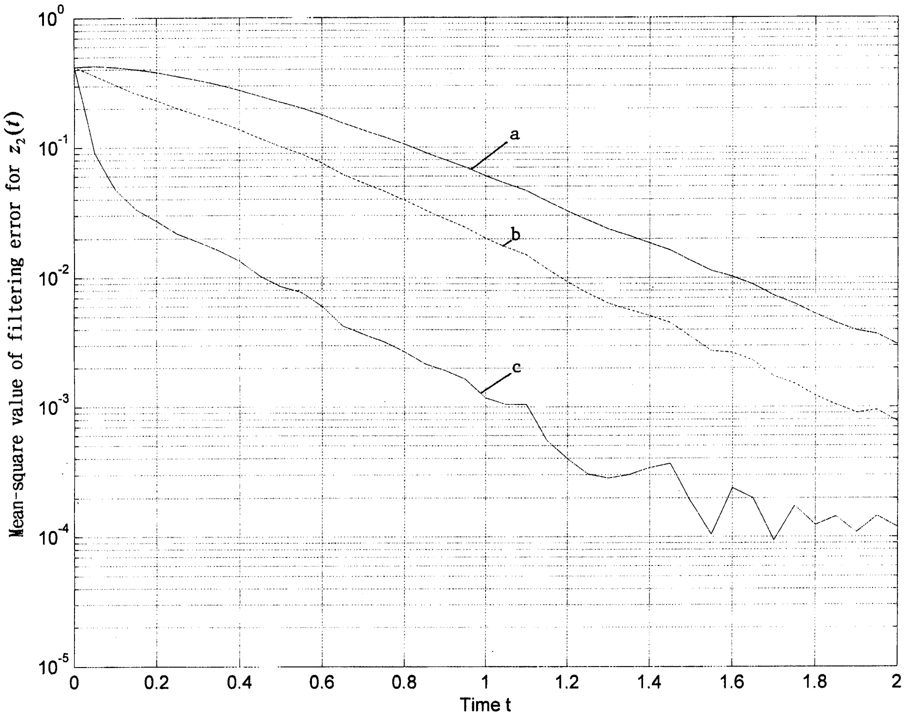

Fig.6 M.S.V. of the filtering error $z_2(t)$- $\hat{z}_2(t,t)$ vs. t for $\gamma^2 = 0.5^2$.
(a)…… M.S.V. of $z_2(t)$- $\hat{z}_2(t,t)$ for the observation noise $N(0,0.5^2)$.
(b)…… M.S.V. of $z_2(t)$- $\hat{z}_2(t,t)$ for the observation noise $N(0,0.3^2)$.
(c)…… M.S.V. of $z_2(t)$- $\hat{z}_2(t,t)$ for the observation noise $N(0,0.1^2)$.

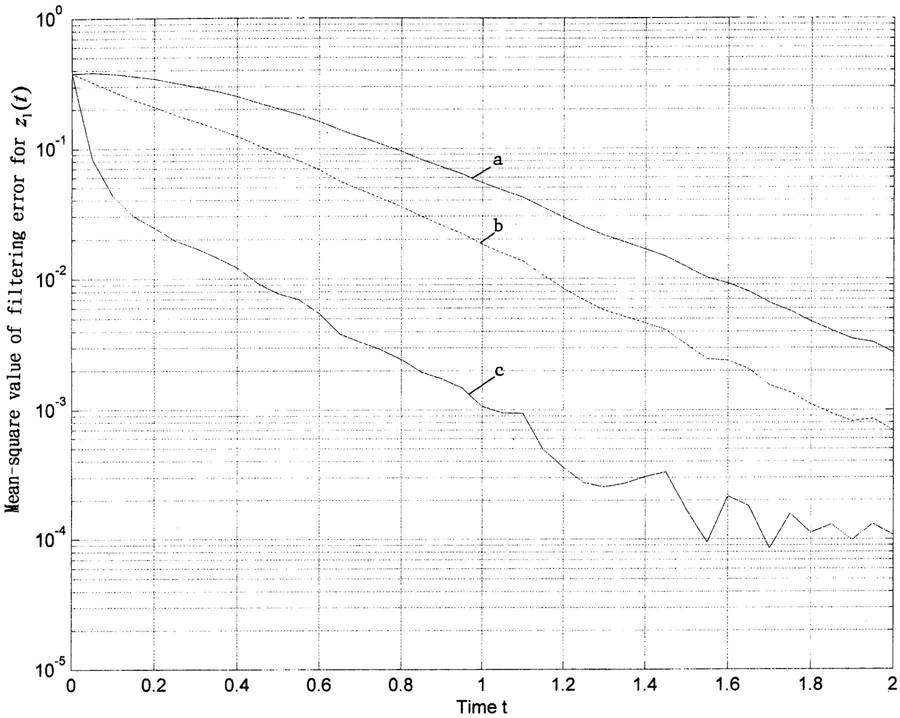

Fig.7 M.S.V. of the filtering error $z_1(t)$- $\hat{z}_1(t,t)$ vs. t for $\gamma^2 = 0.5^2$.
(a)…… M.S.V. of $z_1(t)$- $\hat{z}_1(t,t)$ for the observation noise N$(0,0.5^2)$.
(b)…… M.S.V. of $z_1(t)$- $\hat{z}_1(t,t)$ for the observation noise N$(0,0.3^2)$.
(c)…… M.S.V. of $z_1(t)$- $\hat{z}_1(t,t)$ for the observation noise N$(0,0.1^2)$.

observation noises $N(0,0.1^2)$, $N(0,0.3^2)$ and $N(0,0.5^2)$. The mean-square values are calculated by $\frac{\sum_{i=1}^{2000}(z_2(i\Delta)-\hat{z}_2(i\Delta,i\Delta))^2}{2000}$, $\Delta=0.001$, for the current and RLS filtering estimates and by $\frac{\sum_{i=1}^{2000}\sum_{j=1}^{2500}(z_2(i\Delta)-\hat{z}_2(i\Delta,i\Delta+j\Delta))^2}{5000000}$ for the current and RLS fixed-point smoothing estimates.

For the current filtering and fixed-point smoothing estimates, the mean-square values are evaluated for values of γ^2, 0.5^2, 1, 5^2 and 10^2. The current filtering and fixed-point smoothing estimates for $\gamma^2=\infty$ correspond to the RLS Wiener estimates [1] respectively. Similarly, Table 2 compares the mean-square values of the estimation errors by the current technique with those of

Table 1 Mean-square values of the estimation errors for both the filtering estimate $\hat{z}_2(t,t)$ and the fixed-point smoothing estimate $\hat{z}_2(t,T)$ when the observation noise obeys $N(0,0.1^2)$, $N(0,0.3^2)$ and $N(0,0.5^2)$.

White Gaussian observation noise	Kind of estimation technique	Value of γ^2	M.S.V. of filtering error $z_2(t)-\hat{z}_2(t,t)$	M.S.V. of fixed-point smoothing error $z_2(t)-\hat{z}_2(t,T)$
$N(0,0.1^2)$	Current estimation technique	0.5^2	1.2236227×10^{-2}	3.3813318×10^{-3}
		1	1.2515357×10^{-2}	3.5187265×10^{-3}
		5^2	1.2604574×10^{-2}	3.5629329×10^{-3}
		10^2	1.2607373×10^{-2}	3.5643199×10^{-3}
	RLS estimation technique in [3]	∞	1.2608273×10^{-2}	3.5647758×10^{-3}
$N(0,0.3^2)$	Current estimation technique	0.5^2	9.6362084×10^{-2}	5.0143092×10^{-2}
		1	1.0270526×10^{-1}	5.4761010×10^{-2}
		5^2	1.0461311×10^{-1}	5.6156003×10^{-2}
		10^2	1.0467182×10^{-1}	5.6199026×10^{-2}
	RLS estimation technique in [3]	∞	1.0469137×10^{-1}	5.6213271×10^{-2}
$N(0,0.5^2)$	Current estimation technique	0.5^2	1.7907817×10^{-1}	1.0772996×10^{-1}
		1	1.932574×10^{-1}	1.1858771×10^{-1}
		5^2	1.9706018×10^{-1}	1.2150826×10^{-1}
		10^2	1.9717448×10^{-1}	1.2159609×10^{-1}
	RLS estimation technique in [3]	∞	1.9721242×10^{-1}	1.2162541×10^{-1}

Table 2 Mean-square values of the estimation errors for both the filtering estimate $\hat{z}_1(t,t)$ and the fixed-point smoothing estimate $\hat{z}_1(t,T)$ when the observation noise obeys $N(0,0.1^2)$, $N(0,0.3^2)$ and $N(0,0.5^2)$.

White Gaussian observation noise	Kind of estimation technique	Value of γ^2	M.S.V. of filtering error $z_1(t)-\hat{z}_1(t,t)$	M.S.V. of fixed-point smoothing error $z_1(t)-\hat{z}_1(t,T)$
$N(0,0.1^2)$	Current estimation technique	0.5^2	1.3558144×10^{-2}	3.7466298×10^{-3}
		1	1.3867431×10^{-2}	3.8988703×10^{-3}
		5^2	1.3966287×10^{-2}	3.9478549×10^{-3}
		10^2	1.3969382×10^{-2}	3.9493885×10^{-3}
	RLS estimation technique in [3]	∞	1.3970388×10^{-2}	3.949888×10^{-3}
$N(0,0.3^2)$	Current estimation technique	0.5^2	1.0677241×10^{-1}	5.5560249×10^{-2}
		1	1.1380081×10^{-1}	6.0677015×10^{-2}
		5^2	1.1591481×10^{-1}	6.2222683×10^{-2}
		10^2	1.1597985×10^{-1}	6.2270390×10^{-2}
	RLS estimation technique in [3]	∞	1.1600147×10^{-1}	6.2286269×10^{-2}
$N(0,0.5^2)$	Current estimation technique	0.5^2	1.9842541×10^{-1}	1.1936826×10^{-1}
		1	2.1413564×10^{-1}	1.3139912×10^{-1}
		5^2	2.1834918×10^{-1}	1.3463541×10^{-1}
		10^2	2.1847590×10^{-1}	1.3473251×10^{-1}
	RLS estimation technique in [3]	∞	2.1851786×10^{-1}	1.3476486×10^{-1}

the estimates for $\gamma^2=\infty$ in both cases of the filtering estimate $\hat{z}_1(t,t)$ and the fixed-point smoothing estimate $\hat{z}_1(t,T)$. The mean-square values are shown for the observation noises $N(0,0.1^2)$, $N(0,0.3^2)$ and $N(0,0.5^2)$. The mean-square values for the filtering and smoothing estimates of $z_1(t)$ are calculated similarly with those of $z_2(t)$. For the filtering and fixed-point smoothing estimates, the mean-square values are evaluated for $\gamma^2=0.5^2$, 1, 5^2, 10^2. From Table 1 and Table 2, we find that the estimation accuracy of the smoothing estimates $\hat{z}_2(t,T)$ and $\hat{z}_1(t,T)$ is superior to the filtering estimates $\hat{z}_2(t,t)$ and $\hat{z}_1(t,t)$ respectively. Also, as the variance of the observation noise becomes small, the mean-square values of the filtering errors $z_2(t)$-$\hat{z}_2(t,t)$ and $z_1(t)$-$\hat{z}_1(t,t)$ and the smoothing errors $z_2(t)$-$\hat{z}_2(t,T)$ and $z_1(t)$-$\hat{z}_1(t,T)$ become small. Clearly, the estimation accuracy of the estimates $\hat{z}_2(t,t)$, $\hat{z}_1(t,t)$, $\hat{z}_2(t,T)$ and $\hat{z}_1(t,T)$ by the proposed estimators is superior to that of the estimates for $\gamma^2=\infty$ respectively. As the value of γ^2 increases, the mean-square values of the filtering errors $z_2(t)-\hat{z}_2(t,t)$ and $z_1(t)-\hat{z}_1(t,t)$ and the smoothing errors $z_2(t)-\hat{z}_2(t,T)$ and $z_1(t)-\hat{z}_1(t,T)$ tend to be large.

8. Conclusions

The numerical simulation results have shown that the recursive suboptimal fixed-point smoothing and filtering algorithms in [Theorem 4] are feasible. For $\gamma^2=\infty$, the estimation algorithms for the fixed-point smoothing and filtering estimates in [Theorem 4] are same as those by the RLS Wiener estimators [3] using the covariance information. For $\gamma^2<\infty$, the estimation accuracy of the proposed estimators are preferable to those in [3].

In this paper, the stochastic estimation algorithms have been derived in a unified manner.

By use of the covariance information, the optimal and suboptimal estimators have been proposed respectively in [Theorem 3] and [Theorem 4] for linear continuous-time stochastic systems. The estimation algorithms for the fixed-point smoothing and filtering estimates of $z_2(t)(=Lx(t))$ and $z_1(t)(=Cx(t))$ have been obtained in relation to the deterministic H_∞ estimation technique in the Krein spaces [1],[2].

In [Theorem 5] and [Theorem 6], by use of the state-space parameters, the optimal and suboptimal algorithms for the fixed-point smoothing and filtering estimates have been derived respectively. The suboptimal filtering equations in [Theorem 6] using the state-space parameters are identical with those based on the game theory approach [5] in linear continuous systems. The suboptimal fixed-point smoother in [Theorem 6] is proposed for the first time in this paper.

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