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ON THE STIEFEL-WHITNEY CLASSES OF MAPS AND HAEFLIGER'S OBSTRUCTIONS TO EMBEDDINGS

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Dedicated to Professor Teiichi Kobayashi on his 60th birthday

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Abstract. The purposes of this article are (1) to more explicitly describe Haefliger's obstruction for a map $f : M^n \to N^{2n-k}$ to be homotopic to an embedding, (2) to show some relations among the Stiefel-Whitney classes of a map f and (3) to give methods to determine the Stiefel-Whitney classes of a map.

1. Introduction

Throughout this note, n-manifolds mean n-dimensional connected differentiable manifolds without boundary. The cohomology is understood to have Z_2 as coefficients.

For a map $f : M \to N$ between manifolds, the total Stiefel-Whitney class of $f, w(f)(=\sum_{i\geq 0} w_i(f))$, is defined by the equation

$$w(f) = \bar{w}(M)f^*(w(N)), \tag{1.1}$$

where $w(N)$ and $\bar{w}(M)(=w(M)^{-1})$ are the total Stiefel-Whitney class of N and the total normal Stiefel-Whitney class of M, respectively. For an n-manifold V^n, let $U_V \in H^n(V^2)$ denote the mod 2 Thom class (or the diagonal class) of V as given in [6, p. 125].

Haefliger [1.5 and 3.7] defined a primary obstruction $\phi_2^f \in H^{n+k}(M^*)$ for a map $f : M^n \to N^{n+k}$ to be homotopic to an embedding, where $M^*(=(M \times M - \Delta M)/Z_2)$ denotes the reduced symmetric product of M.

THEOREM (Haefliger). *For a map $f : M^n \to N^{2n-k}$, the primary obstruction ϕ_2^f to being homotopic to an embedding vanishes if and only if* (1) $U_M(1 \otimes w_{n-k}(f)) + (f^2)^* U_N = 0$, *and* (2) $w_i(f) = 0$ *for* $i > n-k$.

The explicit description of $H^*(M^*)$ is given in [7, §2] (see §2). But ϕ_2^f has not been given an explicit description as an element of $H^*(M^*)$. We regard the isomorphism $\rho_M \mid B_M^* \oplus I_M^* : B_M^* \oplus I_M^* \to H^*(M^*)$ in [7, Proposition 2.9(d)] (see Theorem 2.1 in §2) as the identity. Then we have

THEOREM 1.1. *Let M^n be a compact n-manifold. Then for a map $f : M^n \to N^{2n-k}$,*

$$\phi_2^f = U_M(1 \otimes w_{n-k}(f)) + (f^2)^* U_N + \sum_{1 \le j \le [(k-1)/2]} \sum_{0 \le i < j} u^{k-2j} \otimes (Q_M^i w_{n-k+j-i}(f))^2.$$

Here $Q_M^i : H^*(M) \to H^{*+i}(M)(i \ge 0)$ are Yo's operations defined in [11] (see [8, §2]) and satisfy the relation

$$\sum_{i+j=k} Sq^i Q_M^j(x) = x w_k(M) \quad \text{for } x \in H^*(M). \tag{1.2}$$

COROLLARY 1.2. *Under the above assumption, $\phi_2^f = 0$ if and only if (1) $U_M(1 \otimes w_{n-k}(f)) + (f^2)^* U_N = 0$, and (2) $w_i(f) = 0$ for $n-k < i \le n - \left[\frac{k}{2}\right] - 1$.*

This corollary and Haefliger's theorem indicate that $w_i(f)$ for $i \ge n - k/2$ are described by using $w_j(f)(n-k \le j < n - [k/2])$. In fact, we have the following relations, where $\bar{Q}_M = \sum_{i \ge 0} \bar{Q}_M^i$ is the dual to $Q_M = \sum_{i \ge 0} Q_M^i$, that is, $\bar{Q}_M Q_M = Q_M \bar{Q}_M = 1$.

THEOREM 1.3. *Let M^n be a compact n-manifold.*

(1) *For $f : M^n \to N^{2n-(2k-1)}$,*

$$w_{n-(k-1)+i}(f) = \sum_{1 \le j < k} \sum_{0 \le l \le i} \bar{Q}_M^l Q_M^{i+j-l} w_{n-(k-1)-j}(f) \qquad \textit{for } i \ge 0.$$

(2) *For $f : M^n \to N^{2n-2k}$,*

$$w_{n-k}(f) = \sum_{1 \le j \le k} Q_M^j w_{n-k-j}(f) + f^* v_{n-k}(N),$$

$$w_{n-k+i}(f) = \sum_{0 \le j < k} \sum_{0 \le l < i} \bar{Q}_M^l Q_M^{i+j-l} w_{n-k-j}(f) \qquad \textit{for } i \ge 1.$$

Here $v_i(N)$ stands for the i-th Wu class of N.

The relation (1) in Haefliger's Theorem (or in Corollary 1.2) is important in the study of embeddings. The following method helps determine $w_{n-k}(f)$. Let $\bar{S}q(=\sum_{i \ge 0} \bar{S}q^i)$ be the dual operation to $Sq(=\sum_{i \ge 0} Sq^i)$, that is, $\bar{S}q^0 = 1$, $\sum_{0 \le i \le j} Sq^i \bar{S}q^{j-i} = 0 (j > 0)$.

THEOREM 1.4 (cf. Li-Peterson). *For a map $f : M \to N$,*

$$\langle w(f)x, [M] \rangle = \langle \bar{S}q(x) f^*(v(N)), [M] \rangle \qquad \textit{for } x \in H^*(M),$$

or equivalently

$$w_{n-j}(f)x = \sum_{i \ge 0} \bar{S}q^i(x) f^*(v_{n-j-i}(N)) \qquad \textit{for } x \in H^j(M).$$

Applications of these theorems to the existence and the non-existence problems of embeddings will be given in, e.g., [10] and [9], respectively.

This note is organized as follows. §2 is devoted to the explanation of $H^*(M^*)$ by [7], Haefliger's obstruction ϕ_2^f [1] and Yo's operation Q_M^i and its dual [11]. We will prove Theorems 1.1, 1.3 and 1.4 in §3, §4 and §5, respectively.

2. Preliminaries

We adopt the notations and definitions used in [7, §2].

For a manifold M, let M^2 be the product $M \times M$ and ΔM the diagonal of M in M^2. The group Z_2 acts on M^2 by interchanging factors. We set $M^* = (M^2 - \Delta M)/Z_2$. Further, we set $\Gamma M = S^\infty \times_{Z_2} M^2$, where Z_2 acts by the diagonal action. Let $i : S^\infty \times_{Z_2} (M^2 - \Delta M) \subset \Gamma M$ and $p : S^\infty \times_{Z_2} (M^2 - \Delta M) \to M^*$ be the natural inclusion and the natural projection. Then p is a homotopy equivalence and the map $\rho_M = \rho : H^*(\Gamma M) \to H^*(M^*)$ in [7, Theorem 2.1] is given by $\rho_M = p^{*-1} i^*$ [8, (2.2)]. Let t_M be the involution of M^2 which transposes the factors and let

$$I_M^* = (1 + t_M^*) H^*(M^2) \quad \text{and} \quad K_M^* = \{(x^2)(= x \otimes x) \mid x \in H^*(M)\}.$$

Further let $u \in H^1(P^\infty)$ be the generator.

THEOREM 2.1 (Thomas). (1) *There is an $H^*(P^\infty)$-module isomorphism*

$$H^*(\Gamma M) \cong H^*(P^\infty) \otimes K_M^* \oplus I_M^*,$$

where $H^*(P^\infty)$ acts on I_M^* trivially.

(2) *Let B_M^* be the subgroup of $H^*(\Gamma M)$ generated by all elements of the form $u^j \otimes (x)^2$ with $j + \dim x < \dim M$. The restriction of ρ_M to $B_M^* \oplus I_M^*$ is a group isomorphism,*

$$\rho_M \mid B_M^* \oplus I_M^* : B_M^* \oplus I_M^* \cong H^*(M^*).$$

For a map $f : M^n \to N^{2n-k}$, the obstruction $\phi_2^f \in H^{2n-k}(M^*)$ in [1, §3] is described as

$$\phi_2^f = \sum_{0 \le j \le [(2n-k-1)/2]} \rho_M(u^{2n-k-2j} \otimes (f^* v_j(N))^2) + \rho_M((f^2)^* U_N), \tag{2.1}$$

in other words, $\phi_2^f = \rho_M \Gamma(f)^* \varphi_N(1 \otimes 1)$ by [8, Proposition 2.6].

Yo [11] defined operations $Q_M = \sum_{i \ge 0} Q_M^i$, $Q_M^i : H^*(M) \to H^{*+i}(M)$, for compact manifolds M^n, which satisfy such properties as

$$Q_M^0 = 1, \quad Q_M^i x = 0 \quad \text{if } 2i > \dim M - \dim x, \tag{2.2}$$

$$Sq Q_M x = x w(M), \quad \bar{Q}_M x = \bar{w}(M) Sq x. \tag{2.3}$$

There are some other relations.

LEMMA 2.2. *Let M be a compact manifold. Then*

$$\bar{Q}^i_M x = 0 \quad \text{if } i > 0, x \in H^{n-i}(M), \tag{2.4}$$

$$\bar{Q}_M f^* v(N) = w(f) \quad \text{for } f : M \to N. \tag{2.5}$$

PROOF. For $i > 0$, $\bar{Q}^i_M x = \sum_{0 \le j \le i-1} Q^{i-j}_M \bar{Q}^j_M x$ by the definition of $\bar{Q}^i_M$. If $\dim x = n - i$, then $n - \dim \bar{Q}^j x = i - j < 2(i-j)$. Hence $Q^{i-j}_M \bar{Q}^j_M x = 0$ for $0 \le j < i$ by (2.2) and so $\bar{Q}^i_M x = 0$. This shows (2.4). By (2.3) and the relation $w(N) = Sqv(N)$, we have $\bar{Q}_M f^* v(N) = \bar{w}(M) Sq f^* v(N) = \bar{w}(M) f^* Sqv(N) = \bar{w}(M) f^* w(N) = w(f)$. □

3. mod 2 primary obstructions

In this section, we will prove Theorem 1.1, while using the following

LEMMA 3.1. *Let M^n be a compact n-manifold. Then for any $x \in H^{n-i}(M)$,*

$$\rho_M(u^{2i-k} \otimes (x)^2) = \sum_{1 \le j \le [(k-1)/2]} \sum_{0 \le l < j} \rho_M(u^{k-2j} \otimes (Q^l_M \bar{Q}^{i-k+j-l}_M x)^2)$$

$$+ \rho_M(U_M(1 \otimes \bar{Q}^{i-k}_M x)).$$

PROOF. If $[k/2] \le i < k$, then the relation holds trivially. For $i \ge k$, we prove the relation by induction. If $i = k$ and $x \in H^{n-k}(M)$, then $\varphi_M(1 \otimes 1) = \sum_{0 \le j \le [(k-1)/2]} u^{k-2j} \otimes (Q^j_M x)^2 + U_M(1 \otimes x)$ by [8, Proposition 2.6]. Hence

$$\rho_M(u^k \otimes (x)^2) = \sum_{1 \le j \le [(k-1)/2]} \rho_M(u^{k-2j} \otimes (Q^j_M x)^2) + \rho_M(U_M(1 \otimes x))$$

$$= \sum_{1 \le j \le [(k-1)/2]} \sum_{0 \le l < j} \rho_M(u^{k-2j} \otimes (Q^l_M \bar{Q}^{j-l}_M x)^2) + \rho_M(U_M(1 \otimes x)).$$

Assume that the equations hold for $[k/2] \le i \le m$ and consider the case when $i = m+1$. For $x \in H^{n-m-1}(M)$, we have, by [8, Proposition 2.6],

$$\varphi_M(u^{m+1-k} \otimes x_{n-m-1}) = \sum_{0 \le r \le [(m+1)/2]} u^{m+1-k+m+1-2r} \otimes (Q^r_M x)^2.$$

Hence, by the assumption of induction and the fact that $Q^r_M x = 0$ for $2r > n - \dim x = m+1$,

$$\rho_M(u^{2(m+1)-k} \otimes (x)^2)$$

$$= \sum_{1 \le r \le [(m+1)/2]} \rho_M(u^{2(m+1)-k-2r} \otimes (Q^r_M x)^2)$$

$$= \sum_{1\le r\le[(m+1)/2]} \sum_{1\le j\le[(k-1)/2]} \sum_{0\le l<j} \rho_M(u^{k-2j} \otimes (Q_M^l \bar{Q}_M^{m+1-r-k+j-l} Q_M^r x)^2)$$

$$+ \sum_{1\le r\le[(m+1)/2]} \rho_M(U_M(1 \otimes \bar{Q}_M^{m+1-r-k} Q_M^r x))$$

$$= \sum_{1\le j\le[(k-1)/2]} \rho_M\left(u^{k-2j} \otimes \left(\sum_{0\le l<j} \sum_{1\le r} Q_M^l \bar{Q}_M^{m+1-k+j-l-r} Q_M^r x\right)^2\right)$$

$$+ \rho_M\left(U_M\left(1 \otimes \sum_{1\le r} \bar{Q}_M^{m+1-k-r} Q_M^r x\right)\right)$$

$$= \sum_{1\le j\le[(k-1)/2]} \sum_{0\le l<j} \rho_M(u^{k-2j} \otimes (Q_M^l \bar{Q}_M^{m+1-k+j-l} x)^2)$$

$$+ \rho_M(U_M(1 \otimes \bar{Q}_M^{m+1-k} x)).$$

Thus the relation holds for $m+1$. This completes the proof. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. For a map $f : M^n \to N^{2n-k}$, we have

$$\phi_2^f = \sum_{0\le m\le[(2n-k-1)/2]} \rho_M(u^{2n-k-2m} \otimes (f^* v_m(N))^2) + \rho_M((f^2)^* U_N) \quad \text{by (2.1)},$$

$$= \sum_{0\le m\le[(2n-k-1)/2]} \sum_{1\le j\le[(k-1)/2]} \sum_{0\le l<j} \rho_M(u^{k-2j} \otimes (Q_M^l \bar{Q}^{n-m-k+j-l} f^* v_m(N))^2)$$

$$+ \sum_{0\le m\le[(2n-k-1)/2]} \rho_M(U_M(1 \otimes \bar{Q}_M^{n-m-k} f^* v_m(N)))$$

$$+ \rho_M((f^2)^* U_N) \quad \text{by Lemma 3.1}$$

$$= \sum_{1\le j\le[(k-1)/2]} \rho_M\left(u^{k-2j} \otimes \left(\sum_{0\le l<j} Q_M^l \sum_{0\le m\le n-k+j-l} \bar{Q}_M^{n-k+j-l-m} f^* v_m(N)\right)^2\right)$$

$$+ \rho_M\left(U_M\left(1 \otimes \sum_{0\le m\le n-k} \bar{Q}_M^{n-k-m} f^* v_m(N)\right)\right) + \rho_M((f^2)^* U_N)$$

$$= \sum_{1\le j\le[(k-1)/2]} \sum_{0\le l<j} \rho_M(u^{k-2j} \otimes (Q_M^l w_{n-k+j-l}(f))^2) + \rho_M(U_M(1 \otimes w_{n-k}(f)))$$

$$+ \rho_M((f^2)^* U_N) \quad \text{by (1.1) and Lemma 2.2.}$$

Regarding $\rho_M \mid B^* \oplus I^*$ as the identity, we complete the proof. □

EXAMPLE 3.1. *Let* $f : M^n \to N^{2n-k}$. *If* $k=0$, *then* $\phi_2^f = 0$, *because* $H^{2n}(M^*) = 0$. *If* $k > 0$, *then*

$$\phi_2^f = U_M(1 \otimes w_{n-k}(f)) + (f^2)^* U_N + \begin{cases} 0 & \text{for } k = 1, 2, \\ u^{k-2} \otimes (w_{n-k+1}(f))^2 & \text{for } k = 3, 4, \\ u^{k-2} \otimes (w_{n-k+1}(f))^2 & \\ \quad + u^{k-4} \otimes (w_{n-k+2}(f) + (Sq^1 + w_1(M))w_{n-k+1}(f))^2 & \text{for } k = 5, 6. \end{cases}$$

Corollary 1.2 and Haefliger's theorem indicate that if (1) and (2) of Corollary 1.2 are satisfied then $w_i(f) = 0$ for $i \geq n - [k/2]$. This follows immediately from the following lemma. Let $j : PM \to M^*$ be the natural inclusion and set $j^* u = u$. Then

LEMMA 3.2. *For a map* $f : M^n \to N^{2n-k}$,

$$j^* \phi_2^f = \sum_{1 \leq i \leq k} \sum_{0 \leq j < i} u^{n-i} w_j(M) w_{n-k+i-j}(f).$$

PROOF.

$$j^* \phi_2^f = \sum_{0 \leq i \leq [(2n-k-1)/2]} \rho_2 k^* (u^{2n-k-2i} \otimes (f^* v_i(N))^2) + \rho_2 k^* (f^2)^* U_N$$

by (2.1) and the commutative diagram in [7, Theorem 2.1]

$$= \sum_{0 \leq i} \rho_2 k^* (u^{2n-k-2i} \otimes (f^* v_i(N))^2)$$

$$\text{because} \begin{cases} (f^2)^* U_N \in I_M^* & \text{if } k \text{ is odd,} \\ (f^2)^* (U_N + (v_{n-[k/2]}(N))^2) \in I_M^* & \text{if } k \text{ is even,} \end{cases}$$

by [8, Proposition 2.6], and $k^* I_M^* = 0$ by [7, Proposition 2.5],

$$= \sum_{0 \leq j \leq i} u^{2n-k-2i+i-j} Sq^j f^* v_i(N) \quad \text{by [7, Proposition 2.5 and (2.7)]}$$

$$= \sum_{0 \leq l} u^{2n-k-l} \sum_{i+j=l} f^* Sq^j v_i(N) = \sum_{0 \leq l} u^{2n-k-l} f^* w_l(N)$$

$$= \sum_{0 \leq l \leq n-k} u^{2n-k-l} f^* w_l(N) + \sum_{n-k < l \leq n} u^{2n-k-l} f^* w_l(N)$$

$$= \sum_{0 \leq l \leq n-k} \sum_{1 \leq i \leq n} u^{n-i} \sum_{0 \leq j \leq n-k-l} w_{n+i-k+l-j}(M) \bar{w}_j(M) f^* w_l(N)$$

$$+ \sum_{0 \leq i \leq k} u^{n-i} f^* w_{n-k+i}(N) \quad \text{by [2, Proposition 3.1].}$$

Let a_i be the coefficient of u^{n-i}. Then

$$\begin{aligned}
a_i &= \sum_{0\leq l\leq n-k}\ \sum_{0\leq j\leq n-k-l} w_{n+i-k-l-j}(M)\bar{w}_j(M)f^*w_l(N) + f^*w_{n-k+i}(N) \\
&= \sum_{0\leq m\leq n-k}\ \sum_{0\leq l\leq m} w_{i+n-k-m}(M)\bar{w}_{m-l}(M)f^*w_l(N) + f^*w_{n-k+i}(N) \\
&= \sum_{0\leq m\leq n-k} w_{i+n-k-m}(M)w_m(f) + f^*w_{n-k+i}(N) \quad \text{by (1.1)} \\
&= \sum_{0\leq j<i} w_j(M)w_{n+i-k-j}(f) \quad \text{because } w(M)w(f) = f^*w(N) \text{ by (1.1).} \qquad \square
\end{aligned}$$

4. Relations among Stiefel-Whitney classes of a map

The aim of this section is to prove Theorem 1.3. For a map $f : M^n \to N^{2n-k}$, we have

$$\begin{aligned}
j^*\phi_2^f &= j^*\rho_M(U_M(1\otimes w_{n-k}(f)) + (f^2)^*U_N) \\
&\quad + \sum_{0\leq j\leq[(k-1)/2]}\ \sum_{0\leq l<j} j^*\rho_M(u^{k-2j}\otimes(Q_M^l w_{n-k+j-l}(f))^2) \quad \text{by Theorem 1.1,} \\
&= \sum_{0\leq j\leq[(k-1)/2]}\ \sum_{0\leq l<j} j^*\rho_M(u^{k-2j}\otimes(Q_M^l w_{n-k+j-l}(f))^2) \\
&\quad + [\rho_M k^*(1\otimes(Q_M^{k/2}w_{n-k}(f) + (f^2)^*v_{n-k/2}(N))^2)]^1
\end{aligned}$$

by the diagram in [7, Theorem 2.1] and [8, Proposition 2.6].

Therefore, for a map $f : M^n \to N^{2n-k}$, we have, by [7, Proposition 2.5 and (2.7)],

$$j^*\phi_2^f = \sum_{1\leq j\leq[(k-1)/2]}\ \sum_{0\leq l<j}\ \sum_{0\leq i\leq n-k+j} u^{n-j-i}Sq^iQ_M^l w_{n-k+j-l}(f) \tag{4.1}$$
$$+ \left[\sum_{0\leq i\leq n-k/2} u^{n-k/2-i}Sq^i(Q_M^{k/2}w_{n-k}(f) + f^*v_{n-k/2}(N))\right]$$

We begin proving Theorem 1.3. We give the proof of (2) and omit that of (1).

PROOF OF THEOREM 1.3(2). Comparing the coefficients of u^{n-k} of $j^*\phi_2^f$'s

1. brackets appear only when k is even.

in Lemma 3.2 and (4.1), we have

$$
\begin{aligned}
w_{n-k}(f) &= \sum_{1\le j<k} w_j(M)w_{n-2k+k-j}(f) + Q_M^k w_{n-2k}(f) + f^* v_{n-k}(N) \\
&\quad + \sum_{1\le j<k}\sum_{0\le l<j} Sq^{k-j}Q_M^l w_{n-2k+j-l}(f) \\
&= \sum_{1\le j<k} w_j(M)w_{n-k-j}(f) + Q_M^k w_{n-2k}(f) + f^* v_{n-k}(N) \\
&\quad + \sum_{1\le m<k}\sum_{0\le l<m} Sq^{m-l}Q_M^l w_{n-k-m}(f) \quad \text{by setting } k-j+l=m, \\
&= Q_M^k w_{n-2k}(f) + f^* v_{n-k}(N) \\
&\quad + \sum_{1\le m<k}\left(\sum_{0\le l<m} Sq^{m-l}Q_M^l + w_m(M)\right) w_{n-k-m}(f) \\
&= f^* v_{n-k}(N) + \sum_{1\le m<k} Q_M^m w_{n-k-m}(f) + Q_M^k w_{n-2k}(f) \quad \text{by (2.3).}
\end{aligned}
$$

Thus (2) for $i=0$ holds. Next, we consider the coefficients of u^{n-k-1}. Then

$$
\begin{aligned}
w_{n-k+1}(f) &= \sum_{1\le j\le k} w_j(M)w_{n-k+1-j}(f) + Sq^1(Q_M^k w_{n-2k}(f) + f^* v_{n-k}(N)) \\
&\quad + \sum_{1\le j<k}\sum_{0\le l<j} Sq^{k+1-j}Q_M^l w_{n-k-(k-j+l)}(f) \\
&= w_1(M)w_{n-k}(f) + Sq^1(Q_M^k w_{n-2k}(f) + f^* v_{n-k}(N)) \\
&\quad + \sum_{1\le m<k}\left(w_{m+1}(M) + \sum_{0\le l<m} Sq^{m+1-l}Q_M^l\right) w_{n-k-m}(f) \\
&= w_1(M)w_{n-k}(f) + Sq^1(Q_M^k w_{n-2k}(f) + f^* v_{n-k}(N)) \\
&\quad + \sum_{1\le m<k} (Sq^1 Q_M^m + Q_M^{m+1}) w_{n-k-m}(f) \quad \text{by (2.3),} \\
&= w_1(M)w_{n-k}(f) + Sq^1\left(\sum_{1\le m\le k} Q_M^m w_{n-k-m}(f) + f^* v_{n-k}(N)\right) \\
&\quad + \sum_{1\le m<k} Q_M^{m+1} w_{n-k-m}(f) \\
&= Q_M^1 w_{n-k}(f) + \sum_{1\le m<k} Q_M^{m+1} w_{n-k-m}(f) \quad \text{by (2) for } i=0.
\end{aligned}
$$

Thus (2) holds for $i = 1$. We assume that (2) holds for $i = 0$ to i. By comparing the coefficients of $u^{n-k-i-1}$, we have

$$w_{n-k+i+1}(f) + Sq^{i+1}(Q_M^k w_{n-2k}(f) + f^* v_{n-k}(N))$$

$$= \sum_{1\le j\le k+i} w_j(M) w_{n-k+i+1-j}(f) + \sum_{1\le j<k} \sum_{0\le l<j} Sq^{k+i+1-j} Q_M^l w_{n-2k+j-l}(f)$$

$$= \sum_{1\le j\le i} w_j(M) w_{n-k+i+1-j}(f) + \sum_{i+1\le j\le k+i} w_j(M) w_{n-k+i+1-j}(f)$$

$$+ \sum_{1\le m<k} \sum_{0\le l<m} Sq^{i+1+m-l} Q_M^l w_{n-k-m}(f) \quad \text{by setting } k-j+l=m,$$

$$= \sum_{1\le j\le i} w_j(M) \sum_{0\le m<k} \sum_{0\le l\le i-j} \bar{Q}_M^l Q^{i+1-j+m-l} w_{n-k-m}(f)$$

$$+ \sum_{0\le m<k} w_{i+1+m}(M) w_{n-k-m}(f)$$

$$+ \sum_{1\le m<k} \sum_{0\le l<m} Sq^{i+1+m-l} Q_M^l w_{n-k-m}(f) \quad \text{by the assumption of induction,}$$

$$= \sum_{1\le m<k} \left(\sum_{1\le j\le i} \sum_{0\le l\le i-j} w_j(M) \bar{Q}_M^l Q_M^{i+1-j+m-l} + w_{i+1+m}(M) \right.$$

$$\left. + \sum_{0\le l<m} Sq^{i+1+m-l} Q_M^l \right) w_{n-k-m}(f)$$

$$+ \left(\sum_{1\le j\le i} w_j(M) \sum_{0\le l\le i-j} \bar{Q}_M^l Q_M^{i+1-j-l} + w_{i+1}(M) \right) w_{n-k}(f)$$

$$= \sum_{1\le m<k} \left(\sum_{1\le r\le i} \sum_{0\le l<r} w_{r-l}(M) \bar{Q}_M^l Q_M^{i+1+m-r} + Q_M^{i+1+m} \right.$$

$$\left. + \sum_{1\le r\le i+1} Sq^r Q_M^{i+1+m-r} \right) w_{n-k-m}(f)$$

$$+ \left(\sum_{1\le r\le i} \sum_{0\le l<r} w_{r-l}(M) \bar{Q}_M^l Q_M^{i+1-r} + w_{i+1}(M) \right) w_{n-k}(f) \quad \text{by setting } j+l=r,$$

$$= \sum_{1\le m<k}\left(\sum_{1\le r\le i}(Sq^r+\bar{Q}^r_M)Q^{i+1+m-r}+Q^{i+m+1}+Sq^{i+1}Q^m_M\right.$$

$$\left.+\sum_{1\le r\le i}Sq^rQ^{i+1+m-r}_M\right)w_{n-k-m}(f)$$

$$+\left(\sum_{1\le r\le i}(\bar{Q}^r_M+Sq^r)Q^{i+1-r}_M+w_{i+1}(M)\right)w_{n-k}(f) \quad \text{by (2.3)},$$

$$= \sum_{1\le m<k}\left(\sum_{1\le r\le i}\bar{Q}^r_MQ^{i+1+m-r}_M+Q^{i+1+m}_M+Sq^{i+1}Q^m_M\right)w_{n-k-m}(f)$$

$$+\left(\sum_{1\le r\le i}\bar{Q}^r_MQ^{i+1-r}_M+\sum_{1\le r\le i}Sq^rQ^{i+1-r}_M+w_{i+1}(M)\right)w_{n-k}(f)$$

$$= \sum_{0\le m<k}Q^{i+1+m}_Mw_{n-k-m}(f)+\sum_{0\le m<k}\sum_{1\le r\le i}\bar{Q}^r_MQ^{i+1+m-r}_Mw_{n-k-m}(f)$$

$$+Sq^{i+1}\left(w_{n-k}(f)+\sum_{1\le m<k}Q^m_Mw_{n-k-m}(f)\right).$$

Hence, we have, by (2) for $i=0$,

$$w_{n-k+i+1}(f)=\sum_{0\le m<k}\sum_{0\le r\le i}\bar{Q}^r_MQ^{i+1+m-r}_Mw_{n-k-m}(f).$$

This completes the proof of (2). □

5. On the estimate of the Stiefel-Whitney classes of a map

Theorem 1.4, below, was proved implicitly in the proof of [5, Theorem 2.1]. We will give another proof in a way somewhat different from that of [5], while using older methods.

THEOREM 1.4 (cf. Li and Peterson). *For a compact n-manifold M^n and a map $f: M^n \to N^{n+k}$,*

$$\langle w(f)x,[M]\rangle=\langle \bar{S}q(x)f^*(v(N)),[M]\rangle \qquad \text{for } x\in H^*(M).$$

PROOF. Let $f: M^n\to N$ be a map and let $x\in H^*(M)$. Then

$$
\begin{aligned}
\langle w(f)x, [M]\rangle &= \langle \bar{w}(M) f^*(w(N))x, [M]\rangle \quad \text{by (1.1)}, \\
&= \langle \bar{w}(M) f^*(Sq(v(N)))Sq\bar{S}q(x), [M]\rangle \quad \text{because } Sqv(N) = w(N), \\
&= \langle \bar{w}(M) Sq(f^*(v(N))\bar{S}q(x)), [M]\rangle \\
&= \langle \bar{Q}_M(f^*(v(N))\bar{S}q(x)), [M]\rangle \quad \text{by (2.3)}, \\
&= \left\langle \sum_{k=0}^{n} \bar{Q}_M^k [f^*(v(N)\bar{S}q(x)]_{n-k}, [M]\right\rangle,
\end{aligned}
$$

where $[f^*(v(N))\bar{S}q(x)]_{n-k}$ denotes the $(n-k)$-dimensional part of $f^*(v(N))\bar{S}q(x)$. By (2.4), we have

$$\langle w(f)x, [M]\rangle = \langle [f^*v(N)\bar{S}q(x)]_n, [M]\rangle = \langle f^*v(N)\bar{S}q(x), [M]\rangle. \qquad \square$$

COROLLARY 5.1 (cf. Li and Peterson). *The following relations hold:*
(1) $w_{n-1}(f)x = \sum_{i\geq 0} x^{2^i} f^*(v_{n-2^i}(N))$ *for* $x \in H^1(M)$,
(2) $w_{n-2}(f)y = \sum_{i\geq 0} y^{2^i} f^*(v_{n-2^{i+1}}(N))$ *for* $y \in H^2(M)$ *with* $Sq^1 y = 0$.

PROOF. This follows from Theorem 1.4 and the facts that $\bar{S}q(x) = \sum_{i\geq 0} x^{2^i}$ if $\dim x = 1$, and $\bar{S}q(y) = \sum_{i\geq 0} y^{2^i}$ if $\dim y = 2$ with $Sq^1 y = 0$. $\square$

Applications of this corollary to the non-existence problem of immersions and embeddings are given in [5] and [9].

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