Shortest Spherical Network of Pentahedra

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Abstract

We study the problem to divide the spherical surface into five parts of equal area by a network of edges of the shortest total length. It is proved that the regular 3-prism gives the shortest network.

1 Problem and result

Fejes Tóth ([1], [2]) posed the following problem: to divide the surface of the unit sphere into $n(\geq 4)$ parts of equal area, by the shortest possible net of edges. To study it he invented an ingenious method, but even the method could give solutions only for $n = 4, 6$, and 12. In this paper we give a solution for $n = 5$ as follows.

Theorem *Among all networks of pentahedra, the regular 3-prism has the shortest total length of edges.*

(Proof) Note that spherical networks made of pentahedra can have only two topological types, ie prism and pyramid. By Proposition 1 and Proposition 2, it s sufficient to compare the total length of edges of the regular 3-prism L(3-prism) and that of the regular 4-pyramid L(4-pyramid). As is shown in the proofs of these propositions,

$$L(\text{3-prism}) = 3\,\tilde{f}(e_3, e_3), \quad L(\text{4-pyramid}) = 4\,\tilde{g}(e_4),$$

where functions $\tilde{f}, \tilde{g}$ are defined by (6) and (10), and e_n is the common length of sides of the regular n-gon. To find e_3 and e_4 we use Lemma 1 below. Then we can evaluate the two total lengths as $L(\text{3-prism}) \approx 4.28186\pi$ and $L(\text{4-pyramid}) \approx 4.34633\pi$. Thus the theorem is established. (Q.E.D.)

Proposition 1 *Assume $n \leq 4$. Then, among all networks of n-prism type, the regular n-prism has the shortest total length of edges.*

Proposition 2 *Assume $n \leq 5$. Then, among all networks of n-pyramid type, the regular n-pyramid has the shortest total length of edges.*

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Lemma 1 *Let a be the common length of sides of the regular n-gon of area S. Then it is given by*

$$\cos\frac{a}{2} = \frac{\cos\dfrac{\pi}{n}}{\cos\dfrac{2\pi - S}{2n}}.$$

(Proof) Divide the regular n-gon into $2n$ congruent rectangular triangles, and consider one of them. Let z be a side opposite to the rectangle, y be an other side than $a/2$, θ be an angle opposite to y. Then

$$\begin{cases} \theta + \dfrac{\pi}{n} + \dfrac{\pi}{2} - \pi = \dfrac{S}{2n}, \\ \cos z = \cos y \cos\dfrac{a}{2}, \\ \dfrac{\sin z}{\sin\frac{\pi}{2}} = \dfrac{\sin y}{\sin\theta} = \dfrac{\sin\frac{a}{2}}{\sin\frac{\pi}{n}} . \end{cases}$$

Eliminating y, z, θ in the above, we obtain the desired formula. (Q.E.D.)

2 Proof of Proposition 1

Lemma 2.1 *Consider all convex quadrangles where a pair of opposite sides a, b and an area S are fixed. Then the quadrangle that minimizes the sum of other two sides $x + y$, is an isosceles trapezoid, ie $x = y$, that has a circumcircle.*

(Proof)
Step 1 Regard the minimum of $x + y$ as a function of S, and denote it by $h(S)$. We will show that h is a strictly increasing function of S. For any S, consider the minimal quadrangle and denote its four angles by ϕ_i $(i = 1, 2, 3, 4)$. If $\phi_i \leq \pi/2$ for all i, then

$$S = \phi_1 + \phi_2 + \phi_3 + \phi_4 - 2\pi \leq 0 ,$$

which is a contradiction. Hence $\phi_i > \pi/2$ for some i, and thus, without loss of generality, suppose that $\phi_1 > \pi/2$. Then consider a triangle that consists of an angle ϕ_1 and two sides of the quadrangle that emanate from the angle. Without loss of generality we may suppose that these two sides are b and y. Let z be the other side than b, y of the triangle. Then, preserving lengths b and z, and diminishing y continuously by δ, we can diminish area of the triangle and thus area of the quadrangle by ϵ. Consequently, by the definition of h, we can see $h(S - \epsilon) \leq h(S) - \delta < h(S)$. Here note that ϵ can take an arbitrary positive number as long as it is sufficiently small. Therefore h is strictly increasing.

Step 2 Consider the minimal quadrangle Q of area S. Assume that it does not have a circumcircle. It is well-known that the convex quadrangle of given four sides and of the maximal area has a circumcircle. Hence there exists a quadrangle Q' which has the same four sides as Q has, but has a larger area S' than S. Repeating the argument in Step 1, we can deduce that there exists a quadrangle Q'' which has smaller $x + y$ than Q has, but has an area S'' such that $S < S'' < S'$. But this implies $h(S) < h(S'') < h(S') = h(S)$, which is a contradiction. Accordingly the minimal quadrangle has a circumcircle.

Step 3 Let Q be the minimal quadrangle of area S, and R be the radius of its circumcircle. Divide the quadrangle into four isosceles triangles with bases a, b, x, y, and denote them by T_a, T_b, T_x, T_y respectively. Let $\alpha, \beta, \phi, \psi$ be angles of these isosceles at the center of circumcircle.

Consider an isosceles T_a and denote its two angles other than α by θ. Then

$$\cos\alpha = \frac{\cos a - \cos^2 R}{\sin^2 R} \quad \text{and} \quad \cos\theta = \frac{(1-\cos a)\cos R}{\sin a \sin R} .$$

Thus, if we define two functions

$$f(a,R) = \arccos\left(\frac{(1-\cos a)\cos R}{\sin a \sin R}\right) \quad \text{and} \quad g(a,R) = \arccos\left(\frac{\cos a - \cos^2 R}{\sin^2 R}\right),$$

we have $\alpha = g(a,R)$ and area of the isosceles $= g(a,R) + 2f(a,R) - \pi$.

Now note that $\alpha+\beta+\phi+\psi = 2\pi$ and the sum of areas of isosceles T_a, T_b, T_x, T_y equals S. Accordingly we have

$$F(x,y,R) := f(a,R) + f(b,R) + f(x,R) + f(y,R) = \pi + \frac{S}{2} \tag{1}$$

and

$$G(x,y,R) := g(a,R) + g(b,R) + g(x,R) + g(y,R) = 2\pi . \tag{2}$$

Step 4 If we solve (1) and (2) with respect to x and y, while R being regarded as a parameter, we have $x = x(R), y = y(R)$. Since it is required to minimize $x(R) + y(R)$ with respect to R, it must hold

$$\frac{dx}{dR} + \frac{dy}{dR} = 0 . \tag{3}$$

Differentiation of (1) and (2) give

$$\begin{cases} \dfrac{\partial F}{\partial x}\dfrac{dx}{dR} + \dfrac{\partial F}{\partial y}\dfrac{dy}{dR} + \dfrac{\partial F}{\partial R} & = 0 , \\ \dfrac{\partial G}{\partial x}\dfrac{dx}{dR} + \dfrac{\partial G}{\partial y}\dfrac{dy}{dR} + \dfrac{\partial G}{\partial R} & = 0 . \end{cases}$$

Then substitution of them into (3) results in

$$\left(\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y}\right) : \left(\frac{\partial G}{\partial x} - \frac{\partial G}{\partial y}\right) = \frac{\partial F}{\partial R} : \frac{\partial G}{\partial R} \tag{4}$$

Now an elementary computation gives

$$\begin{cases} \dfrac{\partial F}{\partial x} - \dfrac{\partial F}{\partial y} & = -\cos R\left(\dfrac{1}{\sqrt{1+\cos x}\cdot h_x} - \dfrac{1}{\sqrt{1+\cos y}\cdot h_y}\right), \\ \dfrac{\partial G}{\partial x} - \dfrac{\partial G}{\partial y} & = \dfrac{\sqrt{1+\cos x}}{h_x} - \dfrac{\sqrt{1+\cos y}}{h_y} , \\ \qquad \dfrac{\partial F}{\partial R} & = \dfrac{1}{\sin R}\left(\dfrac{\sqrt{1-\cos a}}{h_a} + \dfrac{\sqrt{1-\cos b}}{h_b} + \dfrac{\sqrt{1-\cos x}}{h_x} + \dfrac{\sqrt{1-\cos y}}{h_y}\right), \\ \qquad \dfrac{\partial G}{\partial R} & = -2\cot R\left(\dfrac{\sqrt{1-\cos a}}{h_a} + \dfrac{\sqrt{1-\cos b}}{h_b} + \dfrac{\sqrt{1-\cos x}}{h_x} + \dfrac{\sqrt{1-\cos y}}{h_y}\right), \end{cases}$$

where $h_x = \sqrt{1+\cos x - 2\cos^2 R}$ and h_y, h_a, h_b are defined similarly.
By substitution of them into the condition (4) it can be rewritten as

$$\begin{aligned} &2\cos^2 R \cdot h_y \sqrt{1+\cos y} + h_x \sqrt{1+\cos x} \cdot (1+\cos y) \\ = \;\; &2\cos^2 R \cdot h_x \sqrt{1+\cos x} + h_y \sqrt{1+\cos y} \cdot (1+\cos x) \end{aligned} \tag{5}$$

Step 5 Squaring both the left- and the right-hand side of (5) and subtracting them, we have

$$2\cos^2 R\,(\cos x - \cos y)\,(w_1 - w_2) = 0\ ,$$

where

$$\begin{aligned} w_1 &= 1 - 4\cos^2 R + 4\cos^4 R + \cos x\,(1 - 2\cos^2 R) \\ &\quad + \cos y\,(1 - 2\cos^2 R) + \cos x \cos y \\ w_2 &= 2\sqrt{1+\cos x}\,\sqrt{1+\cos y}\,h_x\,h_y\ . \end{aligned}$$

Suppose that $w_1 = w_2$. Then we have $w_1^2 - w_2^2 = -h_x^2\,h_y^2\,w_3$, where

$$w_3 = 3(1+\cos x)(1+\cos y) + 2\cos^2 R\,(2+\cos x+\cos y) - 4\cos^4 R\ .$$

However, as seen in the definition of h_x, h_y, we have $1+\cos x - 2\cos^2 R > 0$ and $1+\cos y - 2\cos^2 R > 0$. Consequently

$$w_3 > 3\cdot 2\cos^2 R\cdot 2\cos^2 R + 2\cos^2 R\cdot(2\cos^2 R + 2\cos^2 R) - 4\cos^4 R = 16\cos^2 R > 0\ .$$

Thus the hypothesis $w_1 = w_2$ can not be maintained. Therefore we obtain $\cos x - \cos y = 0$, ie. $x = y$. Since the quadrangle is circumscribed by a circle, it must be an isosceles trapezoid. (Q.E.D.)

Consider the minimal isosceles trapezoid in Lemma 2.1. Writing x, y instead of a, b, and regarding half of the minimum, ie the length of one of its two equal sides as a function of x, y, we denote it by $f(x, y)$.

Lemma 2.2 *The function f is given by*

$$f(x,y) = \arccos\left(\frac{-1 + s_x^2 + s_x\,s_y + s_y^2 + k\,c_x\,c_y}{1 + s_x\,s_y - k\,c_x\,c_y}\right)\ ,$$

where

$$c_x = \cos\frac{x}{2},\ c_y = \cos\frac{y}{2},\ s_x = \sin\frac{x}{2},\ s_y = \sin\frac{y}{2},\ \text{and } k = \cos\frac{S}{2}\ .$$

(Proof) We write simply by z instead of $f(x, y)$. Divide the isosceles trapezoid by its symmetry axis, and consider one of the two quadrangles made by the division. Let ϕ be the angle between $x/2$ and z, and ψ be the angle between $y/2$ and z. If we prolong both sides $x/2$ and $y/2$, then a triangle will be made, of which three sides are $\pi/2 - x/2, \pi/2 - y/2, z$, and two of its three angles are $\pi - \phi, \pi - \psi$. Then

$$\begin{cases} \cos\left(\dfrac{\pi}{2} - \dfrac{y}{2}\right) = \cos\left(\dfrac{\pi}{2} - \dfrac{x}{2}\right)\cos z + \sin\left(\dfrac{\pi}{2} - \dfrac{x}{2}\right)\sin z\,\cos(\pi - \phi), \\ \cos\left(\dfrac{\pi}{2} - \dfrac{x}{2}\right) = \cos\left(\dfrac{\pi}{2} - \dfrac{y}{2}\right)\cos z + \sin\left(\dfrac{\pi}{2} - \dfrac{y}{2}\right)\sin z\,\cos(\pi - \psi)\ . \end{cases}$$

Hence

$$\cos\phi = \frac{s_x\,\cos z - s_y}{c_x\,\sin z}, \qquad \sin\phi = \frac{\sqrt{1 - s_x^2 - s_y^2 + 2s_x\,s_y\,\cos z - \cos^2 z}}{c_x\,\sin z},$$

$$\cos\psi = \frac{s_y\,\cos z - s_x}{c_y\,\sin z}, \qquad \sin\psi = \frac{\sqrt{1 - s_y^2 - s_x^2 + 2s_x\,s_y\,\cos z - \cos^2 z}}{c_y\,\sin z}.$$

On the other hand, $\phi + \psi + \pi/2 + \pi/2 - 2\pi = S/2$, ie $\phi + \psi = \pi + S/2$. Then, eliminating ϕ, ψ in $\cos(\phi + \psi) = \cos(\pi + S/2)$, we obtain a quadratic equation for $w = \cos z$,

$$(1 + s_x\,s_y - k\,c_x\,c_y)w^2 - (s_x + s_y)^2 w + (-1 + s_x^2 + s_x\,s_y + s_y^2 + k\,c_x\,c_y) = 0\;.$$

Note that the coefficient of w^2 in the above equation does not vanish, because, if it vanishes, then we have $w = 1$, which is a contradiction. Furthermore note that the quadratic equation always has a root $w = 1$. Hence another root is given by $f(x, y)$. (Q.E.D.)

Lemma 2.3 *The function $f(x, y)$ is strictly convex.*

(Proof)

Step 1 We can see

$$\frac{\partial f}{\partial x} = \frac{n_x}{d_1}, \quad \frac{\partial f}{\partial y} = \frac{n_y}{d_1},$$

where

$$\begin{aligned}
d_1 &= 2\,(1 - k\,c_x\,c_y + s_x\,s_y)\sqrt{c_x^2 + c_y^2 - 2k\,c_x\,c_y}\;,\\
n_x &= k\,(1 + c_x^2)\,c_y - c_x\,(1 + c_y^2) - c_x\,s_x\,s_y + k\,s_x\,c_y\,s_y,\\
n_y &= -(1 + c_x^2)\,c_y + k\,c_x\,(1 + c_y^2) + k\,c_x\,s_x\,s_y - s_x\,c_y\,s_y\;.
\end{aligned}$$

Step 2 Furthermore

$$\frac{\partial^2 f}{\partial x^2} = \frac{n_{xx}}{d_2}, \quad \frac{\partial^2 f}{\partial x\,\partial y} = \frac{n_{xy}}{d_2}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{n_{yy}}{d_2},$$

where

$$\begin{aligned}
d_2 &= 4\,(c_x^2 + c_y^2 - 2k\,c_x\,c_y)^{\frac{3}{2}}\,(1 - k\,c_x\,c_y + s_x\,s_y)^2,\\
n_{xx} &= (1 - k^2)\,(s_x + s_y)\,(1 + c_y^2 + s_x^3\,s_y - 3k\,c_x\,c_y + k\,c_x^3\,c_y),\\
n_{xy} &= -(1 - k^2)\,(s_x + s_y)\,(1 - 2c_x^2 - 2c_y^2 + c_x^2\,c_y^2 + s_x\,s_y + 2kc_xc_y - kc_xs_xc_ys_y),\\
n_{yy} &= (1 - k^2)\,(s_x + s_y)\,(1 + c_x^2 + s_x\,s_y^3 - 3k\,c_x\,c_y + k\,c_x\,c_y^3)\;.
\end{aligned}$$

Hence the Jacobian J becomes

$$J := \frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 = \frac{(1 - k^2)^2\,(s_x + s_y)^4}{16\,(c_x^2 + c_y^2 - 2kc_x\,c_y)^2\,(1 - kc_x\,c_y + s_x\,s_y)^3}\;.$$

Note that if $J > 0$, then the function f is strictly convex. Thus it remains to prove $1 + s_x\, s_y - k\, c_x\, c_y > 0$.

Step 3 Consider again the triangle with three sides $\pi/2 - x/2, \pi/2 - y/2, z$ that appeared in the proof of Lemma 2.2.. It must hold $(\pi/2 - x/2) + (\pi/2 - y/2) > z$ ie. $\pi - z > x/2 + y/2$. Hence, for $w = \cos z$,

$$w > -c_x\, c_y + s_x\, s_y \ .$$

Then the expression for $z = f(x, y)$ given in Lemma 2 becomes

$$\frac{-1 + s_x^2 + s_x\, s_y + s_y^2 + k\, c_x\, c_y}{1 + s_x\, s_y - k\, c_x\, c_y} > -c_x\, c_y + s_x\, s_y \ .$$

Suppose that $1 + s_x\, s_y - k\, c_x\, c_y < 0$. Then, after some computation, we can derive $k < 0$. However, since $S = 4\pi/(n+2)$ with $n \geq 3$, this leads to a contradiction. Thus we have $1 + s_x\, s_y - k\, c_x\, c_y > 0$. (Q.E.D.)

Let us define a function

$$\tilde{f}(x, y) = x + y + f(x, y) \ . \tag{6}$$

Lemma 2.4 *For $n \leq 4$, the function $\tilde{f}(x, y)$ is strictly increasing.*

(Proof) Ask when the following condition holds

$$\frac{\partial}{\partial x}\tilde{f}(x, y) = \frac{\partial}{\partial y}\tilde{f}(x, y) = 0 \ . \tag{7}$$

Then, by the expressions given in Step 1 of the proof of Lemma 2.3, we have

$$d_1 + n_x = d_1 + n_y = 0.$$

Since

$$n_x - n_y = -(1 + k)\,(c_x - c_y)\,(1 - c_x c_y + s_x s_y).$$

Hence we can deduce $x = y$. Then we have

$$c_x = \sqrt{\frac{1}{1+k}\left(2 - \sqrt{\frac{1-k}{2}}\right)} \ .$$

For $n \leq 4$ we see

$$\frac{1}{1+k}\left(2 - \sqrt{\frac{1-k}{2}}\right) \geq 1 \ .$$

Accordingly the condition (7) does not hold. Therefore the proof is completed.

(Q.E.D.)

Proof of Proposition 1

A network of n-prism type consists of n quadrangles $Q_i\,(i=1,2,\cdots,n)$, and two n-gons A and B. Let a_i be the common side of A and Q_i, and b_i be the common side of B and Q_i. Denote the total legth of Q_i by L_i. Then it can be seen that the total length of the network L is given by

$$2L=\sum_{i=1}^{n} L_i+\sum_{i=1}^{n} a_i+\sum_{i=1}^{n} b_i\ .$$

Since, by Lemma 2.1,

$$L_i \geq a_i+b_i+2\,f(a_i,b_i),$$

we have

$$L \geq \sum_{i=1}^{n} f(a_i,b_i)+\sum_{i=1}^{n} a_i+\sum_{i=1}^{n} b_i=\sum_{i=1}^{n} \tilde{f}(a_i,b_i)\ .$$

Then, Lemma 2.3, with aid of Jensen's inequality, shows that

$$\frac{1}{n}\sum_{i=1}^{n} \tilde{f}(a_i,b_i) \geq \tilde{f}(\overline{a},\overline{b})\ ,$$

where

$$\overline{a}=\frac{1}{n}\sum_{i=1}^{n} a_i \quad\text{and}\quad \overline{b}=\frac{1}{n}\sum_{i=1}^{n} b_i\ .$$

Consequently

$$\frac{L}{n} \geq \tilde{f}(\overline{a},\overline{b})\ .$$

Now recall the isoperimetric property of spherical polygons: among all spherical polygons of area S, the regular polygon has the shortest perimeter length. Thus, if e_n stands for the length of one side of the regular n-gon of area S, we have $\overline{a}\geq e_n, \overline{b}\geq e_n$. Then Lemma 2.4 implies $\tilde{f}(\overline{a},\overline{b}) \geq \tilde{f}(e_n,e_n)$. Therefore we obtain

$$L \geq n\,\tilde{f}(e_n,e_n),$$

which proves the theorem. (Q.E.D.)

3 Proof of Proposition 2

Lemma 3.1 *Consider all triangles where a side a and an area S are fixed. Then the triangle that minimizes the sum of other two sides $x+y$ is an isosceles, ie $x=y$.*

(Proof) Consider a triangle satisfying the given conditions, and let R be the radius of its circumcircle. (Note that any triangle has a circumcircle.) Divide the triangle into three isosceles triangles with bases a,x,y, and denote them by T_a,T_x,T_y respectively. By a similar reasoning to that in Step 3 of the proof of Lemma 1.1, we see that it is sufficient to minimize $x+y$ when x,y satisfies both conditions

$$F(x,y,R):=f(a,R)+f(x,R)+f(y,R)=\pi+\frac{S}{2} \tag{8}$$

and

$$G(x,y,R):=g(a,R)+g(x,R)+g(y,R)=2\pi\ . \tag{9}$$

Once F, G were defined, we can repeat the reasoning in Step 4 and Step 5 of the proof of Lemma 1.1 without any change. Thus we come to the conclusion $x = y$, which completes the proof. (Q.E.D.)

Consider the minimal isosceles triangle in Lemma 3.1. Writing x instead of a, and regarding half of the minimum, ie the length of one of its two equal sides as a function of x, we denote it by $g(x)$.

Lemma 3.2 *The function g is given by*

$$g(x) = \arccos\left(\frac{c_x\,(k - c_x)}{1 - k\,c_x}\right),$$

where

$$c_x = \cos\frac{x}{2} \quad \text{and} \quad k = \cos\frac{S}{2}\ .$$

(Proof) We write simply by z instead of $g(x)$. Divide the isosceles triangle by its symmetry axis, and consider one of the two triangles made by the division. Let ϕ be the angle opposite to the side $x/2$, and ψ be the angle between $x/2$ and z. Prolong the side $x/2$ and draw a line which makes an angle $\pi/2 - \phi$ with the side z. Then a triangle will be made, of which three sides are $\pi/2, \pi/2 - y/2, z$, and two of its three angles are $\pi/2 - \phi, \pi - \psi$. Then

$$\begin{cases} \cos\left(\dfrac{\pi}{2} - \dfrac{x}{2}\right) & = \cos\dfrac{\pi}{2}\cos z + \sin\dfrac{\pi}{2}\sin z\,\cos\left(\dfrac{\pi}{2} - \phi\right), \\ \cos\dfrac{\pi}{2} & = \cos\left(\dfrac{\pi}{2} - \dfrac{x}{2}\right)\cos z + \sin\left(\dfrac{\pi}{2} - \dfrac{x}{2}\right)\sin z\,\cos(\pi - \psi)\ . \end{cases}$$

Hence

$$\sin\phi = \frac{s_x}{\sin z}, \qquad \cos\phi = \frac{\sqrt{c_x^2 - \cos^2 z}}{\sin z},$$
$$\cos\psi = \frac{s_x\,\cos z}{c_x\,\sin z}, \qquad \sin\psi = \frac{\sqrt{c_x^2 - \cos^2 z}}{c_x\,\sin z}\ ,$$

where $s_x = \sin\frac{x}{2}$. Now, from the assumption on area, we have $\phi + \psi = \dfrac{\pi}{2} + \dfrac{S}{2}$. Hence follows a quadratic equation for $w = \cos z$,

$$(1 - k\,c_x)\,w^2 - (1 - c_x^2)\,w + c_x\,(k - c_x) = 0\ .$$

It can be factored as

$$(w - 1)\,((1 - k\,c_x)\,w - c_x\,(k - c_x)) = 0\ ,$$

which gives the desired result. (Q.E.D.)

Lemma 3.3 *The function g is strictly convex.*

(Proof) By differentiation we have

$$\frac{\partial g}{\partial x} = \frac{-2c_x + k\,(1+c_x^2)}{2(1-k\,c_x)\sqrt{1+c_x^2-2k\,c_x}},$$
$$\frac{\partial^2 g}{\partial x^2} = \frac{(1-k^2)(2+k\,(C_x^3-3c_x))\,\sqrt{1-c_x^2}}{4(1-k\,c_x)^2(1+c_x^2-2k\,c_x)^{3/2}}.$$

Since $0 > c_x^3 - 3c_x > -2$ for $0 < c_x < 1$, we see $2 + k\,(c_x^3 - 3c_x) > 2 - 1 \cdot 2 = 0$, and thus the second derivative is positive. Thus the proof is completed. (Q.E.D.)

Let us define a function

$$\tilde{g}(x) = x + g(x)\ . \tag{10}$$

Lemma 3.4 *Assume $n \leq 5$. Then the function $\tilde{g}$ is strictly increasing.*

(Proof) Using the expression for the derivative of g given in the proof of Lemma 3.3, from the condition that the derivative of $\tilde{g}$ vanishes, it follows $h(c_x) = 0$, where

$$h(\xi) = 3k^2\,\xi^4 - 4k\,(1+2k^2)\,\xi^3 + 18k^2\,\xi^2 - 12k\,\xi + (4-k^2)\ .$$

Since $h(1) = 4(1-k)^2(1-2k)$, we have $h(1)$ is non-negative when $n \leq 5$. Furthermore,

$$h'(\xi) = -12k(1+\xi^2-2k\,\xi)(1-k\,\xi) < 0$$

for $0 < \xi < 1$. Accordingly $h(\xi) > 0$ for $0 < \xi < 1$. Hence we obtain the conclusion.

(Q.E.D.)

Proof of Proposition 2

A network of n-pyramid type consists of n triangles T_i $(i = 1, 2, \cdots, n)$ and an n-gon A. Let a_i be the common side of A and T_i. Denote by L_i the total length of perimeter of T_i. Then the total length of the network L is given by

$$2L = \sum_{i=1}^{n} L_i + \sum_{i=1}^{n} a_i\ .$$

Since Lemma 3.1 shows that $L_i \geq a_i + 2g(a_i)$, we have

$$L \geq \sum_{i=1}^{n} g(a_i) + \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \tilde{g}(a_i)\ .$$

Accordingly, by convexity of f proved in Lemma 3.3, Jensen's inequality implies

$$\frac{L}{n} \geq \tilde{g}(\overline{a}). \tag{11}$$

Now the isoperimetric inequality shows that $\overline{a} \geq e_n$. Then Lemma 3.4 implies that $\tilde{g}(\overline{a}) \geq \tilde{g}(e_n)$. Therefore we obtain

$$L \geq n\,\tilde{g}(e_n),$$

which completes the proof.

(Q.E.D.)

References

[1] Fejes Tóth, L. (1953) *Lagerungen in der Ebene, auf der Kugel und im Raum*, Springer-Verlag.

[2] Fejes Tóth, L. (1964) *Regular Figures*, Pergamon Press.