APPLICATIONS OF AVERAGING METHODS TO COMPLEX FINSLER GEOMETRY

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Preface

A complex Finsler metric on a manifold is a smooth assignment of a complex Minkowski norm on each tangent space, and thus the class of complex Finsler metrics contains Hermitian metrics as a special sub-class. The geometry of complex Finsler manifolds was started by G.B. Rizza [Ri], and, via the methods of tensor analysis, H. Rund([Ru1], [Ru2]) developed the theory of connections on a complex manifold endowed with a complex Finsler metric, and he derived the connection coefficients and presented the equation of geodesics in close analogue with the real case (see also [Ic2],[Ic3], [Fm], [Ro], [Ai1] and [Ai2]).

The ampleness of holomorphic line bundles is an important notion in algebraic geometry, and this notion is generalized to the case of holomorphic vector bundles of higher rank(see [Ha], [Ko1]). The geometry of complex Finsler vector bundles becames productive after S. Kobayashi [Ko1], in which he suggested the importance of complex Finsler geometry in the study of ampleness of holomorphic vector bundles.

Let $\pi : E \to M$ be a holomorphic vector bundle over a compact complex manifold M. Denoted by E^0 the set of all non-zero elements of E, the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ acts on E^0 by scalar multiplication. Then the projective bundle $\phi : \mathbb{P}(E) \to M$ associated with E is defined by $\mathbb{P}(E) = E^0/\mathbb{C}^*$, and the *tautological line bundle* $\mathbb{L}(E)$ over $\mathbb{P}(E)$ is defined by $\mathbb{L}(E) = \{([v], V) \in \mathbb{P}(E) \times E \mid V \in [v]\}$. Then E is said to be *negative* if $\mathbb{L}(E)$ is negative, i.e., the first Chern class $c_1(\mathbb{L}(E))$ is represented a negative real $(1,1)$-form. A holomorphic vector bundle is said to be *ample* if its dual E^* is negative. By sending any point v in E^0 to a point $([v], v) \in \mathbb{P}(E) \times E^0$, we may identify E^0 with $\mathbb{L}(E)^0$. Thus any Hermitian metric on $\mathbb{L}(E)$ is identified with a complex Finsler metric F on E, and the first Chern class of $\mathbb{L}(E)$ is expressed by $c_1(\mathbb{L}(E)) = \left[\frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial \log F\right]$. Using this fact, Kobayashi [Ko1] showed that the negativity of E is equivalent to the existence of complex Finsler metric on E such that $\sqrt{-1}\bar{\partial}\partial \log F < 0$. If E is negative, we can construct a complex Finsler metric F on E such that $\sqrt{-1}\bar{\partial}\partial \log F < 0$ (cf. [Ai6], [Ha-Ai]). We note that, if E is negative, then $\mathbb{P}(E)$ is a Kähler manifold. The converse is not true, since if M is a compact Kähler manifold, then $\mathbb{P}(E)$ admits a Kähler metric for any holomorphic vector bundle E [Sh-So]. After the epoch-making study[Ko1], many authors investigated complex Finsler geometry, including [Ab-Pa1], [Ca-Wo], [Wo2], [Ai1] and [Pa]. In [Ai1], Aikou showed the local expressions of two types of complex Finsler connection. The first one is the Finsler Hermitian connection which is treated in [Ko1] and the second one is the complex Rund connection which is treated in [Ru2]. Some special Finsler vector bundles were also discussed in [Ai2], [Ai3], [Ai4], [Ai7] and [Ai9].

From the expression (3.28) of $\sqrt{-1}\bar{\partial}\partial \log F$ in Sub-section 3.3.1, it follows that the negativity of E is easier to describe than the ampleness of E from the viewpoint of differential geometry. Hence, in this thesis, we will investigate negative vector bundles instead of ample vector bundles, and we shall show some results obtained from Kobayashi's characterization.

A complex Finsler metric is called a *Rizza metric* if F is strongly pseudo-convex along fibers. If a Rizza metric F is given on a holomorphic vector bundle E over a complex manifold M, then the bundles $\pi : E \to M$ and $\phi : \mathbb{P}(E) \to M$ endow the structure of smooth families of Kähler manifolds. Hence, in this thesis, the author investigates complex Finsler geometry from the view point of differential geometry of smooth families of Kähler manifolds.

On the other hand, G. Schumacher [Sc3] has studied the geometry of a smooth family of compact Kähler-Einstein manifolds, and he showed that his method plays an important role in the study of moduli spaces of Kähler-Einstein manifolds. His fundamental tool is the Lie derivation in a horizontal direction. Defining a *horizontal distribution* from the smooth family of Kähler-Einstein metrics, he applies the Lie derivation in the horizontal direction only to relative tensors, i.e., to differentiable families of tensor fields on the fibers. Such a horizontal distribution is nothing but the *complex non-linear connection* defined in [Ai1] if we investigate the geometry of complex Finsler bundle (E, F), and the Lie derivation in the horizontal direction induces the notion of *partial connection* in the sense of [Ai6]. In this thesis we also apply Lie derivation to smooth families of tensor fields on the fibers $E_z := \pi^{-1}(z)$ or $\mathbb{P}_z = \phi^{-1}(z)$.

The purpose of this thesis is to investigate some negativities of holomorphic vector bundles by using the so-called averaging methods. We say that a holomorphic vector bundle E is *Griffiths-negative* if E admits a Hermitian metric of negative curvature. Also we say that E is *Rizza-negative* if E admits a Rizza metric of negative curvature. To compare these two negativities, we shall introduce the notions of *averaged Hermitian metrics* and *averaged connections* analogous to the real Finsler geometry (see [Ma-Ra-Tr-Ze] and [To-Et]). The contents of this thesis is as follows.

Chapter 1 is the theory of complex vector bundle. First we shall explain the theory of vector bundles in general and the notion of complex vector bundles. Then in Section 1.3 we shall introduce the Hermitian metric and Kähler metric on the complex vector bundle. In Section 1.4 we explain the theory of the sheaf cohomology for holomorphic vector bundle over a complex manifold. Next, Section 1.5 discusses about the characterization of complex line bundles. In the last Section, we list up some vector bundles over complex projective space.

Chapter 2 will be started by the notion of connection and curvature of connection on a smooth complex vector bundle. We shall discuss Hermitian connection and curvature of connection on holomorphic vector bundle. It provides a brief review of Chern classes of complex vector bundles and the ampleness vector bundles is equivalent to the existence of Hermitian metrics of positive curvature. This chapter also presents the notion of Ehresmann connection and shows that the Ehresmann connection is induced by the Hermitian connection on a Hermitian bundle.

Chapter 3 presents the notion of complex Finsler metrics and complex Finsler connections. In Section 3.1, we shall recall the notion of (complex) Minkowski space. Section 3.2 provides the notion of complex Finsler metric on vector bundles and the construction of Rizza metrics on vector bundles which is determined by a pseudo Kähler metric in $\mathbb{P}(E)$. We also introduce the notion of partial connection in vertical sub-bundle and non linear connection. Section 3.3 discusses a characterization of negative holomorphic vector bundles which is given by Kobayashi's theorem. This section also introduces the notion of Rizza-negativity of complex vector bundles and provides a construction of Rizza metrics in negative vector bundles.

Chapter 4 is devoted to the averaged Hermitian metrics and connection on the holomorphic vector bundle. In Section 4.1, we consider for a family of compact Kähler manifolds and introduce the definitions of vertical sub-bundle, horizontal sub-bundle, Hermitian metric in vertical sub-bundle, and a proposition that gives a basic idea for this research. Section 4.2 deals the special case of the previous section and apply Lie derivation to a smooth families of tensor fields on the fiber $\mathbb{P}_z$. Section 4.3 introduces the averaged Hermitian structure and averaged connection. The last section discusses an application of the averaged metric and connection.

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Contents

Chapter 1

Complex vector bundles

This first chapter provides the minimum of the theory of vector bundles and sheaf cohomology which we need in this thesis.

1.1 Vector bundles

Intuitively a vector bundle over a smooth manifold M may be regarded as a smooth family of vector spaces (fibers) parametrized by points of the manifold M. We start with the general notion of vector bundles.

Definition 1.1. Let E and M be two smooth manifolds with a smooth submersion $\pi : E \to M$, and let $\mathbb{V}$ be a r-dimensional vector space. Then E is called a *vector bundle* of rank r if there exists an open covering $\mathcal{U} = \{U, V, \cdots\}$ of M and a family of maps $\{\varphi_U, \varphi_V, \cdots\}$ satisfying the following conditions.

(1) Each map φ_U is a fiber preserving diffeomorphism from $\pi^{-1}(U)$ onto $U \times \mathbb{V}$, i.e.,

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi_U} & U \times \mathbb{V} \\ \pi \big\downarrow & & \big\downarrow p \\ U & \xrightarrow{\text{Id}} & U \end{array}$$

is commutative, where p is the natural projection to the first factor.

(2) The restriction $\varphi_x := \varphi_U\big|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \{x\} \times \mathbb{V} \cong \mathbb{V}$ is a linear isomorphism for every point $x \in U$.

(3) If $U \cap V \neq \emptyset$, the diffeomorphism $\varphi_U \circ \varphi_V^{-1} : (U \cap V) \times \mathbb{V} \to (U \cap V) \times \mathbb{V}$ sends (x, v)

to

$$(\varphi_U \circ \varphi_V^{-1})(x, v) = (x, g_{UV}(x)v),$$

where $g_{UV} : U \cap V \to GL(\mathbb{V})$ is a smooth map which takes values in the general linear group $GL(\mathbb{V})$ of $\mathbb{V}$.

The group $GL(\mathbb{V})$ is called the *structure group* of E. For every point $x \in M$, the inverse image $\pi^{-1}(x) := E_x$ is called the *fiber* over x. The vector space $\mathbb{V}$ which is canonically linear isomorphic to each fiber E_x is called the *canonical fiber*. The dimension of $\mathbb{V}$ is called the *rank* of E and is denoted by $\mathrm{rank}(E)$. If $\mathrm{rank}(E) = 1$, then E is called a *line bundle*.

The the family $\{(U, \varphi_U)\}$ of the pairs (U, φ_U) is called a *local trivialization* of the vector bundle, and the mappings $\{g_{UV}\}$ are called the *transition functions* of the local trivializations $\{(U, \varphi_U)\}$. The transition functions $\{g_{UV}\}$ for E depend on the open the local trivialization $\{(U, \varphi_U)\}$.

Let $\{(U, \varphi_U)\}$ be a local trivialization of a vector bundle E, and let $\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_r\}$ be a basis of the canonical fiber $\mathbb{V}$. Defining $e_j(x) = \varphi_x^{-1}(\boldsymbol{e}_j)$ for every point $x \in U$, the set $e_U = \{e_i, \cdots, e_r\}$ forms a local field of basis of the fiber E_x over each $x \in U$. The family $\{(U, e_U)\}$ is called an *open covering* of E subordinate to the local trivialization $\{(U, \varphi_U)\}$. If $U \cap V \neq \phi$, then e_U and e_V are related by

$$e_V = e_U g_{UV}. \tag{1.1}$$

The transition funcions $\{g_{UV}\}$ satisfy the *cocycle conditions*:

(1) $g_{VU} \cdot g_{UV} = Id$ on $U \cap V$,

(2) $g_{WV} \cdot g_{VU} = g_{WU}$ on $U \cap V \cap W$.

Conversely, if a family $\{g_{UV}\}$ of $GL(\mathbb{V})$-valued functions g_{UV} satisfying the cocycle conditions with respect to an open covering $\mathcal{U}$ of M, we can construct a vector bundle with canonical fiber $\mathbb{V}$ whose transition functions are the given $\{g_{UV}\}$. In fact, we put

$$E := \coprod_{U \in \mathcal{U}} (U \times \mathbb{V}) \,/ \sim,$$

where the equivalent relation $\sim$ is defined by $U \times \mathbb{V} \ni (x, \zeta) \sim (x, g_{UV}(x)\zeta) \in V \times \mathbb{V}$. For an equivalent class $[x, \zeta] \in E$ represented by (x, ζ), we define $\pi([x, \zeta]) = x$. Then E is a vector bundle with canonical fiber $\mathbb{V}$.

Example 1.1. (**Trivial bundle**) Let $\mathbb{V}$ be a vector space. The product space $E = M \times \mathbb{V}$ is a vector bundle over M called *trivial bundle*. The transition function $\{g_{UV}\}$ for $E = M \times \mathbb{V}$ is given by $g_{UV} = 1$ (identity). □

Example 1.2. Let E and $\tilde{E}$ be two vector bundles over a smooth manifold M with canonical fiber $\mathbb{V}$ and $\tilde{\mathbb{V}}$ respectively. We can find a common open covering $\mathcal{U}$ of M so that the transition functions relative to $\mathcal{U}$ are given by $\{g_{UV}\}$ and $\{\tilde{g}_{UV}\}$ respectively.

(1) (**Direct sum**) We define an element G_{UV} of $GL(\mathbb{V} \oplus \tilde{\mathbb{V}})$ by

$$G_{UV} = \begin{pmatrix} g_{UV} & O \\ O & \tilde{g}_{UV} \end{pmatrix}.$$

Then we can easily show that the family $\{G_{UV}\}$ satisfies the cocycle conditions and so it defines a vector bundle which is denoted by $E \oplus \tilde{E}$ and called the *direct sum* of E and $\tilde{E}$.

(2) (**Tensor product**) We define an element H_{UV} of $GL(\mathbb{V} \otimes \tilde{\mathbb{V}})$ by

$$H_{UV}(v \otimes \tilde{v}) = (g_{UV} v) \otimes (\tilde{g}_{UV} \tilde{v}).$$

Then we can easily show that the family $\{H_{UV}\}$ satisfies the cocycle conditions and so it defines a vector bundle which is denoted by $E \otimes \tilde{E}$ and called the *tensor product* of E and $\tilde{E}$. □

Example 1.3. (**Pull-back bundle**) Let f be a smooth map from a smooth manifold N onto a smooth manifold M. Let E be a vector bundle over M with transition functions $\{g_{UV}\}$ relative to an open covering $\mathcal{U}$ of M and canonical fiber $\mathbb{V}$. We can construct a vector bundle f^*E over N with the same canonical fiber $\mathbb{V}$ by attaching to $w \in N$ the fiber $E_{f(w)}$ as follows. The *pull-back bundle* f^*E is defined by

$$f^*E = \{(w, v) \in N \times E \mid f(w) = \pi(v)\}$$

and consider the surjective mapping $f^*\pi : f^*E \to N$ defined by $f^*\pi(w, v) = w$. The following is commutative.

$$\begin{array}{ccc} f^*E & \xrightarrow{\tilde{f}} & E \\ {\scriptstyle f^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{f} & M \end{array}$$

The fiber $(f^*\pi)^{-1}(w)$ over $w \in N$ is given by the fiber $E_{f(w)}$ over $f(w) \in M$. The transition functions $\{f^* g_{UV}\}$ of f^*E are given by the pull-back of $\{g_{UV}\}$, i.e., $(f^*g_{UV})(w) = g_{UV}(f(w))$ at every point $w \in f^{-1}(U \cap V)$. □

Example 1.4. (**Dual bundle**) Let E be a vector bundle over M with canonical fiber $\mathbb{V}$. We take the dual vector space E_x^* of each fiber E_x of E and define

$$E^* = \coprod_{x \in M} E_x^*.$$

Let $\pi : E^* \to M$ be the natural projection defined by $\pi(E_x^*) = x$. Let $\{g_{UV}\}$ be the transition function of E. Then we see that the function g_{UV}^* defined by $g_{UV}^* = {}^t g_{UV}^{-1}$ values in $GL(\mathbb{V}^*)$ and that it satisfies the cocycle conditions. Hence E^* admits a bundle structure with canonical fiber $\mathbb{V}^*$ and transition functions $\{g_{UV}^*\}$. The vector bundle E^* is called the *dual bundle* of E. □

Let $\pi : E \to M$ and $\tilde{\pi} : \tilde{E} \to \tilde{M}$ are vector bundles. A smooth map $\varPhi : E \to \tilde{E}$ is called a *bundle homomorphism* over the base map $f : M \to \tilde{M}$ such that

(1) $\tilde{\pi} \circ \varPhi = f \circ \pi$, i.e., the diagram

$$\begin{array}{ccc} E & \xrightarrow{\varPhi} & \tilde{E} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\ M & \xrightarrow{f} & \tilde{M} \end{array}$$

is commutative,

(2) the restriction map $\varPhi|_{E_x} : E_x \to \tilde{E}_{f(x)}$ is linear for every point $x \in M$.

A bijective bundle homomorphism $\varPhi : E \to \tilde{E}$ is called a *bundle isomorphism* if its inverse $\varPhi^{-1}$ is also a bundle homomorphism and the map f is a diffeomorphism. If there exists a bundle isomorphism between E and $\tilde{E}$, the two bundles are said to be *isomorphic*.

In the special case where both E and $\tilde{E}$ are vector bundles over the same base space M, we denote by $\mathrm{Hom}(E, \tilde{E})$ the space of all bundle homomorphism from E to $\tilde{E}$. Especially, the notation $\mathrm{End}(E)$ denotes the set of all bundle morphism from E to itself.

Let E and $\tilde{E}$ be vector bundles over M. Then E is said to be *isomorphic* to $\tilde{E}$ if there exists a bundle isomorphim $\Phi : E \to \tilde{E}$ such that its base map is the identity:

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & \tilde{E} \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ M & \xrightarrow{Id} & M \end{array}$$

Let $\{g_{UV}\}$ and $\{\tilde{g}_{UV}\}$ be the defining cocycles of E and $\tilde{E}$ respectively, which are adapted to a common open covering $\mathcal{U} = \{U\}$. Then E is isomorphic to $\tilde{E}$ if and only if there exist smooth maps $\Phi_U : U \to GL(r, \mathbb{V})$ on each $U \in \mathbb{U}$ such that

$$\tilde{g}_{UV} = \Phi_U g_{UV} \Phi_V^{-1} \tag{1.2}$$

on $U \cap V$. If E is isomorphic to $\tilde{E}$, we write $E \cong \tilde{E}$ and we do not distinguish isomorphic vector bundles.

Definition 1.2. Let $\pi : E \to M$ be a vector bundle. A sub-manifold G of E is said to be a sub-bundle of E if it satisfies the following conditions.

(1) The restriction $\pi|_G : G \to M$ is a vector bundle.

(2) The inclusion $\iota : G \to E$ is a bundle homomorphism.

Then we have

Proposition 1.1. *Let E and $\tilde{E}$ be two vector bundles over M and $\Phi : E \to \tilde{E}$ a bundle homomorphism. Then*

(1) $\mathrm{Im}(\Phi) := \Phi(E)$ *is a sub-bundle of $\tilde{E}$.*

(2) $\mathrm{Ker}(\Phi) := \{v \in E \mid \Phi(v) = 0\}$ *is a sub-bundle of E.*

We denote by 0 simply the trivial bundle $M \times \{0\}$. Let now E_j $(j = 1, \cdots, k)$ be vector bundles over M and $\Phi_j : E_j \to E_{j+1}$ $(j = 1, \cdots, k-1)$ a bundle homomorphism. The sequence

$$0 \xrightarrow{\iota} E_1 \xrightarrow{\Phi_1} E_2 \xrightarrow{\Phi_2} \cdots \xrightarrow{\Phi_{k-1}} E_k \longrightarrow 0$$

is said to be *exact* if $\mathrm{Im}(\Phi_{j-1}) = \mathrm{Ker}(\Phi_j)$ for each j. The following proposition is useful.

Proposition 1.2. *Let E, F and G be vector bundles over M. We suppose that the sequence*

$$0 \xrightarrow{\iota} E \xrightarrow{\Phi} F \xrightarrow{\Psi} G \longrightarrow 0$$

is exact. Then there exists a splitting λ *of the sequence, i.e., there exists a bundle homomorphism* $\lambda : G \to F$ *such that* $\Psi \circ \lambda = I_G$*, where* I_G *is the identity morphism of* G*. Then* F *is isomorphic to the direct sum* $\mathrm{Im}(\Phi) \oplus \mathrm{Im}(\lambda)$*.*

1.2 Complex vector bundles

In this section we shall introduce the notion of complex vector bundles. Let E be a vector bundle over a smooth manifold M. Then E is said to be a *complex vector bundle* if it satisfies the following conditions:

(1) each fiber E_x is a complex vector space of complex dimension r, i.e., $E_x \cong \mathbb{C}^r$,

(2) the restriction $\varphi_x : E_x \to \{x\} \times \mathbb{V} \cong \mathbb{C}^r$ is a complex linear isomorphism at every point $x \in U$.

The transition functions $g_{UV} : U \cap V \to GL(\mathbb{V})$ take values in $GL(\mathbb{V}) \cong GL(r, \mathbb{C})$.

If $\pi : E \to M$ is a complex vector bundle, there exists an endomorphism $J_E \in \mathrm{End}(E)$ defined by $J_E s = \sqrt{-1}s$. The endomorphism J_E is called the *complex structure* of E, since it satisfies

$$J_E \circ J_E = -I_E. \tag{1.3}$$

Conversely, if a real vector bundle of even rank has a complex structure J_E, E becomes a complex vector bundle by defining

$$(a + \sqrt{-1}b)s = (aI_E + bJ_E)s$$

for every $s \in E$, where $I_E \in \mathrm{End}(E)$ is the identity morphism of E.

If the base manifold M is a complex manifold, then there exists a special class of complex vector bundles.

Definition 1.3. A complex vector bundle E over a complex manifold M is called a *holomorphic vector bundle* if it admits local trivializations $\{(U, \varphi_U)\}$ whose transition functions $g_{UV} : U \cap V \to GL(r, \mathbb{C})$ are holomorphic. In the case of $r = 1$, the bundle E is called a *holomorphic line bundle.*

Example 1.5. (**Holomorphic tangent bundle**) Let M be a complex manifold. Let $\mathcal{O}_p$ be the germ of holomorphic function defined on a neighborhood of a point $p \in M$, which is identified with the ring $\mathbb{C}[z^1, \cdots, z^m]$ of convergent power series of $z^1, \cdots, z^m$. Here we assume that $p = (0, \cdots, 0)$ via a cubic coordinate system $(\Delta^n, (z^1, \cdots, z^m))$ centered at the origin $(0, \cdots, 0)$.

We denote by $T_p^{1,0}M$ the vector space of derivations $D : \mathcal{O}_p \to \mathbb{C}$, where D is said to be a *derivation* if it is complex linear and satisfies Leibniz's rule:

$$D(f \cdot g) = f(z)Dg + g(z)Df.$$

For all $f \in \mathcal{O}_p$ we have $f(z) = f(p) + (\text{linear term}) + h(z)$, where $h(z)$ involves higher order terms. Since $Dh = 0$, we have

$$D = \sum_{\alpha} v^{\alpha} \left(\frac{\partial}{\partial z^{\alpha}} \right)_p$$

for $(v^1, \cdots, v^m) \in \mathbb{C}^m$ defined by $v^{\alpha} = Dz^{\alpha} \in \mathbb{C}$ $(\alpha = 1, \cdots, m)$. Hence we have an isomorphism $T_p^{1,0}M \cong \mathbb{C}^m$. We put

$$T_M = \coprod_{p \in M} T_p^{1,0}M$$

with the natural projection $\pi : T^{1,0}M \to M$ defined by $\pi(T_p^{1,0}M) = p$.

We shall introduce a complex structure on T_M so that the projection π is a holomorphic submersion. The topology of T_M is defined by the standard method so that π is continuous. Let $\mathcal{U}$ be an open covering of M. We introduce an open covering of T_M by $\tilde{\mathcal{U}} = \{\pi^{-1}(U)\}$. Let $z_U = (z_U^1, \cdots, z_U^m)$ be a complex coordinate on $U \in \mathcal{U}$. We define $\tilde{\varphi}_U : \pi^{-1}(U) \to U \times \mathbb{C}^n$ by

$$\tilde{\varphi}_U(D) = (p, \begin{pmatrix} v_U^1 \\ \vdots \\ v_U^m \end{pmatrix}) := (p, (v_U^{\alpha}))$$

for $D = \sum v_U^{\alpha} (\partial / \partial z_U^{\alpha})_p \in T_p^{1,0}M$. Then for each $U \in \mathcal{U}$, these mappings $\tilde{\varphi}_U$ are homeomorphisms, and by definition, the coordinate change on $\pi^{-1}(U \cap V)$ is given by

$$\tilde{\varphi}_U \circ \tilde{\varphi}_V^{-1} : (p, (v_V^{\alpha})) \to \left(p, \left(\sum_{\beta} \frac{\partial z_U^{\alpha}}{\partial z_V^{\beta}}(p) v_V^{\beta} \right) \right).$$

Thus the transition cocycle is given by the Jacobian of the coordinate change of the base manifold M, and hence T_M is a holomorphic vector bundle over M. T_M is called the *holomorphic tangent bundle* of M.

Let $T_{\mathbb{R}}M$ be the real tangent bundle of M. The complex structure $J \in \mathrm{End}(T_{\mathbb{R}}M)$ on M is naturally extended to its complexification $T_{\mathbb{R}}M \otimes \mathbb{C}$. Since $J \circ J = -I$, the eigenvalues of J are $\pm\sqrt{-1}$, and the vector space $T_p^{1,0}M$ and its conjugate $\overline{T_p^{1,0}M}$ are eigenspaces corresponding $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Then we obtain the decomposition of $T_{\mathbb{R}}M$ as follows:

$$T_{\mathbb{R}}M \otimes \mathbb{C} = T_M \oplus \overline{T_M}. \tag{1.4}$$

The bundle $\overline{T_M} = \coprod_{p \in M} \overline{T_p^{1,0}M}$ is called the *anti-holomorphic tangent bundle* of M. □

Example 1.6. (**Holomorphic cotangent bundles**) Let M be a complex manifold. For the holomorphic tangent space $T_p^{1,0}M$ at a point $p \in M$, we denote by $\Omega_{M,p}$ the dual space of $T_p^{1,0}M$. We set

$$\Omega_M := \coprod_{p \in M} \Omega_{M,p},$$

and define the projection $\pi : \Omega_M \to M$ by $\pi(\Omega_{M,p}) = p$. Then $\pi : \Omega_M \to M$ is also a holomorphic vector bundle of rank m called the *holomorphic cotangent bundle* of M. According to the decomposition (1.4), we also obtain the decomposition

$$T_{\mathbb{R}}M^* \otimes \mathbb{C} = \Omega_M \oplus \overline{\Omega_M}.$$

□

Example 1.7. (**Canonical line bundle**) Let M be a complex manifold of $\dim_{\mathbb{C}} M = m$. For the holomorphic cotangent bundle Ω_M, the line bundle K_M defined by

$$K_M := \wedge^m \Omega_M$$

is called the *canonical line bundle* of M. □

For later use we define the notion of sections of vector bundles.

Definition 1.4. Let $\pi : E \to M$ be a complex vector bundle over M. A smooth map $s : M \to E$ satisfying $\pi \circ s = id$ is called a *section* of E. A collection $e_U = (e_1, \cdots, e_r)$ of local sections $e_j : U \to E$ on $U \subset M$ is called a *frame field* on U if $\{e_1(x), \cdots, e_r(x)\}$ is a basis of E_x at each point $x \in U$.

Let $\mathcal{A}(U)$ be the set of all $\mathbb{C}$-valued smooth functions on an open set $U \subset M$. If we take a local frame field $e_U = (e_1, \cdots, e_r)$ on U, any section s of E is expressed uniquely in the form $s = \sum \zeta^j e_j$ for some $r(= \mathrm{rank} E)$ smooth $\mathbb{C}$-valued functions $\zeta^j \in \mathcal{A}(U)$, where

$(\zeta^1, \cdots, \zeta^r)$ is called the *component* of s with respect to $\{(U, e_U)\}$. The correspondence $\varphi_U : \pi^{-1}(U) \ni s \to \big(x, (\zeta^1, \cdots, \zeta^r)\big) \in U \times \mathbb{V}$ defines a local trivialization $\{(U, \varphi_U)\}$ of E.

1.3 Hermitian metrics and Kähler metrics

Let E be a complex vector bundle over M.

Definition 1.5. A *Hermitian metric* h on E is a smooth field of Hermitian inner product on the fibers of E, i.e.,

(H1) $h(s,t)$ is complex linear in s, where $s, t \in E_z$,

(H2) $h(s,t) = \overline{h(t,s)}$,

(H3) $h(s,s) \geq 0$, and the equality holds if and only if $s = 0$,

(H4) $h(s,t)$ is a smooth function on M if $s, t \in \mathcal{A}(E)$.

The pair (E, h) is called a *Hermitian vector bundle.*

Given a local frame field $e_U = (e_1, \cdots, e_r)$ on U, we set $h_{i\bar{j}}(z) = h(e_i, e_j)$ $(1 \leq i, j \leq r)$. The matrix $(h_{i\bar{j}})$ is Hermitian, i.e., $h_{i\bar{j}} = \overline{h_{j\bar{i}}}$. Further $(h_{i\bar{j}})$ is positive-definite, i.e., $\sum h_{i\bar{j}}(z)\zeta^i\overline{\zeta^j} \geq 0$ and the equality holds if and only if $\zeta^1 = \cdots = \zeta^r = 0$.

Since h is Hermitian metric, there exists an open covering $\{(U, e_U)\}$ of E such that $h_{i\bar{j}} = \delta_{i\bar{j}}$. Such a frame field e_U is called a *unitary frame field.* If we take an open covering $\{(U, e_U)\}$ consisting of unitary frame fields e_U, the transformation law show that the transition functions g_{UV} take values in the unitary matrix $U(r)$. Thus the structure group $GL(\mathbb{V}) = GL(r, \mathbb{C})$ is reducible to $U(r)$ if E admits a Hermitian metric. Conversely E admits a Hermitian metric if its structure group $GL(r, \mathbb{C})$ is reducible to $U(r)$.

Remark 1.1. Given a complex vector bundle E, we can introduce a Hermitian metric h on E. Hence the structure group $GL(r, \mathbb{C})$ of a complex vector bundle is always reducible to $U(r)$. □

Let M be a complex manifold. If a Hermitian metric g is given on the holomorphic tangent bundle T_M, then (M, g) is called a *Hermitian manifold.* For the local frame filed $\left\{\dfrac{\partial}{\partial z^1}, \cdots, \dfrac{\partial}{\partial z^m}\right\}$, the metric g is given by the $m \times m$ matrix

$$g_{\alpha\bar{\beta}}(z) := g\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\right).$$

We define a real $(1,1)$-form Π_g by

$$\Pi_g := \frac{\sqrt{-1}}{2}\sum g_{\alpha\bar{\beta}}(z)dz^\alpha \wedge d\bar{z}^\beta. \tag{1.5}$$

We can always define a Hermitian metric on M, but Π_g is not closed generally.

Definition 1.6. A Hermitian manifold (M,g) is called a *Kähler manifold* if $d\Pi_g = 0$. Such a metric g is called a *Kähler metric* on M.

In a local coordinate $(z^1,\cdots,z^m)$ on $U \subset M$, Π_g is closed if and only if

$$\frac{\partial g_{\alpha\bar{\beta}}}{\partial z^\gamma} = \frac{\partial g_{\gamma\bar{\beta}}}{\partial z^\alpha}, \quad \frac{\partial g_{\alpha\bar{\beta}}}{\partial \bar{z}^\gamma} = \frac{\partial g_{\alpha\bar{\gamma}}}{\partial \bar{z}^\beta}. \tag{1.6}$$

In a Kähler manifold (M,g), the form Π_g is closed real two-form. By Poincaré lemma, there exists a local one-form φ_U satisfying $\Pi_g = d\varphi_U$. Since φ_U is real one-form, we can put $\varphi_U = \psi_U + \overline{\psi_U}$. Here ψ_U is $(0,1)$-type. Since Π_g is of $(1,1)$-type and $\Pi_g = (\partial + \bar{\partial})\varphi_U$, we have

$$\partial\overline{\psi_U} = 0,\ \bar{\partial}\psi_U = 0,\ \Pi_g = \partial\psi_U + \bar{\partial}\overline{\psi_U}.$$

Since $\bar{\partial}\psi_U = 0$, by Dolbeault's lemma, there exists a function f_U defined on U such that $\psi_U = \bar{\partial}f_U$. Further we have

$$\Pi_g = \partial\psi_U + \bar{\partial}\bar{\psi}_U = \partial\bar{\partial}f_U + \bar{\partial}\partial\bar{f}_U = \partial\bar{\partial}(f_U - \overline{f_U}),$$

since $\overline{\psi_U} = \overline{\bar{\partial}f_U} = \partial\overline{f_U}$. If we put $K_U = \sqrt{-1}(\overline{f_U} - f_U)$, then K_U is a local real function, and Π_g is given by

$$\Pi_g = \sqrt{-1}\partial\bar{\partial}K_U \tag{1.7}$$

and the matrix $g_{\alpha\bar{\beta}}$ can be written as

$$g_{\alpha\bar{\beta}}(z) = \frac{\partial^2 K_U}{\partial z^\alpha \partial \bar{z}^\beta}. \tag{1.8}$$

The local function K_U is called a *Kähler potentials.*

Conversely, if a metric g on M is given by this form, then its Kähler form Π_g is closed.

Remark 1.2. The Kähler potentials $\{K_U\}$ of a Kähler manifold (M,g) are locally defined, and the Kähler form Π_g can be written as $\Pi_g|_U = \sqrt{-1}\partial\bar{\partial}K_U$ on each U. Then $\sqrt{-1}\partial\bar{\partial}K_U = \sqrt{-1}\partial\bar{\partial}K_V$ implies $\sqrt{-1}\partial\bar{\partial}\,(K_V - K_U) = 0$, i.e., $K_V - K_U$ is *pluri-harmonic.* If we put $\Psi_{UV} := \bar{\partial}(K_V - K_U)$, then since $\partial\Psi_{UV} = 0$ and $\bar{\partial}\Psi_{UV} = 0$, there exists a function

f_{UV} on $U \cap V$ satisfying

$$\Psi_{UV} = df_{UV} = \partial f_{UV} + \bar{\partial} f_{UV}.$$

Since Ψ_{UV} is $(0,1)$-form, $\partial f_{UV} = 0$ shows that f_{UV} is anti-holomorphic. Hence $\bar{\partial}(K_V - K_U - f_{UV}) = 0$, and $g_{UV} := K_V - K_U - f_{UV}$ is a holomorphic function. Since $K_V - K_U = f_{UV} + g_{UV}$ is real, and thus if we put $k_{UV} = (\overline{f_{UV}} + g_{UV})/2$, we have

$$K_V - K_U = k_{UV} + \overline{k_{UV}}$$

for a holomorphic function k_{UV}. □

Example 1.8. (**Riemann surface**) Let M be a Riemann surface. M has the structure of smooth manifold of real dimension two. Since such a manifold is conformally flat, M admits a Riemannian metric ds^2 in the form $ds^2 = f^2(dx \otimes dx + dy \otimes dy)$ for some local function f. Since $dz = dx + \sqrt{-1}dy$ and $d\bar{z} = dx - \sqrt{-1}dy$, the metric ds^2 is written as $ds^2 = f^2 dz \otimes d\bar{z}$. The Kähler form Π_g is given by $\Pi_g = \sqrt{-1} f^2 dz \wedge d\bar{z}$. Since $\dim_{\mathbb{C}} M = 1$ the form Π_g is closed, and so any Riemannian surface is a Kähler manifold. □

1.4 Sheaf cohomology

Let $\mathcal{F}$ be a sheaf of abelian groups over M, and $\mathcal{U} = \{U_j\}$ a locally finite covering of M. In the sequel we use the notation $\Gamma(U_j, \mathcal{F})$ for the set of sections of $\mathcal{F}$ over U_j. For every collection of indices $(i_0, \cdots, i_n)$ we put $U_{i_0 \cdots i_n} := U_{i_0} \cap \cdots \cap U_{i_n}$ and

$$C^q(\mathcal{U}, \mathcal{F}) := \left\{ f_{i_0 \cdots i_q} \in \Gamma(U_{i_0 \cdots i_q}, \mathcal{F}) \right\}.$$

An element $\{f_{i_0 \cdots i_q}\}$ of $C^q(\mathcal{U}, \mathcal{F})$ is called a *q-cochain* of $\mathcal{F}$. We define the *coboundary operator* $\nu : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F})$ by

$$\{\nu(f)\}_{i_0 \cdots i_q} := \sum_{k=0}^{q+1} (-1)^k \rho \left(f_{i_0 \cdots \hat{i_k} \cdots i_{q+1}} \right),$$

where $\hat{i_k}$ denotes the deletion of the index i_k and ρ is the restriction map to the subset $U_{i_0 \cdots i_q} \subset U_{i_0 \cdots \hat{i_k} \cdots i_{q+1}}$. By direct calculations, it is proved that $\nu \circ \nu = 0$, and thus $\{C^q(\mathcal{U}, \mathcal{F}), \nu\}$ is a complex:

$$0 \longrightarrow C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\nu} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\nu} \cdots \xrightarrow{\nu} C^q(\mathcal{U}, \mathcal{F}) \xrightarrow{\nu} \cdots.$$

This complex is not necessary exact. The exactness of this complex is informed by cohomology groups. We set

$$\begin{cases} Z^q(\mathcal{U},\mathcal{F}) := \ker(\nu) = \{f \in C^q(\mathcal{U},\mathcal{F}) \mid \nu(f) = 0\}\,, \\ B^q(\mathcal{U},\mathcal{F}) := \operatorname{Im}(\nu) = \left\{\nu(f) \mid f \in C^{q-1}(\mathcal{U},\mathcal{F})\right\} \text{ and } B^0(\mathcal{U},\mathcal{F}) = \{0\}. \end{cases}$$

Since $\nu \circ \nu = 0$, $B^q(\mathcal{U},\mathcal{F})$ is a submodule of $Z^q(\mathcal{U},\mathcal{F})$. Then the quotient group

$$\check{H}^q(\mathcal{U},\mathcal{F}) := \frac{Z^q(\mathcal{U},\mathcal{F})}{B^q(\mathcal{U},\mathcal{F})}$$

is called the *q-th cohomology group* with respect to the covering $\mathcal{U}$.

Lemma 1.1. *We have $\check{H}^0(\mathcal{U},\mathcal{F}) \cong \Gamma(M,\mathcal{F})$, i.e., $\check{H}^0(\mathcal{U},\mathcal{F})$ consists of global sections of the sheaf $\mathcal{F}$.*

PROOF. By definition $\check{H}^0(\mathcal{U},\mathcal{F}) = Z^0(\mathcal{U},\mathcal{F})$. We have $\nu\{f_i\} = f_j - f_i = 0$ on $U_i \cap U_j$ for $\{f_i\} \in Z^0(\mathcal{U},\mathcal{F})$. Hence we have $f_i = f_j$ on $U_i \cap U_j$ which shows that $\{f_i\}$ is a global section of $\mathcal{F}$.

Conversely, we define a zero-cochain $\{f_i\}$ by $f_i = f|_{U_i}$ for any $f \in \Gamma(M,\mathcal{F})$. Since f is global, we have $f_i = f_j$ on $U_i \cap U_J$, and thus we have $\nu\{f_i\} = 0$. Consequently any element of $\Gamma(M,\mathcal{F})$ defines an element of $Z^0(\mathcal{U},\mathcal{F})$.

Q.E.D.

We denote by $\mathfrak{U}$ the collection of all locally finite open coverings of M. We define a preorder " $<$ " on $\mathfrak{U}$ by $\mathcal{U} < \mathcal{V}$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$. By this preorder the collection $\mathfrak{U}$ is a directed set. Let $\mathcal{U},\mathcal{V} \in \mathfrak{U}$ be two locally finite open covering, and $\pi^{\mathcal{U}}_{\mathcal{V}} : C^q(\mathcal{U},\mathcal{F}) \to C^q(\mathcal{V},\mathcal{F})$ be its restriction morphism. Then the following is commutative diagram at each stage:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\nu} & C^q(\mathcal{U},\mathcal{F}) & \xrightarrow{\nu} & C^{q+1}(\mathcal{U},\mathcal{F}) & \xrightarrow{\nu} & \cdots \\ & & \pi^{\mathcal{U}}_{\mathcal{V}}\big\downarrow & & \pi^{\mathcal{U}}_{\mathcal{V}}\big\downarrow & & \\ \cdots & \xrightarrow{\nu} & C^q(\mathcal{V},\mathcal{F}) & \xrightarrow{\nu} & C^{q+1}(\mathcal{V},\mathcal{F}) & \xrightarrow{\nu} & \cdots . \end{array}$$

This commutative diagram implies

$$\pi^{\mathcal{U}}_{\mathcal{V}}\left(Z^q(\mathcal{U},\mathcal{F})\right) \subset Z^q(\mathcal{V},\mathcal{F}),\ \ \pi^{\mathcal{U}}_{\mathcal{V}}\left(B^q(\mathcal{U},\mathcal{F})\right) \subset B^q(\mathcal{V},\mathcal{F})$$

for every q. Thus it also induces a morphism $\pi^{\mathcal{U}}_{\mathcal{V}} : \check{H}^q(\mathcal{U},\mathcal{F}) \to \check{H}^q(\mathcal{V},\mathcal{F})$.

Definition 1.7. The *q-th cohomology group* $\check{H}^q(M,\mathcal{F})$ with coefficient sheaf $\mathcal{F}$ is defined by the direct limit

$$\check{H}^q(M,\mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^q(\mathcal{U},\mathcal{F}). \tag{1.9}$$

Lemma 1.1 implies

$$\check{H}^0(M,\mathcal{F}) \cong \Gamma(M,\mathcal{F}). \tag{1.10}$$

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups. Each $\varphi_U : \Gamma(U,\mathcal{F}) \to \Gamma(U,\mathcal{G})$ induces morphisms $\varphi : C^q(\mathcal{U},\mathcal{F}) \to C^q(\mathcal{U},\mathcal{G})$ which sends a cochain $\{f_{i_0\cdots i_q}\}$ to $\{\varphi\left(f_{i_0\cdots i_q}\right)\}$. From the commutativity $\varphi\circ\nu = \nu\circ\varphi$, the sheaf morphism φ sends cocycles to cocycles and coboundaries to coboundaries. Therefore the sheaf morphism $\varphi : \mathcal{F} \to \mathcal{G}$ induces the morphism of cohomology groups $\varphi : H^q(\mathcal{U},\mathcal{F}) \to \check{H}^q(\mathcal{U},\mathcal{G})$. Let $\mathcal{V}$ be a refinement of $\mathcal{U}$. Then the commutative diagram

$$\begin{array}{ccc} \check{H}^q(\mathcal{U},\mathcal{F}) & \xrightarrow{\varphi} & \check{H}^q(\mathcal{U},\mathcal{G}) \\ {\scriptstyle \pi^{\mathcal{U}}_{\mathcal{V}}}\downarrow & & {\scriptstyle \pi^{\mathcal{U}}_{\mathcal{V}}}\downarrow \\ \check{H}^q(\mathcal{V},\mathcal{F}) & \xrightarrow{\varphi} & \check{H}^q(\mathcal{V},\mathcal{G}) \end{array}$$

induces a morphism $\varphi : \check{H}^q(M,\mathcal{F}) \to \check{H}^q(M,\mathcal{G})$ of cohomology groups.

Let

$$0 \longrightarrow \mathcal{F} \xrightarrow{\ \iota\ } \mathcal{G} \xrightarrow{\ \varphi\ } \mathcal{H} \longrightarrow 0$$

be a short exact sequence of abelian groups over M which implies the short exact sequence

$$0 \longrightarrow C^q(\mathcal{U},\mathcal{F}) \xrightarrow{\ \iota^q\ } C^q(\mathcal{U},\mathcal{G}) \xrightarrow{\ \varphi^q\ } C^q(\mathcal{U},\mathcal{H}) \longrightarrow 0.$$

For any $[h] \in \check{H}^q(M,\mathcal{H})$ with $h \in Z^q(\mathcal{U},\mathcal{H})$, we choose a refinement $\mathcal{V} > \mathcal{U}$ and $g \in C^q(\mathcal{V},\mathcal{G})$ such that $\pi^{\mathcal{U}}_{\mathcal{V}}(h) = \varphi^q(g)$ if we need.

$$\begin{array}{ccc} C^q(\mathcal{U},\mathcal{G}) & \xrightarrow{\varphi^q} & C^q(\mathcal{U},\mathcal{H}) \ni h \\ {\scriptstyle \pi^{\mathcal{U}}_{\mathcal{V}}}\downarrow & & {\scriptstyle \pi^{\mathcal{U}}_{\mathcal{V}}}\downarrow \\ C^q(\mathcal{V},\mathcal{G}) \ni g & \xrightarrow{\varphi^q} & C^q(\mathcal{V},\mathcal{H}) \ni \varphi(g) \end{array}$$

Then, since $\nu(h) = 0$, we obtain

$$\varphi^{q+1}(\nu(g)) = \nu(\varphi^q(g)) = \nu(\pi^{\mathcal{U}}_{\mathcal{V}}(h)) = \pi^{\mathcal{U}}_{\mathcal{V}}(\nu(h)) = 0.$$

Thus $\nu(g)$ comes from $C^{q+1}(\mathcal{V},\mathcal{F})$, i.e., there exists $f \in C^{q+1}(\mathcal{V},\mathcal{F})$ such that $\nu(g) =$

$\iota^{q+1}(f)$.

$$\begin{array}{ccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
C^q(\mathcal{V},\mathcal{F}) & \xrightarrow{\nu} & C^{q+1}(\mathcal{V},\mathcal{F}) \ni f & \xrightarrow{\nu} & C^{q+2}(\mathcal{V},\mathcal{F}) \ni \nu(f) \\
{\scriptstyle \iota^q}\downarrow & & {\scriptstyle \iota^{q+1}}\downarrow & & {\scriptstyle \iota^{q+2}}\downarrow \\
C^q(\mathcal{V},\mathcal{G}) \ni g & \xrightarrow{\nu} & C^{q+1}(\mathcal{V},\mathcal{G}) \ni \nu(g) & \xrightarrow{\nu} & C^{q+2}(\mathcal{V},\mathcal{G}) \ni 0 \\
{\scriptstyle \varphi^q}\downarrow & & {\scriptstyle \varphi^{q+1}}\downarrow & & {\scriptstyle \varphi^{q+2}}\downarrow \\
C^q(\mathcal{V},\mathcal{H}) \ni h & \xrightarrow{\nu} & C^{q+1}(\mathcal{V},\mathcal{H}) \ni 0 & \xrightarrow{\nu} & C^{q+2}(\mathcal{V},\mathcal{H}) \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

Further we obtain

$$(\iota^{q+2} \circ \nu)(f) = (\nu \circ \iota^{q+1})(f) = \nu(\nu(g)) = 0.$$

Since ι^{q+2} is injective, we have $\nu(f) = 0$, i.e., $f \in Z^{q+1}(\mathcal{V},\mathcal{F})$. Hence f defines a cohomology class $[f] \in \check{H}^{q+1}(\mathcal{U},\mathcal{F})$. Then we define $\nu^* : \check{H}^q(\mathcal{U},\mathcal{H}) \to \check{H}^{q+1}(\mathcal{U},\mathcal{F})$ by

$$\nu^*([h]) = [f]$$

whch induces a morphism $\nu^* : \check{H}^q(M,\mathcal{H}) \to \check{H}^{q+1}(M,\mathcal{F})$ $(q \geq 0)$.

Definition 1.8. The morphism $\nu^* : \check{H}^q(M,\mathcal{H}) \to \check{H}^{q+1}(M,\mathcal{F})$ is called the *connecting morphism.*

The following theorem is basic in cohomology theory.

Theorem 1.1. *Given a short exact sequence* $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ *of sheaves over* M*, there arises a long exact sequence of cohomology groups:*

$$\cdots \longrightarrow \check{H}^{q-1}(M,\mathcal{H}) \xrightarrow{\nu^*} \check{H}^q(M,\mathcal{F}) \longrightarrow \check{H}^q(M,\mathcal{G}) \to \check{H}^q(M,\mathcal{H}) \xrightarrow{\nu^*} \cdots .$$

Let E be a holomorphic vector bundle over a complex manifold M. We denote by $\Gamma(U,E)$ the set of all sections of E over an open set U. Then, for any $s,t \in \Gamma(U,E)$, we define $(s+t)(x) := s(x)+t(x)$ and $(f \cdot s)(x) := f(x)s(x)$ for every smooth function f at each $x \in U$. The collection $\{\Gamma(U,E)\}$ $(U \in \mathcal{U})$ forms a sheaf $\mathcal{A}(E)$ of germs of smooth sections of E. We also denote by $\mathcal{O}(E)$ the sheaf of germs of holomorphic sections of E.

Remark 1.3. From (1.10) we have $\check{H}^0(M, \mathcal{A}(E)) = \Gamma(M, E)$. Since the structure sheaf $\mathcal{A}$ of any smooth manifold M is fine, the sheaf $\mathcal{A}(E)$ is fine. Then $\check{H}^q(M, \mathcal{A}(E)) = 0$ $(q \geq 1)$. On the other hand, if E is a holomorphic vector bundle over a complex manifold M, the sheaf $\mathcal{O}(E)$ of germs of holomorphic sections is not fine, since the structure sheaf $\mathcal{O}$ is not fine. □

Let E be a holomorphic vector bundle of rank r over a compact Hermitian manifold M. We denote by $\mathcal{A}^{p,q}(E)$ the sheaf of germs of E-valued smooth (p,q)-forms. Any section φ of $\mathcal{A}^{p,q}(E)$ is of the form $\varphi = \sum e_i \otimes \varphi^i$ for some $\varphi^i \in \mathcal{A}^{p,q}$. The Dolbeault operator $\bar{\partial}$ is extended to $\bar{\partial} : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q+1}(E)$ by

$$\bar{\partial}\varphi = \sum e_i \otimes \bar{\partial}\varphi^i.$$

Since E is holomorphic, this definition is well-defined.[1] Denoted by $\Omega^p(E)$ the sheaf of germs of E-valued holomorphic p-forms, a fine resolution of $\Omega^p(E)$ is given by

$$0 \longrightarrow \Omega^p(E) \stackrel{\iota}{\longrightarrow} \mathcal{A}^{p,0}(E) \stackrel{\bar{\partial}}{\longrightarrow} \mathcal{A}^{p,1}(E) \stackrel{\bar{\partial}}{\longrightarrow} \mathcal{A}^{p,2}(E) \stackrel{\bar{\partial}}{\longrightarrow} \cdots .$$

Then we have the *Dolbeault theorem* for E-valued holomorphic forms:

$$\check{H}^p(M, \Omega^q(E)) \cong H^{p,q}_{\bar{\partial}}(M, E), \tag{1.11}$$

where (p,q)-th *Dolbeault cohomology group* $H^{p,q}_{\bar{\partial}}(M, E)$ is defined by

$$H^{p,q}_{\bar{\partial}}(M, E) := \frac{\ker\left\{\bar{\partial} : \Gamma(M, \mathcal{A}^{p,q}(E)) \to \Gamma(M, \mathcal{A}^{p,q+1}(E))\right\}}{\bar{\partial}\Gamma(M, \mathcal{A}^{p,q-1}(E))}.$$

We shall state Hodge's theorem for holomorphic vector bundles. If a Hermitian metric h is given on E, then its dual E^* admits a natural Hermitian metric. Then Hodge star operator $* : \mathcal{A}^{p,q} \to \mathcal{A}^{n-p,n-q}$ is extended to $* : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{n-p,n-q}(E^*)$ by $*\varphi := \sum \left(*\varphi^i\right) \otimes e_i^*$, where $\{e_i^*\}$ is the dual frame field of $\{e_i\}$. Then an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{A}^{p,q}(E)$ is defined by

$$\langle \varphi, \psi \rangle = \int_M \varphi \wedge *\psi.$$

With respect to this metric, the space $\mathcal{A}^{p,q}(E)$ is a pre-Hilbert space. The formal adjoint $\bar{\partial}^* : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q-1}(E)$ is defined by $\bar{\partial}^* = -\left(* \circ \bar{\partial} \circ *\right)$, and the $\bar{\partial}$-Laplacian Δ :

[1] If E is a complex vector bundle, the operator $\bar{\partial}$ is not well-defined, since we can not use any holomorphic frame field e_U on U. Even if E is holomorphic, we can not define $\partial\varphi$ by $\partial\varphi = \sum e_i \otimes \partial\varphi^i$.

$\mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q}(E)$ is also defined by $\Delta = \bar{\partial} \circ \bar{\partial}^* + \bar{\partial}^* \circ \bar{\partial}$. We set

$$\mathcal{H}^{p,q}(E) = \ker \{\Delta : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q}(E)\} = \{\varphi \in \mathcal{A}^{p,q}(E); \Delta\varphi = 0\} .$$

Then there exists an operator $G : \mathcal{A}^{p,q}(E) \to \mathcal{H}^{p,q}(E)^{\perp}$ such that G is commutative with the both $\bar{\partial}$ and $\bar{\partial}^*$, and moreover Hodge's decomposition holds:

$$\mathcal{A}^{p,q}(E) = \mathcal{H}^{p,q}(E) \oplus \Delta G \left(\mathcal{A}^{p,q}(E)\right) .$$

Theorem 1.2. (**Hodge's Theorem**) Let E be a holomorphic vector bundle over a compact Hermitian manifold M. Then $\dim_{\mathbb{C}} \mathcal{H}^{p,q}(E) < +\infty$ and the following isomorphism holds:

$$\mathcal{H}^{p,q}(E) \cong H^{p,q}_{\bar{\partial}}(M, E) \cong \check{H}^q(M, \Omega^p(E)).$$

1.5 Line bundles

Let L and $\tilde{L}$ be complex line bundles over a smooth manifold M. With respect to an open covering $\mathcal{U}$ of M, we denote by $\{(U, \varphi_U)\}$ and $\{(U, \tilde{\varphi}_U)\}$) the local trivialization of L and $\tilde{L}$ respectively. Then L and $\tilde{L}$ are isomorphic if and only if there exists a smooth function $\Phi_U : U \to \mathcal{A}^*$ satisfying $\tilde{g}_{UV} = \Phi_U \cdot g_{UV} \cdot {\Phi_V}^{-1}$, where $\{g_{UV}\} \in Z^1(\mathcal{U}, \mathcal{A}^*)$ and $\{\tilde{g}_{UV}\} \in Z^1(\mathcal{U}, \mathcal{A}^*)$ denote the transition functions of L and $\tilde{L}$ respectively, and $\mathcal{A}^*$ is the multiplicative sheaf of germs of non-vanishing complex-valued smooth functions on M. From

$$\tilde{g}_{UV} \cdot g_{UV}^{-1} = \frac{\Phi_V}{\Phi_U}$$

we have $\{\tilde{g}_{UV} \cdot g_{UV}^{-1}\} = \{\nu(\Phi)\}_{UV}$. Hence the cocycles $\{g_{UV}\}$ and $\{\tilde{g}_{UV}\}$ define a cohomology class in $\check{H}^1(M, \mathcal{A}^*)$. Consequently the set of isomorphic class of complex line bundles is naturally identified with the cohomology group $\check{H}^1(M, \mathcal{A}^*)$:

$$\{\text{Complex line bundles}\}/\{\text{isomorphic}\} \cong \check{H}^1(M, \mathcal{A}^*).$$

Thus there exists a one-to-one correspondence between the set of all equivalent classes of complex line bundles over M and the cohomology group $\check{H}^1(M, \mathcal{A}^*)$.

Proposition 1.3. *The equivalence class of complex line bundles over a smooth manifold M is naturally identified with the cohomology group $\check{H}^1(M, \mathcal{A}^*)$.*

Remark 1.4. If M is a complex manifold, replacing the sheaf $\mathcal{A}^*$ by $\mathcal{O}^*$, the equivalence class of holomorphic line bundles over a complex manifold M is naturally identified with the cohomology group $\check{H}^1(M, \mathcal{O}^*)$:

$$\{\text{Holomorphic line bundles}\}/_{\{\text{isomorphic}\}} \cong \check{H}^1(M, \mathcal{O}^*).$$

Under this identification, we can consider $\check{H}^1(M, \mathcal{O}^*)$ as an abelian group by defining

(1) $[L] \cdot [\tilde{L}] = [L \otimes \tilde{L}]$,

(2) $[L]^{-1} = [L^*]$

for all $[L], [\tilde{L}] \in \check{H}^1(M, \mathcal{O}^*)$. This abelian group $\check{H}^1(M, \mathcal{O}^*)$ is called the *Picard group* of M, and is sometimes denoted by $\mathrm{Pic}(M)$. $\square$

Let M be a smooth manifold. We are concerned with the exponential sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathcal{A} \xrightarrow{e} \mathcal{A}^* \longrightarrow 0, \tag{1.12}$$

where $e : \mathcal{A} \to \mathcal{A}^*$ is defined by $e(f) = \exp(2\pi\sqrt{-1}f)$. This short sequence induces the long exact sequence of cohomology groups

$$\cdots \longrightarrow \check{H}^1(M, \mathcal{A}) \longrightarrow \check{H}^1(M, \mathcal{A}^*) \xrightarrow{\nu^*} \check{H}^2(M, \mathbb{Z}) \longrightarrow \check{H}^2(M, \mathcal{A}) \longrightarrow \cdots.$$

Since $\mathcal{A}$ is a fine sheaf, we have $\check{H}^1(M, \mathcal{A}) = \check{H}^2(M, \mathcal{A}) = 0$, and thus the connecting map $\nu^* : \check{H}^1(M, \mathcal{A}^*) \ni [L] \to \nu^*([L]) \in \check{H}^2(M, \mathbb{Z})$ is an isomorphism. Therefore we obtain the identification

$$\check{H}^1(M, \mathcal{A}^*) \cong \check{H}^2(M, \mathbb{Z}). \tag{1.13}$$

Hence any complex line bundle L over M is determined by the class $\nu^*([L]) \in \check{H}^2(M, \mathbb{Z})$.

Definition 1.9. The class $c_1(L) \in \check{H}^2(M, \mathbb{Z})$ defined by

$$c_1(L) = -\nu^*([L]). \tag{1.14}$$

is called the *first Chern class* of L.

Therefore the first Chern class $c_1(L)$ defines a characterization of complex line bundles.

Proposition 1.4. *If $c_1(L) = c_1(\tilde{L})$, then $\tilde{L}$ is isomorphic to L as a complex line bundle.*

If L is a holomorphic line bundle over a complex manifold M, the exponential sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{e} \mathcal{O}^* \longrightarrow 0, \tag{1.15}$$

induces the long exact sequence of cohomology groups

$$\cdots \longrightarrow \check{H}^1(M, \mathcal{O}) \longrightarrow \check{H}^1(M, \mathcal{O}^*) \xrightarrow{\nu^*} \check{H}^2(M, \mathbb{Z}) \longrightarrow \check{H}^2(M, \mathcal{O}) \longrightarrow \cdots .$$

The first Chern class $c_1(L)$ of a holomorphic line bundle L is also defined by (1.14) for the connecting map $\nu^* : \check{H}(M, \mathcal{O}^*) \to \check{H}^2(M, \mathbb{Z})$. Then we obtain

Proposition 1.5. *Let L and $\tilde{L}$ be two holomorphic line bundles over a compact complex manifold M. If L is isomorphic to $\tilde{L}$, then we have $c_1(L) = c_1(\tilde{L})$ in $\check{H}^2(M, \mathbb{Z})$.*

Remark 1.5. Since the sheaf $\mathcal{O}$ is not fine, $\nu^* : \check{H}^1(M, \mathcal{O}^*) \to \check{H}^2(M, \mathbb{Z})$ is not isomorphic. Thus the converse of Proposition 1.5 is no true. □

1.6 Vector bundles over complex projective spaces

In this section, we shall list up some vector bundles over the projective space $\mathbb{P}^n$ for later discussions. Let $\rho : \widehat{\mathbb{C}}^{n+1} \to \mathbb{P}^n$ denote the natural projection, where $\widehat{\mathbb{C}}^{n+1} := \mathbb{C}^{n+1} \setminus \{0\}$. For the natural coordinate system $(\zeta^0, \zeta^1, \cdots, \zeta^n)$ on $\mathbb{C}^{n+1}$, we shall write the projection as $\rho(\zeta^0, \zeta^1, \cdots, \zeta^n) = [\zeta^0 : \zeta^1 : \cdots : \zeta^n]$. Then $\rho : \widehat{\mathbb{C}}^{n+1} \to \mathbb{P}^n$ is a $\mathbb{C}^*$-bundle.

If we set $U_i := \left\{[\zeta^0 : \cdots : \zeta^n] \in \mathbb{P}^n \,\middle|\, \zeta^i \neq 0\right\}$, the collection $\mathcal{U} = \{U_i\}_{0 \le i \le n}$ is an open covering of $\mathbb{P}^n$.

Example 1.9. (**Tangent bundle** and **Fubini-Study metric**) Setting $\eta^i = \zeta^i/\zeta^0$ $(i \neq 0)$ on U_0, the projection ρ is a submersion given by $\rho(\zeta^0, \zeta^1, \cdots, \zeta^n) = (\eta^1, \cdots, \eta^n)$ on U_0, and the derivative $d\rho$ is given by

- If $i \ge 1$, then

$$d\rho\left(\frac{\partial}{\partial \zeta^i}\right) = \sum_{l=1}^n \frac{\partial(\zeta^l/\zeta^0)}{\partial \zeta^i}\frac{\partial}{\partial \eta^l} = \sum_{l=1}^n \frac{\delta_i^l}{\zeta^0}\frac{\partial}{\partial \eta^l} = \frac{1}{\zeta^0}\frac{\partial}{\partial \eta^i}$$

- If $i = 0$, then

$$d\rho\left(\frac{\partial}{\partial \zeta^0}\right) = \sum_{l=1}^n \frac{\partial \eta^l}{\partial \zeta^0}\frac{\partial}{\partial \eta^l} = \sum_{l=1}^n \left(-\frac{\zeta^l}{(\zeta^0)^2}\right)\frac{\partial}{\partial \eta^l}$$

Hence we have

$$d\rho\left(\sum_i \zeta^i\left(\frac{\partial}{\partial\zeta^i}\right)\right)=0, \tag{1.16}$$

and thus the holomorphic tangent bundle $T_{\mathbb{P}^n}$ over $\mathbb{P}^n$ is spanned by

$$\left\{d\rho\left(\frac{\partial}{\partial\zeta^0}\right),\cdots,d\rho\left(\frac{\partial}{\partial\zeta^n}\right)\right\}$$

with the relation (1.16). If we put

$$\mathcal{E}=\sum\zeta^i\left(\frac{\partial}{\partial\zeta^i}\right),$$

the line bundle $\ker(d\rho)$ is spanned by $\mathcal{E}$. In deed

$$d\rho\left(\sum_{i=0}^n X^i\frac{\partial}{\partial\zeta^i}\right)=\sum_{l=1}^r\frac{1}{\zeta^0}\left(X^l-X^0\frac{\zeta^l}{\zeta^0}\right)\frac{\partial}{\partial\eta^l}$$

implies

$$\begin{aligned}\sum_{i=0}^n X^i\frac{\partial}{\partial\zeta^i}\in\ker(d\rho) \quad&\Longleftrightarrow\quad \frac{X^0}{\zeta^0}\frac{\zeta^l}{\zeta^0}=\frac{X^l}{\zeta^0}\\ &\Longleftrightarrow\quad X^0:X^1:\cdots:X^n=\zeta^0:\zeta^1:\cdots:\zeta^n\\ &\Longleftrightarrow\quad \sum_{i=0}^n X^i\frac{\partial}{\partial\zeta^i}=\lambda\sum_{i=0}^n\zeta^i\frac{\partial}{\partial\zeta^i}=\lambda\cdot\mathcal{E}\end{aligned}$$

We shall show that the natural Hermitian metric $\langle\bullet,\bullet\rangle$ on $\mathbb{C}^{n+1}$ induces a Kähler metric on $\mathbb{P}^n$. We define a Hermitian metric δ on $T_\zeta\widehat{\mathbb{C}}^{n+1}$ by

$$\delta=\frac{1}{\|\mathcal{E}\|^2}\sum d\zeta^i\otimes d\bar{\zeta}^i,$$

where we put $\|\mathcal{E}\|=\sqrt{\langle\mathcal{E},\mathcal{E}\rangle}=\sum\left|\zeta^i\right|^2$. Denoted by $\mu:\mathbb{C}^*\times\widehat{\mathbb{C}}^{n+1}\ni(\lambda,\zeta)\longmapsto\lambda\cdot\zeta:=\mu_\lambda(\zeta)\in\widehat{\mathbb{C}}^{n+1}$ the action of the multiplicative group $\mathbb{C}^*:=\mathbb{C}\setminus\{0\}$ on $\widehat{\mathbb{C}}^{n+1}$, the Hermitian metric δ is invariant by the action μ, i.e., $\mu_\lambda^*\delta=\delta$ for any $\lambda\in\mathbb{C}^*$. Therefore there exists a Hermitian metric $\mathfrak{g}$ on $T_{\mathbb{P}^n}$ such that $\delta=\rho^*\mathfrak{g}$.

Let $\mathcal{E}_\zeta^\perp$ be the δ-orthogonal complement of $\mathcal{E}$ in $T_\zeta\widehat{\mathbb{C}}^{n+1}$. We denote by $p:T_\zeta\widehat{\mathbb{C}}^{n+1}\to$

$\mathcal{E}_\zeta^\perp$ the orthogonal projection, i.e.,

$$p(Y) = Y - \delta(Y, \mathcal{E})\ \mathcal{E} := Y^\perp.$$

Since $\ker(d\rho)$ is spanned by $\mathcal{E}$, each $\mathcal{E}_\zeta^\perp$ is naturally identified with $T_{[\zeta]}\mathbb{P}^n$, where $[\zeta] = \rho(\zeta)$.

$$\begin{array}{ccc} & T_\zeta \widehat{\mathbb{C}}^{n+1} & \\ {}^{p}\swarrow & & \searrow^{d\rho} \\ \mathcal{E}_\zeta^\perp & \xrightarrow{\cong} & T_{[\zeta]}\mathbb{P}^n \end{array}$$

Further there exists $Y \in T_\zeta\widehat{\mathbb{C}}^{n+1}$ for any $\widetilde{Y} \in T_{[\zeta]}\mathbb{P}^n$ such that $\widetilde{Y}$ is identified with $Y^\perp$, since $d\rho_\zeta : T_\zeta\widehat{\mathbb{C}}^{n+1} \to T_{[\zeta]}\mathbb{P}^n$ is surjective. Then a Hermitian metric $\mathfrak{g}$ on $\mathbb{P}^n$ is defined by

$$\begin{aligned}
\rho^*\mathfrak{g}(\widetilde{Y}, \widetilde{Z}) &= \delta\left(Y^\perp, Z^\perp\right) \\
&= \frac{1}{\|\mathcal{E}\|^2}\Big\langle Y - \delta(Y,\mathcal{E})\mathcal{E}, Z - \delta(Z,\mathcal{E})\mathcal{E}\Big\rangle \\
&= \frac{1}{\|\mathcal{E}\|^4}\Big[\langle\mathcal{E},\mathcal{E}\rangle\langle Y, Z\rangle - \langle Y,\mathcal{E}\rangle\langle\mathcal{E}, Z\rangle\Big].
\end{aligned}$$

Thus $\mathfrak{g}$ is given by

$$\mathfrak{g} = \frac{\|\mathcal{E}\|^2\sum d\zeta^i \otimes d\bar{\zeta}^i - \left(\sum\bar{\zeta}^i d\zeta^i\right) \otimes \left(\sum \zeta^j d\bar{\zeta}^j\right)}{\|\mathcal{E}\|^4}$$

and the form $\Pi_\mathfrak{g}$ associated with $\mathfrak{g}$ is given by

$$\begin{aligned}
\Pi_\mathfrak{g} &= \frac{\sqrt{-1}}{2}\frac{\|\mathcal{E}\|^2\sum d\zeta^i\wedge d\bar{\zeta}^i - \sum\bar{\zeta}^i d\zeta^i \wedge \zeta^i d\bar{\zeta}^i}{\|\mathcal{E}\|^4} \\
&= \frac{\sqrt{-1}}{2}\frac{(|\zeta^0|^2 + |\zeta^1|^2 + \cdots + |\zeta^r|^2)\sum d\zeta^i\wedge d\bar{\zeta}^i - \sum\bar{\zeta}^i d\zeta^i \wedge \zeta^i d\bar{\zeta}^i}{(|\zeta^0|^2 + |\zeta^1|^2 + \cdots + |\zeta^r|^2)^2} \\
&= \frac{\sqrt{-1}}{2}\frac{(1 + |\eta^1|^2 + \cdots + |\eta^r|^2)\sum d\eta^i\wedge d\bar{\eta}^i - \sum\bar{\eta}^i d\eta^i \wedge \eta^i d\bar{\eta}^i}{(1 + |\eta^1|^2 + \cdots + |\eta^r|^2)^2} \\
&= \frac{\sqrt{-1}}{2}\frac{(1 + \|\eta\|^2)\sum d\eta^i\wedge d\bar{\eta}^i - \sum\bar{\eta}^i d\eta^i \wedge \eta^i d\bar{\eta}^i}{(1 + \|\eta\|^2)^2} \\
&= \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log(1 + \|\eta\|^2)
\end{aligned}$$

which is nothing but the Fubini-Study metric Π_{FS}. □

Example 1.10. (**Hyperplane bundle**) Let $F = F(\zeta^0, \cdots, \zeta^n)$ be a homogeneous polynomial of degree k on $\widehat{\mathbb{C}}^{n+1}$, and $V(F)$ the zero-set of F:

$$V(F) = \left\{ (\zeta^0, \cdots, \zeta^n) \in \widehat{\mathbb{C}}^{n+1} \,\middle|\, F(\zeta^0, \cdots, \zeta^n) = 0 \right\}.$$

For the natural projection $\rho : \widehat{\mathbb{C}}^{n+1} \to \mathbb{P}^n$, the subset $Z = \rho(V(F)) \subset \mathbb{P}^n$ is called a *analytic hypersuface* of degree k. A *hyplerplane* is a hypersurface of degree one, and a *hyperquadric* is a hypersurface of degree two. For a hypersurface Z of degree k, we set

$$Z \cap U_i = \left\{ [\zeta^0 : \cdots : \zeta^n] \in U_i \,\middle|\, R_i := \frac{F(\zeta^0, \cdots, \zeta^n)}{(\zeta^i)^k} = 0 \right\}.$$

Then we have

$$\frac{R_i}{R_j} = \left(\frac{\zeta^j}{\zeta^i} \right)^k \in \mathcal{O}^*(U_i \cap U_j),$$

and therefore $\{R_i\}$ defines a holomorphic line bundle over $\mathbb{P}^n$ with the transition functions

$$h^k_{(ij)}(\zeta) = \left(\frac{\zeta^j}{\zeta^i} \right)^k$$

with respect to the open covering $\mathcal{U} = \{U_i\}$. Such a line bundle is usually denoted by $\mathcal{O}(k)$. In particular, any hypersurface $H = V(F)$ defined by a polynomial F of degree one is isomorphic to $\widehat{\mathbb{C}}^{n+1}$, and it defines the line bundle $\mathcal{O}(1)$. We denote this line bundle by $\mathbb{H}$, and we call $\mathbb{H}$ the *hyperplane bundle* over $\mathbb{P}^n$. It is trivial that the k-th tensor power of $\mathbb{H}$ is given by $\mathcal{O}(k) := \mathbb{H}^{\otimes k}$. All hyperplanes are linearly equivalent to each other as divisors so that $\mathbb{H}$ is well-defined (In fact, the line bundle $\mathbb{H}$ is defined by the transition functions $h_{(ij)} := h^1_{(ij)}$ which is independent of the choice of polynomial $F(\zeta)$).

For a homogeneous polynomial F of degree k on $\widehat{\mathbb{C}}^{n+1}$, we set $s^F_i([\zeta]) := R_i([\zeta]) = F(\zeta^0, \cdots, \zeta^n)/(\zeta^i)^k$ on the open set U_i. Then, on the intersection $U_i \cap U_j \neq \phi$, we have

$$s^F_i = s^F_j \cdot h^k_{(ij)}([\zeta]).$$

Therefore $s^F = \{s^F_i\}$ defines a global holomorphic section of the line bundle $\mathbb{H}^{\otimes k}$. It is trivial that the zero set $\{[\zeta] = [\zeta^0 : \cdots : \zeta^n] \in \mathbb{P}^n \,|\, s^F([\zeta]) = 0\}$ is given by $F(\zeta^0, \cdots, \zeta^n) = 0$. Therefore an analytic hyersurface is defined as the zero set of a global section s^F of $\mathbb{H}^{\otimes k}$.

Conversely, let $s = \{s_i\}$ be an arbitrary global holomorphic section of $\mathbb{H}^{\otimes k}$. Then s

satisfies $s_i([\zeta]) = s_j([\zeta]) \cdot h^k_{(ij)}([\zeta])$ on $U_i \cap U_j$, i.e.,

$$\left(\zeta^j\right)^k \cdot s_j([\zeta^0 : \cdots : \zeta^k]) = \left(\zeta^i\right)^k \cdot s_i([\zeta^0 : \cdots : \zeta^k]).$$

This shows that

$$F(\zeta^0, \cdots, \zeta^n) := \left(\zeta^i\right)^k \cdot s_i([\zeta^0 : \cdots : \zeta^k])$$

defines a holomorphic function F on $\widehat{\mathbb{C}}^{n+1}$ satisfying homogeneity condition $F(\lambda \cdot \zeta) = \lambda^k \cdot F(\zeta)$. Therefore F must be a homogeneous polynomial of degree k. □

Proposition 1.6. *The space $\Gamma\left(\mathbb{H}^{\otimes k}\right)$ of global holomorphic sections of $\mathbb{H}^{\otimes k}$ is identified with the space of homogeneous polynomials of degree k.*

Example 1.11. (**Tautological line bundle**) Let $\mathbb{L}$ be the disjoint union of lines in $\mathbb{C}^{n+1}$. For a line l_ζ defined by vector $\zeta \in \widehat{\mathbb{C}}^{n+1}$, we define $\pi : \mathbb{L} \to \mathbb{P}^n$ by $\pi(l_\zeta) = \rho(\zeta)$, where $\rho : \widehat{\mathbb{C}}^{n+1} \to \mathbb{P}^n$ is the natural projection. In another way, $\mathbb{L}$ is defined by

$$\mathbb{L} = \left\{([\zeta], V) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid \zeta \in V\right\}.$$

The fiber $\pi^{-1}([\zeta])$ over $[\zeta] = l_\zeta \in \mathbb{P}^n$ is given by the line $l_\zeta \subset \mathbb{C}^{n+1}$.

We show that $\pi : \mathbb{L} \to \mathbb{P}^n$ is a holomorphic line bundle. Since any point of $\mathbb{L}$ is represented uniquely in the form

$$(t\zeta^0, \cdots, t\zeta^n) = t(\zeta^0, \cdots, \zeta^n) \in \mathbb{C}^{n+1}$$

for $(\zeta^0, \cdots, \zeta^n) \in \widehat{\mathbb{C}}^{n+1}$ and $t \in \mathbb{C}$, we have

$$\pi^{-1}(U_j) = \left\{t(\zeta^0, \cdots, \zeta^n) \in \mathbb{C}^{n+1} \mid t \in \mathbb{C}, \zeta_j \neq 0\right\}$$

on U_j. If we set $t_j = t\zeta^j$ on $\pi^{-1}(U_j)$, then t_j is uniquely determined by the element in $\pi^{-1}(U_j)$. Then, since $t(\zeta^0, \cdots, \zeta^n) \cong t_j \times (\zeta^0 : \cdots : \zeta^n) \in \mathbb{C} \times U_j$, we define a homeomorphism $\varphi_j : \pi^{-1}(U_j) \to U_j \times \mathbb{C}$ by

$$\varphi_j(t(\zeta^0, \cdots, \zeta^n)) = \left((\zeta^0 : \cdots : \zeta^n), t_j\right).$$

It is trivial that φ_i is $\mathbb{C}$-linear on fibers. If $t(\zeta^0, \cdots, \zeta^n) \in \pi^{-1}(U_{ij})$, where $U_{ij} := U_i \cap U_j$, $t_j = t\zeta^j$ and $t_i = t\zeta^i$ lead to

$$t_i = \frac{\zeta^i}{\zeta^j} t_j.$$

This shows that the coordinate change $\varphi_i \circ \varphi_j^{-1}$ is holomorphic, and thus $\pi : \mathbb{L} \to \mathbb{P}^n$ is a holomorphic line bundle. $\mathbb{L}$ is called the *tautological line bundle* over $\mathbb{P}^n$. The transition functions $\{l_{(ij)}\}$ of $\mathbb{L}$ with respect to the covering $\{U_j\}$ of $\mathbb{P}^n$ is given by

$$l_{(ij)}([\zeta]) = \frac{\zeta^i}{\zeta^j} = h^1_{(ij)}([\zeta])^{-1}. \tag{1.17}$$

This relation shows that $\mathbb{L}$ is the dual of $\mathbb{H}$. □

Proposition 1.7. *The tautological line bundle $\mathbb{L}$ is the dual of the hyperplane bundle $\mathbb{H}$:*

$$\mathbb{L} = \mathbb{H}^*. \tag{1.18}$$

Remark 1.6. The hyperplane bundle $\mathbb{H}$ has many global holomorphic sections, but the tautological line bundle $\mathbb{L}$ has no non-zero global holomorphic section:

$$\check{H}^0(\mathbb{P}^n, \mathcal{O}(\mathbb{L})) = 0.$$

In fact, if we suppose that $\mathbb{L}$ has a global section τ, then, for every point $[\zeta] \in \mathbb{P}^n$, τ defines a point $\big(\tau^0([\zeta]), \cdots, \tau^n([\zeta])\big) \in \mathbb{C}^{n+1}$ which lies on the line l_ζ. By projecting to the j-th component, we obtain a holomorphic function $f^j : \mathbb{P}^n \to \mathbb{C}$, i.e., $f^j([\zeta]) = \tau^j([\zeta])$. Since $\mathbb{P}^n$ is compact, and so $\mathbb{P}^n$ has no non-constant holomorphic function. Hence this function is constant. The functions $f^0, \cdots, f^n$ defined in this way are constant. The constant point f_0 defined by the functions should be the origin, since the point lying on all lines through the origin is the origin itself. Hence $\mathbb{L}$ has no non-zero global holomorphic sections. □

Example 1.12. (**Euler sequence**) Let $\mathbb{H}$ be the hyperplane bundle over the projective space $\mathbb{P}^n$. By Proposition 1.6, any holomorphic section of $\mathbb{H}$ is naturally identified with a linear functional on $\mathbb{C}^{n+1}$. For holomorphic sections $\sigma^0, \cdots, \sigma^n$ of $\mathbb{H}$, we consider a $(1,0)$-type vector field

$$\sigma(\zeta) = \sum \sigma^j(\zeta)\frac{\partial}{\partial \zeta^j}, \tag{1.19}$$

on $\mathbb{P}^n$. Since $d\rho(\sigma(\lambda\zeta)) = d\rho(\sigma(\zeta))$ for all $\lambda \in \mathbb{C}$, the definition $d\rho(\sigma)([\zeta]) = d\rho(\sigma(\zeta))$ is well-defined. Then we define a bundle morphism

$$\mathcal{P}(\sigma^0, \cdots, \sigma^n) := d\rho\left(\sum \sigma^j(\zeta)\frac{\partial}{\partial \zeta^j}\right). \tag{1.20}$$

This map $\mathcal{P}$ is surjective. Furthermore the kernel of $\mathcal{P}$ is the trivial line bundle spanned by the section $\mathcal{E} = (\zeta^0, \cdots, \zeta^n) \in \mathbb{H}^{\oplus(n+1)}$. Thus we have an exact sequence

$$0 \longrightarrow 1_{\mathbb{P}^n} \longrightarrow \mathbb{H}^{\oplus(n+1)} \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0, \tag{1.21}$$

Tensoring this sequence with the tautological line bundle $\mathbb{L}$ over $\mathbb{P}^n$, we obtain the so-called *Euler sequence* (cf. [Zh]):

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathcal{O}^{\oplus(n+1)} \longrightarrow \mathbb{L} \otimes T_{\mathbb{P}^n} \longrightarrow 0. \tag{1.22}$$

Chapter 2

Hermitian connections

In this chapter we shall introduce the concepts of connections on complex vector bundles. In section 2.2, we will discuss Hermitian metrics, connections and curvatures on holomorphic vector bundles. In section 2.3, we shall express the first Chern class of a complex vector bundle. The last section is devoted to show that the Hermitian connection on a Hermitian bundle defines an Ehresmann connection.

2.1 Connection and curvature

Let E be a smooth complex vector bundle of rank r over a smooth manifold M. We denote by $\mathcal{A}^p(E)$ the sheaf of germs of smooth E-valued p-forms on M, especially $\mathcal{A}(E) := \mathcal{A}^0(E)$.

Definition 2.1. A *connection* ∇ on E is a homomorphism $\nabla : \mathcal{A}(E) \to \mathcal{A}^1(E)$ satisfying the Leibniz rule:

$$\nabla(f \cdot s) = df \otimes s + f\nabla s \tag{2.1}$$

for all $f \in \mathcal{A}$ and for all $s \in \mathcal{A}(E)$.

Let J_E be a complex structure of a complex vector bundle E. We shall extend the definition of connection to the case of complex vector bundles. A connection ∇ on (E, J_E) is required to satisfy the Leibniz rule (2.1) for every $s \in \mathcal{A}(E)$ and complex-valued function $f \in \mathcal{A}$. This assumption is equivalent to that ∇ satisfies (2.1) and $\nabla\sqrt{-1}s = \sqrt{-1}\nabla s$. Since $\sqrt{-1}s = J_E s$, this condition is equivalent to

$$\nabla J_E = 0. \tag{2.2}$$

In the sequel, we are concerned with a connection ∇ on (E, J_E) satisfying (2.2). Such a connection ∇ is called a *complex connection* on E.

We shall give a local description of a complex connection ∇. Let $e_U = \{e_1, \cdots, e_r\}$ be a local frame field of E over open set U. The *connection form* $\omega = \left(\omega^i_j\right)$ of ∇ with respect to e_U is defined by

$$\nabla e_j = \sum e_i \otimes \omega^i_j, \ (1 \leq j \leq r) \tag{2.3}$$

or, simply by $\nabla e = e \otimes \omega$ in matrix notation. Each $\omega^i_j \in \Gamma(U, \mathcal{A})$ is a local one-form on $U \subset M$. For an arbitrary $s = \sum \zeta^i e_i \in \mathcal{A}(E)$, its covariant derivative ∇s is given by

$$\nabla s = \sum e_i \otimes (d\zeta^i + \sum \omega^i_j \zeta^j). \tag{2.4}$$

We can extend ∇ to a homomorphism $\nabla : \mathcal{A}^k(E) \to \mathcal{A}^{k+1}(E)$ by requiring

$$\nabla(\varphi \otimes s) = d\varphi \otimes s + (-1)^k \varphi \wedge \nabla s \tag{2.5}$$

for every $\varphi \in \mathcal{A}^k$ and $s \in \mathcal{A}(E)$. From (2.1) we have

$$\nabla^2(fs) = \nabla\left(df \otimes s + f\nabla s\right) = -df \wedge \nabla s + df \wedge \nabla s + f\nabla^2 s = f\nabla^2 s$$

for every $f \in \mathcal{A}$ and $s \in \mathcal{A}(E)$, which shows that $R = \nabla^2 : \mathcal{A}(E) \to \mathcal{A}^2(E)$ is a homomorphism. Hence we can identify R as a section of $\mathcal{A}^2(\mathrm{End}(E))$. We set

$$Re_j = \sum e_i \otimes \Omega^i_j. \tag{2.6}$$

Definition 2.2. The operator $R = \nabla \circ \nabla$ is called the *curvature* of ∇. The $\mathrm{End}(E)$-valued two-form $\Omega = (\Omega^i_j)$ is called the *curvature form* of ∇.

By direct calculations, we have $Re_j = \sum e_i \otimes (d\omega^i_j + \sum \omega^i_m \wedge \omega^m_j)$. Hence the curvature form $\Omega = (\Omega^i_j)$ is given by $\Omega^i_j = d\omega^i_j + \sum \omega^i_m \wedge \omega^m_j$:

$$\Omega = d\omega + \omega \wedge \omega. \tag{2.7}$$

Let $(\tilde{e}_1, \cdots, \tilde{e}_r)$ and $(e_1, \cdots, e_r)$ be two local frame fields over U. Then there exists a smooth local function $A = (A^i_j) : U \to GL(r, \mathbb{C})$ satisfying $\tilde{e} = e \cdot A$:

$$\tilde{e}_j(x) = \sum e_i(x) A^i_j(x). \tag{2.8}$$

Applying (2.5) to this relation, we have

$$\nabla \tilde{e}_j = \sum \left(\nabla e_i \, A^i_j + e_i \otimes dA^i_j\right) = \sum e_i \otimes \left(\sum \omega^i_k A^k_j + dA^i_j\right).$$

If we put $\nabla\tilde{e}_j = \sum \tilde{e}_i \otimes \tilde{\omega}^i_j$, we have

$$\tilde{\omega}^i_j = \sum (A^{-1})^i_k \left(dA^k_j + \sum \omega^k_m A^m_j \right).$$

From (2.8), the curvature forms Ω^i_j and $\tilde{\Omega}^i_j$ relative to e and $\tilde{e}$ also satisfy the relation

$$\tilde{\Omega}^i_j = \sum (A^{-1})^i_l \Omega^l_m A^m_j.$$

Hence we obtain

Proposition 2.1. *Let e and $\tilde{e}$ be two local frame fields with common domain. The connection forms ω and $\tilde{\omega}$ of a connection ∇ relative e and $\tilde{e}$ are related by*

$$\tilde{\omega} = A^{-1}(dA + \omega A). \tag{2.9}$$

The curvature forms Ω and $\tilde{\Omega}$ of ∇ relative to e and $\tilde{e}$ are related by

$$\tilde{\Omega} = A^{-1}\Omega A. \tag{2.10}$$

We shall list up the connections on some associated vector bundles with the given vector bundle.

Example 2.1. (**Connection on trivial bundle**) Let $E = M \times \mathbb{C}^r$ be the trivial bundle. In this case, E admits a natural flat connection. In fact, for a frame filed $e = (e_1, \cdots, e_r)$, the connection ∇ is defined by $\nabla e_j = 0$. □

Example 2.2. Let E and $\tilde{E}$ be two vector bundles over a smooth manifold M with connections ∇ and $\tilde{\nabla}$ respectively.

(1) (**Connection on direct sum**) The connections ∇ and $\tilde{\nabla}$ define a connection $\nabla \oplus \tilde{\nabla}$ on the direct sum $E \oplus \tilde{E}$ by

$$(\nabla \oplus \tilde{\nabla})(s \oplus \tilde{s}) = (\nabla s) \oplus (\tilde{\nabla}\tilde{s})$$

for every section s of E and $\tilde{s}$ of $\tilde{E}$ respectively. The connection form of $\nabla \oplus \tilde{\nabla}$ is given by $\begin{pmatrix} \omega & O \\ O & \tilde{\omega} \end{pmatrix}$, and its curvature form of $\nabla \oplus \tilde{\nabla}$ is given by $\begin{pmatrix} \Omega & O \\ O & \tilde{\Omega} \end{pmatrix}$.

(2) (**Connection on tensor product**) In terms of the same notations in (1), the

connection $\nabla \otimes \tilde{\nabla}$ induced on the tensor product $E \otimes \tilde{E}$ is defined by

$$(\nabla \otimes \tilde{\nabla})(s \otimes \tilde{s}) = (\nabla s) \otimes \tilde{s} + s \otimes (\tilde{\nabla} \tilde{s})$$

for every section s of E and $\tilde{s}$ of $\tilde{E}$ respectively. The connection form of $\nabla \otimes \tilde{\nabla}$ is given by $\omega \otimes I_{\tilde{E}} + I_E \otimes \tilde{\omega}$, and its curvature form of $\nabla \otimes \tilde{\nabla}$ is given by $\Omega \otimes I_{\tilde{E}} + I_E \otimes \tilde{\Omega}$.

(3) (**Connection on determinant bundle**) The r-the exterior product $\wedge^r E = E \wedge \cdots \wedge E$, $r = \operatorname{rank}(E)$, is called the *determinant bundle* of E and denoted by $\det E$. This line bundle is defined by $\det(g_{UV})$ for the transition function $\{g_{UV}\}$, and is locally spanned by $e_1 \wedge \cdots \wedge e_r$ for a local holomorphic frame field $\{e_1, \cdots, e_r\}$ of E. If a connection ∇ is given on E, then ∇ induces a connection D by

$$D(e_1 \wedge \cdots \wedge e_r) = (\nabla e_1) \wedge \cdots \wedge e_r + \cdots + e_1 \wedge \cdots \wedge \nabla e_r.$$

Therefore the connection form for D on $\det E$ is given by the trace of ω:

$$\operatorname{tr.}(\omega) = \sum \omega_i^i.$$

Also the curvature form for D on $\det E$ is given by the trace of Ω:

$$\operatorname{tr.}(\Omega) = \sum \Omega_i^i.$$

□

Example 2.3. (**Connection on pull-back bundle**) Let $f : N \to M$ be a smooth map between smooth manifolds N and M. Let E be a vector bundle over M. A connection on E induces a connection $f^*\nabla$ on the pull-back bundle f^*E by

$$(f^*\nabla) f^*s = f^*(\nabla s)$$

for any $s \in \Gamma(E)$. The connection form of $f^*\nabla$ is given by the pull-back $f^*\omega$ of the connection form ω of ∇. The curvature form of $f^*\nabla$ is also given by the pull-back $f^*\Omega$ of the curvature form Ω. □

Example 2.4. (**Connection on dual bundle**) Let E^* be the dual bundle of a vector bundle E over M. A connection ∇ on E induces a dual connection ∇^* on E^* as follows. Let $\langle \cdot, \cdot \rangle$ the natural pairing between E and E^*. For a frame field $e = (e_1, \cdots, e_r)$ of E, its dual frame field $e^* = \left(e^1, \cdots, e^r\right)$ is defined by $\left\langle e_j, e^i \right\rangle = \delta_j^i$. The dual connection ∇^*

of E^* is defined by $d\langle s, s^*\rangle = \langle \nabla s, s^*\rangle + \langle s, \nabla^* s^*\rangle$ for every section s of E and s^* of E^* respectively. Since $\left\langle \nabla e_j, e^i \right\rangle + \left\langle e_j, \nabla^* e^i \right\rangle = 0$, the connection form ω^* of ∇^* is given by

$$\omega^* = -{}^t\omega,$$

and its curvature form Ω^* is given by

$$\Omega^* = -{}^t\Omega.$$

The curvature tensor $R = \sum \Omega^i_j e_i \otimes e^j$ of ∇ is a section of $\mathrm{End}(E) = E \otimes E^*$. For the connection D induced on $\mathrm{End}(E)$ from ∇, (2.7) implies

$$\begin{aligned} DR &= \sum \left(d\Omega^i_j \; e_i \otimes e^j + \Omega^i_j \wedge \nabla e_i \otimes e^j + \Omega^i_j \otimes e_i \wedge \nabla e^j \right) \\ &= \sum \left(d\Omega^i_j + \sum \Omega^m_j \wedge \omega^i_m - \sum \Omega^i_m \wedge \omega^m_j \right) e_i \otimes e^j \\ &= 0. \end{aligned}$$

Therefore we obtain the so called *Bianchi identity*

$$DR = 0. \tag{2.11}$$

□

A connection ∇ on a vector bundle E is said to be *flat* if the curvature R of ∇ vanishes identically, i.e., $\nabla \circ \nabla \equiv 0$. Then we have

Proposition 2.2. *A connection ∇ on E is flat if and only if there exists an open covering $\{(U, e_U)\}$ of E such that the frame field e_U is parallel, i.e., $\nabla e_U = 0$.*

PROOF. We take an open covering $\{(U, e_U)\}$ of E. On each U, we take another frame field $\tilde{e}_U$. Then there exists a smooth function $A : U \to GL(r, \mathbb{C})$ such that $\tilde{e}_U = e_U A$. From (2.8) the respective connection forms $\tilde{\omega}$ and ω of ∇ relative to $\tilde{e}_U$ and e_U are related by (2.9). The condition $\nabla \tilde{e}_U = 0$ is equivalent to $\tilde{\omega} \equiv 0$, i.e., the differential equation $dA + \omega A = 0$. Then we have

$$0 = d(dA) = -d(\omega A) = -d\omega A + \omega \wedge dA = -d\omega A - \omega \wedge \omega A = -\Omega A,$$

which shows that the equation $dA + \omega A = 0$ is completely integrable if and only if $\Omega \equiv 0$.

Q.E.D.

A vector bundle E is said to be *flat* if E admits an open covering $\{(U, e_U)\}$ such that its transition functions $\{g_{UV}\}$ are locally constant in $GL(r, \mathbb{C})$. If E is flat, such an open covering $\{(U, e_U)\}$ is called a *flat structure* of E.

We suppose that E admits a flat connection ∇. Then E admits an open covering $\{(U, e_U)\}$ with parallel fields $\{e_U\}$. Then the relation

$$\omega_V = g_{UV}^{-1} dg_{UV} + g_{UV}^{-1} \omega_U g_{UV} \tag{2.12}$$

implies $\omega_U = \omega_V = 0$ on $U \cap V$. Hence we have $dg_{UV} = 0$, i.e., the open covering $\{(U, e_U)\}$ is a flat structure on E.

Conversely, if E admits a flat structure $\{(U, e_U)\}$, then we define a connection ∇ by $\nabla e_U = 0$ on each U. Then, because of flatness of $\{(U, e_U)\}$ and (2.12), ∇ is a well-defined connection on E. Hence we have

Proposition 2.3. *A vector bundle E is flat if and only if E admits a flat connection ∇.*

2.2 Hermitian connections

Let E be a holomorphic vector bundle of rank r over a complex manifold M. According to the decomposition (1.4), the sheaf $\mathcal{A}^1(E)$ is also decomposed as $\mathcal{A}^1(E) = \mathcal{A}^{1,0}(E) \oplus \mathcal{A}^{0,1}(E)$. Hence the connection ∇ is also decomposed as $\nabla = \nabla^{1,0} + \nabla^{0,1}$, where $\nabla^{1,0} : \mathcal{A}(E) \to \mathcal{A}^{1,0}(E)$ and $\nabla^{0,1} : \mathcal{A}(E) \to \mathcal{A}^{0,1}(E)$. A connection ∇ on a holomorphic vector bundle E is said to be *of* $(1,0)$*-type* if the connection forms ω^i_j of ∇ with respect to a holomorphic open covering $\{(U, s_U)\}$ are $(1,0)$-forms, i.e.,

$$\nabla^{0,1} = \bar{\partial}. \tag{2.13}$$

Proposition 2.4. *Let E be a holomorphic vector bundle over a complex manifold M. Then E admits a $(1,0)$-type connection. Such a connection ∇ satisfies $\nabla^{0,1} \circ \nabla^{0,1} = 0$.*

Proof. For a holomorphic open covering $\{(U_\alpha, s_\alpha)\}$ of E, we denote by $\nabla^{(\alpha)}$ the trivial connection on each $E|_{U_\alpha} = U_\alpha \times \mathbb{C}^r$ defined by $\nabla^{(\alpha)} s_\alpha = 0$. Such a family $\{\nabla^{(\alpha)}\}$ of local flat connections forms a zero-cochain with values in $\Omega^1(\mathrm{End}(E))$. Then $\Lambda_{(\beta\alpha)} := \nabla^{(\beta)} - \nabla^{(\alpha)}$ determines a one-cocycle, and thus $\Lambda_{(\beta\alpha)}$ determines a cohomology class $[\Lambda_{(\beta\alpha)}] \in \check{H}^1(M, \Omega^1(\mathrm{End}(E))$. Since $\Omega^1(\mathrm{End}(E))$ is a sub-sheaf of $A^{1,0}(\mathrm{End}(E))$, and $A^{1,0}(\mathrm{End}(E))$ is fine, there exists $\{\Lambda_{(\alpha)}\} \in A^{1,0}(\mathrm{End}(E))$ such that

$$\Lambda_{(\beta\alpha)} = \Lambda_{(\beta)} - \Lambda_{(\alpha)}$$

on $U_\alpha \cap U_\beta \neq \emptyset$. Hence we obtain

$$\nabla^{(\alpha)} + \Lambda_{(\alpha)} = \nabla^{(\beta)} + \Lambda_{(\beta)}$$

on $U_\alpha \cap U_\beta$. Consequently we obtain a globally defined $(1,0)$-connection $\nabla = \{\nabla^{(\alpha)} + \Lambda_{(\alpha)}\}$, i.e., $\nabla^{0,1} = \overline{\partial}$.

Q.E.D.

The converse is also true (see e.g., [Ko2]).

Proposition 2.5. *If a complex vector bundle E admits a complex connection ∇ such that $\nabla^{0,1} \circ \nabla^{0,1} = 0$, then there exists a holomorphic vector bundle structure in E such that ∇ is of $(1,0)$-type.*

Let (E,h) be a Hermitian vector bundle, and let $e_U = (e_1, \cdots, e_r)$ be a local frame field of E over U. The smoothness of assignment $M \ni z \longmapsto h(z)$ means that $h(\zeta, \eta)$ is a smooth function on M for all $\zeta, \eta \in \mathcal{A}(E)$. We put $h_{i\bar{j}} = h(e_i, e_j)$ on U. Then, since h is Hermitian, we have $h_{i\bar{j}} = \overline{h_{j\bar{i}}}$.

Definition 2.3. Let (E,h) be a Hermitian bundle over a complex manifold M. A connection ∇ on E is said to be *metrical* if it satisfies

$$dh(u,v) = h(\nabla u, v) + h(u, \nabla v) \tag{2.14}$$

for all $u,\ v \in \mathcal{A}(E)$.

In the sequel we assume that E is a holomorphic vector bundle.

Theorem 2.1. *Let (E,h) be a Hermitian vector bundle over a complex manifold M. There exists a unique metrical connection ∇ of $(1,0)$-type on (E,h).*

Proof. Let $e_U = (e_1, \cdots, e_n)$ be a local holomorphic frame field on an open set U. The assumption (2.14) is given by $dh_{i\bar{j}} = \sum(h_{m\bar{j}}\omega_i^m + h_{i\bar{m}}\overline{\omega_j^m})$. Since $dh_{i\bar{j}} = \partial h_{i\bar{j}} + \bar{\partial} h_{i\bar{j}}$ and ω is $(1,0)$-form, we obtain $\omega_j^i = \sum h^{i\bar{m}} \partial h_{j\bar{m}}$. Hence such a connection ∇ is uniquely determined.

Q.E.D.

The unique connection ∇ determined in Theorem 2.1 is called the *Hermitian connection* on (E,h). The connection form $\omega = (\omega_j^i)$ of Hermitian connection ∇ of (E,h) is given by

$$\omega = h^{-1}\partial h \tag{2.15}$$

in matrix notation. Since

$$\Omega(e_j) = \nabla \circ \nabla^{1,0} e_j = \nabla^{1,0} \circ \nabla^{1,0} e_j + \nabla^{0,1} \circ \nabla^{1,0} e_j,$$

the curvature form $\Omega = (\Omega^i_j)$ must be $(1,1)$-type. Consequently $\Omega = \bar{\partial}\omega + \partial\omega + \omega \wedge \omega$ implies $\partial\omega + \omega \wedge \omega = 0$ and

$$\Omega = \bar{\partial}\omega. \tag{2.16}$$

We write $\omega^i_j = \sum \Gamma^i_{j\alpha}(z) dz^\alpha$ so that $\Gamma^i_{j\alpha}$ are given by

$$\Gamma^i_{j\alpha}(z) = \sum h^{i\bar{m}} \frac{\partial h_{j\bar{m}}}{\partial z^\alpha}. \tag{2.17}$$

We also write $\Omega^i_j = \sum R^i_{j\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$ so that $R^i_{j\alpha\bar{\beta}}$ are given by

$$R^i_{j\alpha\bar{\beta}} = -\sum h^{i\bar{m}} \frac{\partial^2 h_{j\bar{m}}}{\partial z^\alpha \partial \bar{z}^\beta} + \sum h^{i\bar{m}} h^{p\bar{q}} \frac{\partial h_{p\bar{m}}}{\partial z^\alpha} \frac{\partial h_{j\bar{q}}}{\partial \bar{z}^\beta}. \tag{2.18}$$

Let L be a holomorphic line bundle over a complex manifold M. Suppose that L admits a Hermitian metric h. Let $\{(U, e_U)\}$ be an open covering of L, where e_U is a local holomorphic frame field of L over U. The local function $h_U(z) := h(e_U, e_U)$ defined on U is smooth and positive. Denoted by $\{g_{UV}\}$ the transition functions of L with respect to $\{(U, e_U)\}$, the local functions $\{h_U\}$ are related by

$$h_U = |g_{UV}|^2 h_V \tag{2.19}$$

on $U \cap V \neq \emptyset$.

From (2.15) the connection form $\omega = (\omega_U)$ of the Hermitian connection ∇ on (L, h_L) with respect to e_U is given by

$$\omega_U = h_U^{-1} \partial h_U = \partial \log h_U, \tag{2.20}$$

and from (2.16), the curvature form $\Omega = (\Omega_U)$ is given by

$$\Omega_U = \bar{\partial}\omega_U = \bar{\partial}\partial \log h_U. \tag{2.21}$$

Example 2.5. Let $\mathbb{L}$ be the tautological line bundle over the complex projective space $\mathbb{P}^n$. We use the notations in §1.6. For the open covering $\mathcal{U} = \{U_j\}_{0 \le j \le n}$ of $\mathbb{P}^n$, the transition functions $\{l_{(ij)}\}$ of $\mathbb{L}$ are given by $l_{(ij)} = \zeta^i/\zeta^j$, where $(\zeta^0, \zeta^1, \cdots, \zeta^n)$ is the homogeneous coordinate system on $\mathbb{P}^n$. From (2.19) any Hermitian metric h on $\mathbb{L}$ is given by a family of positive functions $\{h_{U_i} : U_i \to \mathbb{R}\}$ satisfying $\left|\zeta^i\right|^2 h_{U_i}([\zeta]) = \left|\zeta^j\right|^2 h_{U_j}([\zeta])$ on $U_i \cap U_j \neq \emptyset$. If we define h_{U_j} by

$$h_{U_j}([\zeta]) = \left|\frac{\zeta^0}{\zeta^j}\right|^2 + \cdots + 1 + \cdots + \left|\frac{\zeta^n}{\zeta^j}\right|^2 = \sum_{k=0}^{n} \left|\frac{\zeta^k}{\zeta^j}\right|^2 := K_j$$

on each U_j, then $\{h_{U_j}\}$ defines a Hermitian metric $h_{\mathbb{L}}$ on $\mathbb{L}$. Hence the connection form ω of the Hermitian connection ∇ on $(\mathbb{L}, h_{\mathbb{L}})$ is given by $\omega = \partial \log K_j$, and the curvature form Ω of ∇ is given by $\Omega = \bar{\partial}\partial \log K_j$. Therefore

$$\sqrt{-1}\Omega = -\sqrt{-1}\partial\bar{\partial} \log K_j = -2\Pi_{FS}$$

shows that $\sqrt{-1}\Omega_{U_j}$ is negative, i.e., $(\mathbb{L}, h_{\mathbb{L}})$ is of negative curvature. □

Example 2.6. Let $h = \sum h_{i\bar{j}}(z) e^i \otimes \overline{e^j}$ be a Hermitian metric on E. Then h induces a natural Hermitian metric $\det(h)$ on the determinant bundle $\det E$ by

$$\det(h)\left(e_1 \wedge \cdots \wedge e_r, e_1 \wedge \cdots \wedge e_r\right) = \det\left(h_{i\bar{j}}\right).$$

The Hermitian connection ∇ on (E, h) induces the natural connection D on $(\det E, \det(h))$ as shown in Example 2.2. The connection form for D is given by

$$\operatorname{tr}(\omega) = \sum \omega_i^i = \sum h^{i\bar{m}} \frac{\partial h_{j\bar{m}}}{\partial z^\alpha} dz^\alpha = \partial \log \det\left(h_{i\bar{j}}\right),$$

and the curvature form Ω is given by

$$\operatorname{tr}(\Omega) = \sum \Omega_i^i = \bar{\partial}\partial \log \det\left(h_{i\bar{j}}\right).$$

□

Definition 2.4. The curvature $\operatorname{tr}(\Omega) = \sum \Omega_i^i$ of the determinant bundle $(\det E, \det(h))$ is called the *Ricci curvature* of (E, h). The real $(1,1)$-form $\operatorname{Ric}(h)$ defined by

$$\operatorname{Ric}(h) = \sqrt{-1}\ \bar{\partial}\partial \log \det\left(h_{i\bar{j}}\right) \tag{2.22}$$

is called the *Ricci form* of (E, h).

If we put $\operatorname{Ric}(h) = \sqrt{-1} \sum R_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$, then the Ricci curvature $R_{\alpha\bar{\beta}}$ of $\operatorname{Ric}(h)$ is given by

$$R_{\alpha\bar{\beta}} = -\frac{\partial^2 \log \det\left(h_{i\bar{j}}\right)}{\partial z^\alpha \partial \bar{z}^\beta}.$$

Example 2.7. Let M be a Riemannian surface with a Kähler metric

$$ds^2 = 2g(z)dz \otimes d\bar{z}. \tag{2.23}$$

Since T_M is a holomorphic line bundle, its connection form ω is given by $\omega = g^{-1}\partial g = \partial \log g(z)$, and its curvature Ω is given by $\Omega = \bar{\partial}\omega = \bar{\partial}\partial \log g(z)$. The curvature tensor $R^1_{11\bar{1}}$ is given by

$$R^1_{11\bar{1}} = -\frac{\partial^2 \log g(z)}{\partial z \partial \bar{z}}. \tag{2.24}$$

Hence its Ricci tensor $R_{1\bar{1}}$ is given by

$$R_{1\bar{1}} = -\frac{1}{2g(z)} \frac{\partial^2 \log g(z)}{\partial z \partial \bar{z}}.$$

□

Let (M, g) be a Hermitian manifold with its Hermitian connection ∇. The connection form ω^α_β of (M, g) is given by $\omega^\alpha_\beta = \sum \Gamma^\alpha_{\beta\gamma} dz^\gamma$ with coefficients

$$\Gamma^\alpha_{\beta\gamma} = \sum g^{\alpha\bar{\delta}} \frac{\partial g_{\beta\bar{\delta}}}{\partial z^\gamma}. \tag{2.25}$$

Hence

Proposition 2.6. *A Hermitian manifold (M, g) is a Kähler manifold if and only if its Hermitian connection ∇ satisfies the condition*

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}. \tag{2.26}$$

We put $R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \sum g_{\rho\bar{\beta}} R^\rho_{\alpha\gamma\bar{\delta}}$. Then we have

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = -\frac{\partial^2 g_{\alpha\bar{\beta}}}{\partial z^\gamma \partial \bar{z}^\delta} + \sum g^{\rho\bar{\epsilon}} \frac{\partial g_{\alpha\bar{\epsilon}}}{\partial z^\gamma} \frac{\partial g_{\rho\bar{\beta}}}{\partial \bar{z}^\delta}. \tag{2.27}$$

For an arbitrary point $(z, \zeta) \in T_M \setminus \{0\}$, we put

$$H_g(z, \zeta) = \frac{2}{\|\zeta\|^4} \sum R_{\alpha\bar{\beta}\gamma\bar{\delta}}(z) \zeta^\alpha \bar{\zeta}^\beta \zeta^\gamma \bar{\zeta}^\delta. \tag{2.28}$$

Definition 2.5. The function $H_g : T_M \setminus \{0\} \to \mathbb{R}$ defined by (2.28) is called the *holomorphic sectional curvature* of (M, g) at $z \in M$ in the direction $\zeta \in T_M$.

Example 2.8. (**Riemannian surface**) Let M be a Riemannian surface with a Kähler metric $ds^2 = 2g(z)dz \otimes d\bar{z}$ in (2.23). Since its curvature tensor $R^1_{11\bar{1}}$ is given by (2.24), its sectional curvature $H_g(z,\zeta)$ is given by

$$H_g(z,\zeta) = \frac{2}{4g(z)^2|\zeta|^4}\left(-2g(z)\frac{\partial^2 \log g(z)}{\partial z \partial \bar{z}}|\zeta|^4\right) = -\frac{1}{g(z)}\frac{\partial^2 \log g(z)}{\partial z \partial \bar{z}}. \tag{2.29}$$

Hence the sectional curvature of a Riemannian surface is independent of the direction ζ and is nothing but the *Gaussian curvature* $K_g(z)$ of (M,g). If M is compact, then by Gauss-Bonnet theorem, the characteristic $\chi(M)$ is given by

$$\chi(M) = \frac{1}{2\pi}\int_M K_g dV_g,$$

where the volume form dV_g is given by $dV_g = \sqrt{-1}g(z)dz \wedge d\bar{z}$. Hence we have

$$\chi(M) = \int_M \frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial \log g(z) = \int_M \frac{\sqrt{-1}}{2\pi}\mathrm{R}ic(g) = \int_M c_1(M), \tag{2.30}$$

where $c_1(M) = c_1(T_M)$ is the first Chern class of T_M (see later section). □

Example 2.9. (**Poincaré disk**) Let $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disk in $\mathbb{C}$ with the Poincaré metric

$$g_\Delta = \frac{1}{(1-|z|^2)^2}dz \otimes d\bar{z}. \tag{2.31}$$

This manifold (Δ, g_Δ) is a Kähler manifold of $\dim_{\mathbb{C}} \Delta = 1$. Then, its holomorphic sectional curvature (or Gaussian curvature) is given by (2.29) with $g(z) = \dfrac{1}{2(1-|z|^2)^2}$. By direct calculations, we have

$$\frac{\partial^2}{\partial z \partial \bar{z}}\log\left(\frac{1}{2(1-|z|^2)^2}\right) = \frac{2}{(1-|z|^2)^2}.$$

Consequently we get $H_{g_\Delta}(z,\zeta) = -4$, i.e., the holomorphic sectional curvature H_{g_Δ} is negative constant. This shows that (Δ, g_Δ) is a *hyperbolic* manifold. □

Definition 2.6. A Hermitian vector bundle (E,h) is said to be *flat* if its Hermitian connection ∇ is flat.

To characterize the flatness of Hermitian metrics, we define a function $F_h : E \to \mathbb{R}$ by

$$F_h(z,\zeta) = \sum h_{i\bar{j}}(z)\zeta^i\bar{\zeta}^j \tag{2.32}$$

for the components $h_{i\bar{j}} = h(e_i, e_j)$. This function F_h is smooth on the whole of the total space E.

Proposition 2.7. *A Hermitian bundle (E, h) is flat if and only if there exists an open covering $\{(U, e_U)\}$ with respect to which the function F_h is independent of the base point $z \in M$.*

PROOF. We suppose that the Hermitian connection ∇ of (E, h) is flat. Then, by Proposition 2.3, there exists a flat structure $\{(U, e_U)\}$ satisfying $\nabla e = 0$. From $\nabla e = 0$ and the compatibility condition (2.14), we have $dh(e_i, e_j) = 0$, i.e., the components $h_{i\bar{j}}$ relative to $\{(U, e_U)\}$ are all constants. Hence the function F_h is independent of the base point $z \in M$. Therefore the flatness of (E, h) is equivalent to the existence of an open covering $\{(U, e_U)\}$ of E relative to which the function F_h is independent on the base point $z \in M$.

Q.E.D.

2.3 Chern classes

2.3.1 First Chern class of complex line bundles

From Proposition 1.4 any complex line bundle over a smooth manifold M is determined by its first Chern class since $\check{H}^1(M, \mathcal{A}^*)$ is naturally identified with $\check{H}^2(M, \mathbb{Z})$ via connecting map $\nu^* : \check{H}^1(M, \mathcal{A}^*) \to \check{H}^2(M, \mathbb{Z})$. In this sub-section, we shall express the first Chern class $c_1(L) = -\nu^*(L)$ of L as a class in the de Rham cohomology group $H^2_{DR}(M, \mathbb{C})$ in terms of the curvature of a connection ∇ on L. For this purpose, we shall keep in mind two resolutions of constant sheaf $\mathbb{C}$ on M given by the de Rham complex and the Čech complex respectively.

Let L be a complex line bundle with a Hermitian metric h over a smooth manifold M. We may assume that we are concerned with an open covering $\{(U, s_U)\}$ of L such that s_U is a unitary frame field on U, i.e., $h(s_U, s_U) = 1$. If $U \cap V \neq \emptyset$, we may put $s_U = \exp(2\pi\sqrt{-1}k_{UV})s_V$ for some $\{k_{UV}\} \in C^1(\mathcal{U}, \mathcal{A})$, and the transition functions $\{g_{UV}\} \in Z^1(\mathcal{U}, \mathcal{A}^*)$ are given by $g_{UV} = \exp(2\pi\sqrt{-1}k_{UV}) \in U(1)$. The relation

$$s_U = \exp(2\pi\sqrt{-1}k_{UV})\exp(2\pi\sqrt{-1}k_{VW})s_W = \exp(2\pi\sqrt{-1}(k_{UV} + k_{VW}))s_W$$

shows that $k_{VW} - k_{UW} + k_{UV}$ takes values in $\mathbb{Z}$ on $U \cap V \cap W \neq \emptyset$. If we set

$$c_{UVW} := k_{VW} - k_{UW} + k_{UV} = \nu(k)_{UVW},$$

then $\{c_{UVW}\} \in Z^2(\mathcal{U}, \mathbb{Z})$ determines a cohomology class $[c_{UVW}] \in \check{H}^2(M, \mathbb{Z})$:

$$\begin{array}{ccc}
 & & c_{UVW} \\
 & & {\scriptstyle i}\big\downarrow \\
k_{UV} & \xrightarrow{\nu} & c_{UVW} \\
{\scriptstyle e}\big\downarrow & & {\scriptstyle e}\big\downarrow \\
g_{UV} & \xrightarrow{\nu} & 1
\end{array}$$

Hence the image $\nu^*([L])$ of $[L] = [g_{UV}] \in \check{H}(M, \mathcal{A}^*)$ is given by the class $[c_{UVW}] \in \check{H}^2(M, \mathbb{Z})$ and the first Chern class of L is given by $c_1(L) = -[c_{UVW}]$.

In the sequel we shall consider the class $[c_{UVW}]$ as a class in $\check{H}^2(M, \mathbb{C}) \cong H^2_{DR}(M, \mathbb{C})$. For this purpose we are concerned with an arbitrary connection ∇ of L. Denoted by $\omega = (\omega_U)$ the connection form of ∇ with respect to $\{(U, s_U)\}$, the curvature form Ω of ∇ is given by $\Omega = d\omega + \omega \wedge \omega = d\omega = (d\omega_U)$, since L is a line bundle. Hence $d\Omega = 0$ implies that Ω determines a cohomology class $[\Omega] \in H^2_{DR}(M, \mathbb{C})$.

The relation $\omega_V = g_{UV}^{-1} dg_{UV} + g_{UV}^{-1} \omega_U g_{UV} = g_{UV}^{-1} dg_{UV} + \omega_U$ on $U \cap V \neq \emptyset$ implies $\omega_V - \omega_U = g_{UV}^{-1} dg_{UV} = 2\pi\sqrt{-1} dk_{UV}$, i.e.,

$$dk_{UV} = \frac{1}{2\pi\sqrt{-1}}(\omega_V - \omega_U).$$

Then the commutative diagram

$$\begin{array}{ccccccc}
 & & & & \dfrac{1}{2\pi\sqrt{-1}}\omega_U & \xrightarrow{d} & \dfrac{1}{2\pi\sqrt{-1}} d\omega_U = \dfrac{1}{2\pi\sqrt{-1}}\Omega \\
 & & & & {\scriptstyle \nu}\big\downarrow & & {\scriptstyle \nu}\big\downarrow \\
 & & k_{UV} & \xrightarrow{d} & dk_{UV} = \dfrac{1}{2\pi\sqrt{-1}}\nu(\omega)_{UV} & \xrightarrow{d} & 0 \\
 & & {\scriptstyle \nu}\big\downarrow & & {\scriptstyle \nu}\big\downarrow & & \\
c_{UVW} & \xrightarrow{i} & c_{UVW} & \xrightarrow{d} & 0 & & \\
 & & {\scriptstyle \nu}\big\downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

shows that $\nu^*([L]) = [c_{UVW}]$ is represented by $\left[\dfrac{1}{2\pi\sqrt{-1}}\Omega\right] = -\left[\dfrac{\sqrt{-1}}{2\pi}\Omega\right]$, i.e.,

$$c_1(L) = \left[\frac{\sqrt{-1}}{2\pi}\Omega\right]. \tag{2.33}$$

Proposition 2.8. *Let Ω be the curvature form of a connection ∇ on a complex line bundle L over a smooth manifold M. Then the first Chern class $c_1(L)$ is given by $\left[\frac{\sqrt{-1}}{2\pi}\Omega\right]$.*

Remark 2.1. If L is a holomorphic line bundle over a complex manifold M. The curvature Ω of a Hermitian metric h on L is given by (2.21). Hence $c_1(L)$ is given by

$$c_1(L) = \left[\frac{\sqrt{-1}}{2\pi}\Omega\right] = \left[\frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial \log h\right].$$

□

The natural inclusion $j : \mathbb{Z} \hookrightarrow \mathbb{R}$ induces an inclusion $j^* : \check{H}^2(M, \mathbb{Z}) \hookrightarrow \check{H}^2(M, \mathbb{R}) \cong H^2_{DR}(M)$. A cohomology class in $\check{H}^2(M, \mathbb{C}) \cong H^2_{DR}(M, \mathbb{C})$ is said to be *integral* if it lies in the image $j^*\check{H}^2(M, \mathbb{Z})$. Thus $\left[\frac{\sqrt{-1}}{2\pi}\Omega\right]$ for the curvature Ω of ∇ is integral.

Conversely we suppose that a closed two-form Π such that $\left[\frac{\sqrt{-1}}{2\pi}\Pi\right] \in H^2_{DR}(M, \mathbb{C}) \cong \check{H}^2(M, \mathbb{C})$ is integral, i.e., $\left[\frac{\sqrt{-1}}{2\pi}\Pi\right] = j^*[c_{UVW}]$ for some $[c_{UVW}] \in \check{H}^2(M, \mathbb{Z})$. We put $\Pi = d\Pi_U$ on $U \subset M$. By tracing the reverse of the steps above, there exists $\{k_{UV}\} \in C^1(\mathcal{U}, \mathcal{A})$ satisfying $dk_{UV} = \sqrt{-1}\Pi_V - \sqrt{-1}\Pi_U = \sqrt{-1}\{\nu(\Pi)\}_{UV}$ and

$$\frac{1}{2\pi}(k_{VW} - k_{UW} + k_{UV}) := c_{UVW} \in Z^2(\mathcal{U}, \mathbb{Z}).$$

If we define $g_{UV} \in C^1(\mathcal{U}, \mathcal{A}^*)$ by $g_{UV} := \exp(-\sqrt{-1}k_{UV})$, then we obtain

$$g_{VW} \cdot g_{UW}^{-1} \cdot g_{UV} = \exp(-\sqrt{-1}(k_{VW} - k_{UW} + k_{UV})) = \exp(-2\pi\sqrt{-1} \times c_{UVW}) = 1,$$

i.e., $\{g_{UV}\} \in Z^1(\mathcal{U}, \mathcal{A}^*)$ and $\{g_{UV}\}$ determines a complex line bundle $L \in \check{H}^1(M, \mathcal{A}^*)$. Let $\{(U, e_U)\}$ be an open covering of L with transition functions $\{g_{UV}\}$. Then, for an arbitrary connection ∇ on L, we put $\nabla e_U = e_U \otimes \omega_U$. The relation betwenn ω_U and ω_V implies

$$\omega_V - \omega_U = g_{UV}^{-1}dg_{UV} = -\sqrt{-1}dk_{UV} = \{\nu(\Pi)\}_{UV},$$

and the class $[\Pi]$ coincides with the curvature class $[d\omega_U] = [\Omega]$ in $\check{H}^2(M, \mathbb{C}) \cong H^2_{DR}(M, \mathbb{C})$.

Proposition 2.9. *If there exists a closed two-form Π on M such that $\left[\frac{\sqrt{-1}}{2\pi}\Pi\right]$ is integral, then it gives the first Chern class $c_1(L)$ of a complex line bundle L over M.*

2.3.2 Chern forms and Chern classes of vector bundles

In this subsection, we shall give a brief review of Chern classes of complex vector bundles. Let $\pi : E \to M$ be a complex vector bundle of rank r over a smooth manifold M. We take a complex connection ∇ on E with curvature Ω^{∇}. Then we put

$$\det\left(\frac{\sqrt{-1}}{2\pi}\Omega + tI_E\right) = \sum_{k=0}^{i} c_k(E,\nabla)t^k. \tag{2.34}$$

Then it is verified that the form $c_k(E,\nabla)$ is a well-defined closed $2k$-form on M and so it defines a cohomology class $[c_k(E,\nabla)] \in \check{H}^{2k}(M,\mathbb{C})$. The form $c_k(E,\nabla)$ is called the *k-th Chern form*. The Chern forms $c_k(E,\nabla)$ are defined by a complex connection ∇ on E. For a proof of the following lemma, see p. 161 of [Zh].

Lemma 2.1. *The cohomology classes $[c_k(E,\nabla)]$ are independent on the choice of a complex connection ∇ of E.*

Thus we can compute the Chern forms $c_k(E,\nabla)$ in terms of the curvature Ω^{∇} of the Hermitian connection ∇ of a suitable Hermitian metric g of E. Then it is easily proved that the Chern forms $c_k(E,\nabla)$ are real forms, i.e., $\overline{c_k(E,\nabla)} = c_k(E,\nabla)$. Hence it defines a de Rham cohomology class $c_k(E) = [c_k(E,\nabla)] \in H^{2k}_{DR}(M,\mathbb{R})$ under the natural inclusion $\check{H}^{2k}(M,\mathbb{R}) \subset \check{H}^{2k}(M,\mathbb{C})$. (More strictly, it is proved that the Chern classes $c_k(E)$ are integral, i.e., the classes $c_k(E)$ are contained in the image of the natural inclusion $\check{H}^{2k}(M,\mathbb{Z}) \subset \check{H}^{2k}(M,\mathbb{R})$.) It is trivial that $c_0(E) = 1 \in H^0_{DR}(M,\mathbb{R})$. Other important classes are given by

$$c_1(E) = \left[\frac{\sqrt{-1}}{2\pi}\sum_j \Omega^j_j\right],$$

$$c_2(E) = \left[\frac{-1}{4\pi^2}\sum_{j<k}\left(\Omega^j_j \wedge \Omega^k_k - \Omega^j_k \wedge \Omega^k_j\right)\right],$$

$$\vdots$$

$$c_r(E) = \left[\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \det\left(\Omega^j_j\right)\right]$$

in terms of curvature $\Omega = (\Omega^j_j)$ of ∇. The *total Chern form* $c(E,\nabla)$ of E is defined by

$$c(E,\nabla) = \sum c_k(E,\nabla) = \det\left(\frac{\sqrt{-1}}{2\pi}\Omega^{\nabla} + I\right) \tag{2.35}$$

and the *total Chern class* of E is defined by

$$c(E) = \sum c_k(E) \in H^*_{DR}(M, \mathbb{R}), \tag{2.36}$$

where $H^*_{DR}(M, \mathbb{R})$ is the de Rham cohomology ring. We shall list up some basic properties of Chern classes (cf. [We]).

Proposition 2.10. *Let $f : N \to M$ be a smooth map between smooth manifolds. For a complex vector bundle E over M, the Chern classes $c_k(f^*E)$ of the pull-back bundle f^*E are given by*

$$c_k(f^*E) = f^*c_k(E). \tag{2.37}$$

PROOF. In fact, for a connection ∇ on E, the induced connection $f^*\nabla$ defines a connection on the pull-back bundle f^*E, and the curvature $\Omega^{f^*\nabla}$ is given by $\Omega^{f^*\nabla} = f^*\Omega^{\nabla}$. Hence the Chern forms $c_k(f^*E, f^*\nabla)$ are given by $c_k(f^*E, f^*\nabla) = f^*c_k(E, \nabla)$, which implies $c_k(f^*E) = f^*c_k(E)$.

Q.E.D.

Proposition 2.11. *Let E and $\tilde{E}$ be complex vector bundles over M. Then*

$$c(E \oplus \tilde{E}) = c(E) \cdot c(\tilde{E}). \tag{2.38}$$

PROOF. To prove this proposition, let ∇ and $\tilde{\nabla}$ be connections on E and $\tilde{E}$ respectively. The curvature of $\nabla \oplus \tilde{\nabla}$ on the direct sum $E \oplus \tilde{E}$ is given by $\begin{pmatrix} \Omega & O \\ O & \tilde{\Omega} \end{pmatrix}$. This implies

$$\det \begin{pmatrix} \frac{\sqrt{-1}}{2\pi}\Omega + I_E & O \\ O & \frac{\sqrt{-1}}{2\pi}\tilde{\Omega} + I_{\tilde{E}} \end{pmatrix} = \det\left(\frac{\sqrt{-1}}{2\pi}\Omega + I_E\right) \wedge \det\left(\frac{\sqrt{-1}}{2\pi}\tilde{\Omega} + I_{\tilde{E}}\right),$$

which implies $c(E \oplus \tilde{E}, \nabla \oplus \tilde{\nabla}) = c(E, \nabla) \wedge c(\tilde{E}, \tilde{\nabla})$, and thus we have $c(E \oplus \tilde{E}) = c(E) \cdot c(\tilde{E})$.

Q.E.D.

Proposition 2.12. *Let E^* be the dual bundle of a complex vector bundle over M. Then*

$$c_k(E^*) = (-1)^k c_k(E). \tag{2.39}$$

Proof. In fact, a connection ∇ on E induces a connection ∇^* on E^* by $\omega^* = -{}^t\omega$. Hence the curvature Ω^{∇^*} is given by $\Omega^{\nabla^*} = -{}^t\Omega^{\nabla}$. Hence we have $c_k(E^*, \nabla^*) = (-1)^k c_k(E, \nabla)$, which implies $c_k(E^*) = (-1)^k c_k(E)$.

Q.E.D.

Proposition 2.13. *Let E be a complex vector bundle of rank r over a smooth manifold M, and L a line bundle over M. Then*

$$c_1(E \otimes L) = c_1(E) + rc_1(L). \tag{2.40}$$

Proof. In fact, for a connection ∇^E on E and a connection ∇^L on L, the induced connection $\nabla^{E\otimes L}$ is defined by $\nabla^{E\otimes L} = \nabla^E \otimes 1 + I \otimes \nabla^L$, and its curvature $\Omega^{E\otimes L}$ is given by $\Omega^{E\otimes L} = \Omega^E \otimes 1 + I \otimes \Omega^L$. Hence we have

$$c_1(E \otimes L, \nabla^E \otimes \nabla^L) = \frac{\sqrt{-1}}{2\pi} \sum_j \left(\Omega^{E\otimes L}\right)^j_j = c_1(E, \nabla^E) + rc_1(L, \nabla^L),$$

which implies (2.40).

Q.E.D.

Let (E, h) be a holomorphic Hermitian vector bundle over a complex manifold M. If we denote by ∇ the Hermitian connection on (E, h), the k-th Chern form $c_k(E, \nabla)$ is of (k, k)-type. Especially the first Chern form $c_1(E, \nabla)$ is given by the Ricci form $\mathrm{Ric}(h)$ of (E, h):

$$c_1(E, \nabla) = \frac{\sqrt{-1}}{2\pi} \sum_j \Omega^j_j = \frac{\sqrt{-1}}{2\pi} \bar{\partial}\partial \log \det \left(h_{i\bar{j}}\right) = \frac{\sqrt{-1}}{2\pi} \sum R_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta.$$

Example 2.10. We shall compute the first Chern class $c_1(T_{\mathbb{P}^n})$ of the tangent bundle $T_{\mathbb{P}^n}$ of complex projective space $\mathbb{P}^n$. For this purpose, we shall consider the Fubini-Study metric Π_{FS} on $\mathbb{P}^n$. Because of

$$\Pi_{FS} = \frac{\sqrt{-1}}{2} \partial\bar{\partial} \log \left(1 + \|\zeta\|^2\right),$$

the volume element dV of $(\mathbb{P}^n, \Pi_{FS})$ is given by

$$dV = \frac{1}{n!} \Pi^n_{FS} = \left(\frac{\sqrt{-1}}{2}\right)^n \frac{1}{(1 + \|\zeta\|^2)^{n+1}} d\zeta^1 \wedge d\bar{\zeta}^1 \cdots \wedge d\zeta^n \wedge d\bar{\zeta}^n.$$

Thus (2.22) implies

$$\mathrm{Ric}(\Pi_{FS}) = (n+1)\Pi_{FS},$$

namely, Fubini-Study metrics satisfies the so-called *Einstein condition*. From this we get

$$c_1(T_{\mathbb{P}^n}) = (n+1)[\Pi_{FS}].$$

On the other hand, by the Euler sequence (1.22), if we take a Hermitian metric on $\mathbb{H}^{\oplus(n+1)}$, we have an orthogonal decomposition $\mathbb{H}^{\oplus(n+1)} = T_{\mathbb{P}^n} \oplus 1_{\mathbb{P}^n}$ for the hyperplane bundle $\mathbb{H}$ over $\mathbb{P}^n$. Thus the total Chern class of $\mathbb{H}^{\oplus(n+1)}$ is given by $c\left(\mathbb{H}^{\oplus(n+1)}\right) = c(T_{\mathbb{P}^n}) \cdot c(1_{\mathbb{P}^n}) = c(T_{\mathbb{P}^n}) \cdot 1$. Since $c\left(\mathbb{H}^{\oplus(n+1)}\right) = c(\mathbb{H})^{n+1}$ and $c(\mathbb{H}) = 1 + c_1(\mathbb{H})$, we have

$$c(T_{\mathbb{P}^n}) = (1 + c_1(\mathbb{H}))^{n+1}.$$

Because of $c(T_{\mathbb{P}^n}) = \sum c_k(T_{\mathbb{P}^n})$, we have

$$c_1(\mathbb{H}) = \frac{1}{n+1} c_1(T_{\mathbb{P}^n}) = \frac{1}{\pi}[\Pi_{FS}], \tag{2.41}$$

which is the positive generator of the cohomology group $\check{H}^1(\mathbb{P}^1, \mathbb{Z}) \cong \mathbb{Z}$. □

2.3.3 Positive lines bundles and ample line bundles

Let L be a holomorphic line bundle with a Hermitian metric h, and let $\{(U, e_U)\}$ be an open covering of L with transition functions $\{g_{UV}\}$. If we put $h(e_U, e_U) = h_U(z)$ on each U, the local function h_U is smooth and positive, and moreover it satisfies $h_V = h_U|g_{UV}|^2$ on $U \cap V$. The Hermitian connection ∇ of (L, h) is given by the local $(1,0)$-form $\omega = \partial \log h$ and its curvature Ω is given by $\Omega = \bar{\partial}\partial \log h_U$. The first Chern class $c_1(L)$ is represented by

$$c_1(L, \nabla) = \frac{\sqrt{-1}}{2\pi}\Omega.$$

Definition 2.7. A holomorphic line bundle L is said to be *positive* if its first Chern class $c_1(L)$ is represented by a positive real $(1,1)$-form.

By this definition, a holomorphic line bundle L is positive if and only if L admits a Hermitian metric h whose curvature $\sqrt{-1}\Omega = \sqrt{-1}\bar{\partial}\partial \log h$ is positive-definite. Then the form $-\sqrt{-1}\Omega = \sqrt{-1}\partial\bar{\partial} \log h$ defines a Kähler metric on M.

Example 2.11. Let $\mathbb{H}$ be the hyperplane bundle over a complex projective space $\mathbb{P}^n$. Since $c_1(\mathbb{H})$ is given by (2.41), $\mathbb{H}$ is positive. □

Let (M, g) be a compact Kähler manifold. Since the Kähler form Π_g defined by (1.5) is closed, it determines a cohomology class $[\Pi_g] \in H^2_{DR}(M, \mathbb{R}) \subset H^2_{DR}(M, \mathbb{C})$. We take an open cover $\mathcal{U}$ of M so that Π_g is expressed as $\Pi_g = \sqrt{-1}\partial\bar{\partial}K_U$ on each $U \in \mathcal{U}$, where K_U is the Kähler potential for g on U. If we put $\alpha_U = -\sqrt{-1}\partial K_U$, then we have

$$d\alpha_U = d\left(-\sqrt{-1}\partial K_U\right) = \sqrt{-1}\partial\bar{\partial}K_U = \Pi_g\big|_U.$$

Therefore $[d\alpha_U]$ represents the class $[\Pi_g]$ in $H^2_{DR}(M, \mathbb{C})$. From Remark 1.2 there exists $\{k_{UV}\} \in C^1(\mathcal{U}, \mathcal{O})$ satisfying $K_V - K_U = k_{UV} + \overline{k_{UV}}$. Then we have

$$\alpha_V - \alpha_U = -\sqrt{-1}\ \partial\left(K_V - K_U\right) = -\sqrt{-1}\ \partial\left(k_{UV} + \overline{k_{UV}}\right) = d\left(-\sqrt{-1}k_{UV}\right)$$

since k_{UV} is holomorphic. We set $h_{UV} = -\sqrt{-1}k_{UV}$, i.e., $\nu(\alpha)_{UV} = d\left(h_{UV}\right)$.

Now we suppose that $[\Pi_g]$ is integral, i.e., $[\Pi_g] = j^*[c_{UVW}]$ for some $[c_{UVW}] \in \check{H}^2(M, \mathbb{Z})$.

$$\begin{array}{ccccccc}
 & & & & \alpha_U & \xrightarrow{d} & d\alpha_U = \Pi_g \\
 & & & & \nu\big\downarrow & & \nu\big\downarrow \\
 & & h_{UV} & \xrightarrow{d} & d(h_{UV}) & \xrightarrow{d} & 0 \\
 & & \nu\big\downarrow & & \nu\big\downarrow & & \\
c_{UVW} & \xrightarrow{i} & c_{UVW} & \xrightarrow{d} & 0 & & \\
 & & \nu\big\downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

We set $g_{UV} := \exp(2\pi\sqrt{-1}h_{UV})$. Then $\{g_{UV}\} \in C^1(\mathcal{U}, \mathcal{O}^*)$ satisfies

$$\begin{aligned}
g_{VW} \cdot g_{UW}^{-1} \cdot g_{UV} &= \exp(2\pi\sqrt{-1}h_{VW}) \cdot \exp(-2\pi\sqrt{-1}h_{UW}) \cdot \exp(2\pi\sqrt{-1}h_{UV}) \\
&= \exp(2\pi\sqrt{-1}c_{UVW}) \\
&= 1.
\end{aligned}$$

Hence $\{g_{UV}\} \in Z^1(\mathcal{U}, \mathcal{O}^*)$ determines a holomorphic line bundle $L \in \check{H}^1(M, \mathcal{O}^*)$. Further, from Proposition 2.9, the first Chern class $c_1(L)$ is represented by the positive real $(1,1)$-form $\dfrac{1}{2\pi}\Pi_g$. Thus L is positive.

Conversely, if there exists a positive holomorphic line bundle L over M, then its first

Chern class $c_1(L)$ is represented by the curvature Ω of a Hermitian metric h on L:

$$c_1(L) = \left[\frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial \log h\right] \in \check{H}^2(M, \mathbb{Z}).$$

Then the closed form $\sqrt{-1}(\bar{\partial}\partial \log h)/2\pi$ defines a Kähler metric on M which is integral.

A compact Kähler manifold (M, g) is called a *Hodge manifold* if the Kähler class $[\Pi_g]$ is integral. Therefore we obtain

Theorem 2.2. ([Mo-Ko]) *A compact complex manifold admits a positive holomorphic line bundle if and only if M is a Hodge manifold.*

Let L be a holomorphic line bundle over a compact complex manifold M. Since M is compact, $\dim_{\mathbb{C}} \check{H}^0(M, \mathcal{O}(L))$ is finite. Let $\{s_0, \cdots, s_N\}$ be a set of linear independent sections of L of the complex vector space of global sections. The vector space is called a *linear system* on M. If the vector space consists of all global sections of L, it is called a *complete linear system* on X. Then a rational map $\varphi_{|L|} : M \to \mathbb{P}^N$ is defined by

$$\varphi_{|L|}(z) = [f^0(z) : \cdots : f^N(z)], \tag{2.42}$$

where we put $\varphi_U(s_i) = \left(z^\alpha, f^i\right) \in U \times \mathbb{C}$ for a local trivialization $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}$. This rational map is defined on the open set in M which is the complementary to the common zero-set of the sections s_i $(0 \leq i \leq N)$. It is verified that the rational map $\tilde{\varphi}_{|L|}$ obtained from another basis $\{\tilde{s}_0, \cdots, \tilde{s}_N\}$ is transformed by an automorphism of $\mathbb{P}^n$.

Definition 2.8. A holomorphic line bundle L over M is said to be *very ample* if the rational map $\varphi_{|L|} : M \to \mathbb{P}^N$ determined by its complete linear system $|L|$ is a holomorphic embedding. L is said to be *ample* if there exists an integer $m \in \mathbb{Z}$ such that $L^{\otimes m}$ is very ample.

Let $\{U_j\}$ be the open covering of $\mathbb{P}^N$ defined in §.1.4. Suppose that L is a very ample line bundle over a compact complex manifold M with a basis $\{s_0, \cdots, s_N\}$ of $\check{H}^0(M, \mathcal{O}(L))$ which defines a holomorphic embedding $\varphi_{|L|} : M \to \mathbb{P}^N$. Under this embedding, we can think of $[f^0 : \cdots : f^N]$ as a coordinate system on the embedded M in $\mathbb{P}^N$. We define an open covering $\{V_j\}$ of M by $V_j = \{z \in M \mid s_j(z) \neq 0\} = \varphi^{-1}_{|L|}(U_j) \cap M$. With respect to this covering, the local trivialization $\varphi_i : \pi^{-1}(V_j) \to V_j \times \mathbb{C}$ of L is given by $\varphi_j(s_i) = (z^\alpha_{(j)}, f^i_{(j)})$, and the transition functions $\{g_{jk}\}$ are given by $g_{jk}(z) = f^i_{(k)}(z)/f^i_{(j)}(z)$.

On the other hand, the transition functions $\{h_{(jk)}\}$ of the hyperplane bundle $\mathbb{H}$ over $\mathbb{P}^N$ is given by the form $h_{(jk)} = \dfrac{\zeta^k}{\zeta^j}$ for the covering $\{U_j\}$ of $\mathbb{P}^N$ (cf. Example 1.11). Hence

$\{h_{(jk)}\}$ satisfy the relations

$$h_{(jk)} \circ \varphi_{|L|} = \frac{f^i_{(k)}}{f^i_{(j)}} = g_{jk}$$

and thus we have $L = \varphi^*_{|L|}\mathbb{H}$.

Lemma 2.2. *Let L be a very ample line bundle over a complex manifold. Then L is isomorphic to the pull-back bundle $\varphi^*_{|L|}\mathbb{H}$ of the hyperplane bundle $\mathbb{H}$ over the target space $\mathbb{P}^N$ of $\varphi_{|L|}$.*

The following well-known theorem shows that any Hodge manifold M is *projective algebraic*, i.e., M is embedded into a projective space $\mathbb{P}^N$.

Theorem 2.3. (**Kodaira's embedding theorem**) *Let L be a holomorphic line bundle over a compact complex manifold. If L is positive, then it is ample, i.e., there exists an integer n_0 such that for all $N \geq n_0$ the map $\varphi_{|L|} : M \to \mathbb{P}^N$ is a holomorphic embedding.*

The converse of this theorem is also true, i.e., we have

Proposition 2.14. *A holomorphic line bundle L over a compact complex manifold M is positive if and only if L is ample.*

Proof. We suppose that L is ample. Then there exists a basis $\{s_0, \cdots, s_N\}$ of $\check{H}^0(M, \mathcal{O}(L^m))$ such that $\varphi_{|L^m|}(z) : M \to \mathbb{P}^N$ defined by (2.42) is a holomorphic embedding. By Lemma 2.2, the line bundle L^m is identified with $\varphi^*_{|L^m|}\mathbb{H}$. Thus there exists a Hermitian metric g on L^m such that

$$c_1(L^m) = mc_1(L) = \left[\frac{1}{2\pi\sqrt{-1}}\bar{\partial}\partial \log g(z)\right],$$

where $g(z)$ is defined by $g = \sum_{i=0}^N \left|f^i(z)\right|^2$. Since $\mathbb{H}$ is positive, the $(1,1)$-form $\sqrt{-1}\bar{\partial}\partial\log g(z)$ is positive, and thus

$$c_1(L) = \frac{1}{m}\left[\frac{1}{2\pi\sqrt{-1}}\bar{\partial}\partial \log g(z)\right]$$

is positive. Consequently L is positive.

Q.E.D.

Remark 2.2. From the proof above, an ample line bundle L admits a Hermitian metric of the form

$$g(z) = \sqrt[m]{\sum_{i=0}^N |f^i(z)|^2} \tag{2.43}$$

for some sections $s_i(z) = \left(z^\alpha, f^i(z)\right) \in \check{H}^0(M, \mathcal{O}(L^m))$. $\square$

2.3.4 Compact Riemannian surfaces

Let E be a holomorphic vector bundle over a compact complex manifold M. The Serre duality $\check{H}^{p,q}(M,E) \cong \check{H}^{n-p,n-q}(M,E^*)^*$ and Dolbeault isomorphism $\check{H}^{p,q}(M,E) \cong \check{H}^q(M,\Omega^p(E))$ imply

$$\check{H}^q(M,\Omega^p(E)) \cong \check{H}^{n-q}(M,\Omega^{n-p}(E^*))^*.$$

Putting $p=q=0$, we have $\check{H}^0(M,\mathcal{O}(E)) \cong \check{H}^n(M,\Omega^n(E^*))^*$ (see, e.g., [Ko2]).

Now we suppose that M is a compact Riemannian surface. Since $\dim_{\mathbb{C}} M = 1$, we have

$$\check{H}^0(M,\mathcal{O}(E)) \cong \check{H}^1(M,\Omega^1(E^*))^* = \check{H}^1(M,\mathcal{O}(E^* \otimes K_M))^*$$

and

$$\dim_{\mathbb{C}} \check{H}^0(M,\mathcal{O}(E)) = \dim_{\mathbb{C}} \check{H}^1(M,\mathcal{O}(E^* \otimes K_M))$$

for the canonical line bundle $K_M = \Omega_M$ of M. The integer

$$g := \dim_{\mathbb{C}} \check{H}^1\left(M,\mathcal{O}_M\right) = \dim_{\mathbb{C}} \check{H}^0\left(M,\mathcal{O}(K_M)\right)$$

is the *genus* of M.

The *degree* $\deg(L)$ of a holomorphic line bundle is defined by $\deg(L) = \int_M c_1(L) \in \mathbb{Z}$. If we apply the well-known *Riemann-Roch theorem*

$$\dim_{\mathbb{C}} \check{H}^0(M,\mathcal{O}(L)) - \dim_{\mathbb{C}} \check{H}^1(M,\mathcal{O}(L)) = \deg(L) + 1 - g$$

to the case of $L = K_M$, we have

$$\dim_{\mathbb{C}} \check{H}^1(M,\mathcal{O}(K_M)) = \dim_{\mathbb{C}} \check{H}^0(M,\Omega^1(K_M^*)) = \dim_{\mathbb{C}} \check{H}^0(M,\mathcal{O}_M) = 1$$

since M is compact and thus $\check{H}^0(M,\mathcal{O}_M) = \mathbb{C}$. Consequently we have

$$\deg(K_M) = 2g - 2.$$

The Euler characteristic is given by

$$\chi(M) = \int_M c_1(T_M) = -\deg K_M = 2 - 2g.$$

Any compact Riemannian surface M is determined completely by its genus g:

(1) if $g = 0$, then M is holomorphically isometric to the Riemannian sphere $\mathbb{P}^1 =$

$\mathbb{C} \cup \{\infty\}$,

(2) if $g = 1$, then M is holomorphically isometric to a torus $\mathbb{C}/\Lambda$,

(3) if $g > 1$, then M is hyperbolic, i.e., M admits a Kähler metric of negative curvature.

In the case of $g = 0$, i.e., $M = \mathbb{P}^1$, then since $c_1(T_M) > 0$, its tangent bundle is positive (or equivalently ample). In the case of $g > 1$, then its tangent bundle T_M is negative since $c_1(T_M) < 0$.

2.4 Negative vector bundles and Griffiths-negativity

The ampleness of holomorphic line bundles is an important notion in algebraic geometry, and it is equivalent to the positivity in the sense of differential geometry. It is natural to generalize the notion of ampleness to the case of higher rank.

Let M be a compact complex manifold with a complex coordinate system $\{U, (z^\alpha)\}$, and let $\pi : E \to M$ be a holomorphic vector bundle. Since we shall work in an open set $U \subset M$, we fix a local holomorphic frame field $(e_1, \cdots, e_r)$ on U. Since any $v \in \pi^{-1}(U)$ is represented as $v = \sum \zeta^i e_i$, we shall think $(z, \zeta) = (z^1, \cdots, z^m, \zeta^1, \cdots, \zeta^r)$ as a local holomorphic coordinate system in $\pi^{-1}(U)$, where $(\zeta^1, \cdots, \zeta^r)$ is the fiber coordinate in E_z.

Let $\phi : \mathbb{P}(E) \to M$ be the projective bundle associated with E. Denoted by $[v]$ the point of $\mathbb{P}(E)$ corresponding to $v = (z, \zeta) \in E$, the *tautological line bundle* $\mathbb{L}(E) \to \mathbb{P}(E)$ is defined by

$$\mathbb{L}(E) = \{([v], V) \in \mathbb{P}(E) \times E \mid [v] \in V\}. \tag{2.44}$$

Definition 2.9. ([Ko1]) A holomorphic vector bundle E over a compact complex manifold M is said to be *negative* if the tautological line bundle $\mathbb{L}(E)$ is negative, i.e., $c_1(\mathbb{L}(E)) < 0$. A holomorphic vector bundle E over M is said to be *ample* if its dual E^* is negative.

Suppose that E admits a Hermitian metric $h = \sum h_{i\bar{j}} e^i \otimes \bar{e}^j$. For all $u \in E_z$ and $X \in T_z M$, we set

$$R(u \otimes X) = \sum R_{i\bar{j}\alpha\bar{\beta}}(z) u^i \overline{u^j} X^\alpha \overline{X^\beta}$$

for the curvature tensor $R_{i\bar{j}\alpha\bar{\beta}}$ of (E, h) where $R_{i\bar{j}\alpha\bar{\beta}} := \sum h_{l\bar{j}} R^l_{i\alpha\bar{\beta}}$.

Definition 2.10. A holomorphic vector bundle is said to be *Griffiths-negative* if E admits a Hermitian metric h of negative curvature, i.e., $R(u \otimes X) < 0$ at any point $z \in M$ for all non-zero $u \in E_z$ and non-zero $X \in T_z M$.

Remark 2.3. Let E be a holomorphic vector bundle endowed with a Hermitian metric h. Denoted by Ω the curvature form of the Hermitian connection ∇ on (E, h), the curvature form Ω^* of the induced connection ∇^* on the dual bundle E^* is given by $\Omega^* = -{}^t\Omega$ (cf. Example 2.4). Therefore the curvature R^* of ∇^* is given by

$$R^*(u^* \otimes X) = -\sum R^{j\bar{k}}{}_{\alpha\bar{\beta}} u_j X^\alpha \overline{u_k X^\beta},$$

where $R^{j\bar{k}}{}_{\alpha\bar{\beta}} = \sum h^{j\bar{l}} h^{m\bar{k}} R_{m\bar{l}\alpha\bar{\beta}}$. Hence

$$\begin{aligned} R^*(v^* \otimes X) &= -\sum R^{j\bar{k}}{}_{\alpha\bar{\beta}} u_j X^\alpha \overline{u_k X^\beta} \\ &= -\sum R_{m\bar{l}\alpha\bar{\beta}} u^m X^\alpha \overline{u^l X^\beta} \\ &= -R(u \otimes X). \end{aligned}$$

Thus E is Griffiths-negative if and only if E^* is Griffiths-positive. □

We show a sufficient condition for the negativity of Hermitian bundles.

Theorem 2.4. *If a holomorphic vector bundle E over a compact complex manifold M is Griffith-negative, then E is negative.*

Proof. Let h be a Hermitian metric on E of negative curvature. We define a function F_h on E^0 by $F_h(z, \zeta) = h(v, v) = \sum h_{i\bar{j}}(z)\zeta^i\bar{\zeta}^j$. If we set $F_j = F_h/|\zeta^j|^2$ on each U_j, then $\{F_j\}$ satisfy the relation

$$F_j = \left(\frac{|\zeta^i|}{|\zeta^j|}\right)^2 F_i = \left|l_{(ij)}\right|^2 F_i.$$

Hence the family $\{F_j\}$ defines a Hermitian metric on $\mathbb{L}(E)$. Some direct calculations imply that the curvature form $\bar{\partial}\partial \log F_j = \bar{\partial}\partial \log F$ of this metric is given by

$$\begin{aligned} \bar{\partial}\partial \log F_h &= \begin{pmatrix} \dfrac{1}{F_h}\sum R_{i\bar{j}\alpha\bar{\beta}}\zeta^i\bar{\zeta}^j & O \\ O & -(\log F_h)_{i\bar{j}} \end{pmatrix} \\ &= \frac{1}{\|\zeta\|^2}\sum (R_{i\bar{j}\alpha\bar{\beta}}(z)\zeta^i\bar{\zeta}^j) dz^\alpha \wedge d\bar{z}^\beta - \sum \frac{\partial^2 \log F_h}{\partial\zeta^i \partial\bar{\zeta}^j} \nabla\zeta^i \wedge \overline{\nabla\zeta^j} \\ &= \frac{1}{\|\zeta\|^2} h(R(v), v) - \sum \frac{\partial^2 \log F_h}{\partial\zeta^i \partial\bar{\zeta}^j} \nabla\zeta^i \wedge \overline{\nabla\zeta^j}. \end{aligned}$$

Hence $R(v \otimes X) < 0$ implies that $\mathbb{L}(E)$ is negative, and thus E is negative.

Q.E.D.

Remark 2.4. Let E^0 (resp. $\mathbb{L}(E)^0$) denote the open sub-manifold of E (resp. $\mathbb{L}(E)$) consisting of all non-zero elements. We define an open covering $\{U_j\}$ of $\mathbb{P}(E)$ by $U_j = \phi^{-1}(U) \cap \{[v] \in \mathbb{P}(E) \mid \zeta^j \neq 0\}$. Further we define $t_j : U_j \to U_j \times \mathbb{C}^r$ by

$$t_j([v]) := \left([v], \left(\frac{\zeta^1}{\zeta^j}, \cdots, \frac{\zeta^r}{\zeta^j}\right)\right).$$

Then $\{U_j, t_j\}$ defines a local trivialization $\varphi_j : U_j \times \mathbb{C} \to \mathbb{L}(E)_{|U_j}$ of $\mathbb{L}(E)$ by $\varphi_j([v], \lambda) = \lambda t_j([v])$. Using this local trivialization, we define a map $\tau : E^0 \to \mathbb{P}(E) \times E$ by

$$\tau(v) := ([v], v) = \zeta^j t_j([v]) = \varphi_j([v], \zeta^j).$$

This holomorphic map τ maps E^0 biholomorphically onto $\mathbb{L}(E)^0$. Then, for any Hermitian metric $h_{\mathbb{L}(E)}$ on $\mathbb{L}(E)$, we define the norm $\|v\|_E$ of $v \in E^0$ by $\|v\|_E = \sqrt{h_{\mathbb{L}(E)}(\tau(v), \tau(v))}$. Extending this definition continuously on E, we obtain a function $F : E \to \mathbb{R}$ by

$$F(z, \zeta) = \|v\|_E^2 .$$

This function F defines a *complex Finsler metric* on E, not Hermitian metric in general. Hence the converse of Theorem 2.4 is an open problem. Kobayashi[Ko1] characterized negativity of holomorphic vector bundles in terms of complex Finsler metrics (see Theorem 3.1 in the next chapter). □

2.5 Ehresmann connections

Let E be a holomorphic vector bundle over a complex manifold M. We define the *vertical sub-bundle* V of the total space T_E by $V = \ker(\widetilde{d\pi})$ with $\widetilde{d\pi} = (\pi, d\pi)$ for the derivative $d\pi : T_{(z,\zeta)}E \to T_zM$ at $v = (z, \zeta) \in E$. The fiber $V_v \subset T_vE$ over $v \in E$ is the tangent space of the fiber $\pi^{-1}(\pi(v))$ at $v \in E$, i.e., $V_v = \{Y \in T_vE \mid d\pi(Y) = 0\}$. The vertical sub-bundle $V \cong \widetilde{E}$ is a holomorphic vector bundle over E with standard fiber $\mathbb{C}^r$. Since the projection π is a holomorphic submersion, we have $(\widetilde{TM})_v \cong T_{\pi(v)}M = \mathrm{Im}(d\pi_v) \cong T_vE/\ker(d\pi_v) = T_vE/V_v$ which induces the exact sequence of holomorphic vector bundles over E:

$$\mathbb{O} \longrightarrow V \xrightarrow{\ i\ } T_E \xrightarrow{\ \widetilde{d\pi}\ } \widetilde{T_M} \longrightarrow \mathbb{O}, \tag{2.45}$$

where $\widetilde{T_M} = \pi^* T_M$.

A subspace $H_v \subset T_vE$ is called the *horizontal subspace* at $v \in E$ if H_v is complementary to V_v, i.e., $T_vE = V_v \oplus H_v$. Although the vertical space V_v is uniquely determined at each point $v \in E$, the horizontal subspace H_v is not canonically determined.

The multiplier group $\mathbb{C}^* \cong \{cI \in GL(r,\mathbb{C});\ c \in \mathbb{C}^*\} \subset GL(r,\mathbb{C})$ acts on the total space as a sub-algebra of $\mathrm{End}(E)$ by the rule $\mu_\lambda : E \ni v \mapsto \mu_\lambda(v) := (z, \lambda \cdot \zeta) \in E$.

Definition 2.11. An *Ehresmann connection* on E is a smooth distribution $H : E \ni v \longmapsto H_v \subset T_E$ of a horizontal subspace H_v at each point $v \in E$ in a $\mathbb{C}^*$-invariant way, i.e., the selection is required that

$$H_{\lambda\cdot v} = d\mu_\lambda(H_v) \tag{2.46}$$

for all $\lambda \in \mathbb{C}^*$ and $v \in E$, and H_v depends on $v \in E$ smoothly.

An Ehresmann connection H on E is also called a *horizontal sub-bundle* of T_E. Given an Ehresmann connection H on E, T_E splits into a C^∞ decomposition

$$T_E = V \oplus H. \tag{2.47}$$

In another word, an Ehresmann connection H is a smooth distribution which assigns to each point $v \in E$ a linear sub-space $H_v \subset T_vE$ such that $\dim_{\mathbb{C}} H_v = \dim_{\mathbb{C}} M$ and $H_v \cap V_v = \{0\}$. Further $d\pi$ is an isomorphism on H, i.e., $d\pi_v(H_v) = T_{\pi(v)}M$. Thus an Ehresmann connection H is equivalent to determine a C^∞ splitting ψ of the short exact sequence (2.45), namely, ψ is a C^∞ bundle morphism $\psi : \widetilde{T_M} \to T_E$ satisfying $\widetilde{d\pi} \circ \psi = id$:

$$\mathbb{O} \longrightarrow V \xrightarrow{\ \iota\ } T_E \underset{\psi}{\overset{\widetilde{d\pi}}{\rightleftarrows}} \widetilde{T_M} \longrightarrow \mathbb{O}.$$

The splitting (2.47) is written as $T_E = V \oplus \psi(\widetilde{T_M})$.

For any section $X \in \mathcal{A}(T_M)$, there exists a unique $X^H \in \mathcal{A}(H)$ such that $d\pi(X^H) = X$. Such a section X^H is called the *horizontal lift* of X. Then $\pi \circ \mu_\lambda = \pi$ implies

$$d\pi_{\lambda\cdot v}\left(X^H(\lambda\cdot v)\right) = \left(d\pi_v \circ d\mu_{\lambda^{-1}}\right)\left(X^H(\lambda\cdot v)\right) = d\pi_v\left(d\mu_{\lambda^{-1}}\left(X^H(\lambda\cdot v)\right)\right).$$

The assumption (2.46) shows $d\mu_{\lambda^{-1}}\left(X^H(\lambda\cdot v)\right) = X^H(v)$. Consequently any horizontal lift X^H satisfies

$$X^H(\lambda\cdot v) = d\mu_\lambda\left(X^H(v)\right).$$

An alternative definition of Ehresmann connection is given by a left splitting P of the sequence (2.45), i.e., V-valued $(1,0)$-form P on E satisfying $\iota(Z)P = Z$ for every $Z \in \mathcal{A}(V)$, and

$$P_v = \mu_\lambda^* P_{\lambda\cdot v} \tag{2.48}$$

for every $\lambda \in \mathbb{C}^*$. Then the horizontal sub-bundle H is defined by $H = \ker(P)$.

Definition 2.12. For a smooth curve $c : I = [0,1] \to M$ in the base manifold M, a curve $\tilde{c}_v : I \to E$ starting a point $v \in E_{c(0)}$ is called the *horizontal lift* of c with respect to H if it satisfies $\tilde{c}_v(0) = v$, $\pi \circ \tilde{c}_v(t) = c(t)$ and

$$\tilde{c}_v^* P = 0. \tag{2.49}$$

Since $\dim I = 1$, this differential equation is integrable, and it has a unique solution $\tilde{c}_\zeta$ with the initial condition $v = \tilde{c}_v(0)$. Consequently the horizontal lift $\tilde{c}_v = (c(t), X(t))$ exists for every curve $c = c(t)$ in M and for any point $v \in E_{c(0)}$. Then we define a map $\tau_c : E_{c(0)} \to E_{c(1)}$ by

$$\tau_c(v) = \widetilde{c}_v(1), \quad v \in E_{c(0)}. \tag{2.50}$$

Since the solution of (2.49) depends smoothly on the initial condition v, the map τ_c is a diffeomorphism between the fibers. For any $\lambda \in \mathbb{C}^*$, $\lambda \cdot \widetilde{c}_v(t) = \mu_\lambda(\widetilde{c}_v(t)) = (c(t), \lambda X(t))$. Hence (2.48) implies

$$\begin{aligned} P_{\lambda \cdot \widetilde{c}_v}\left(\frac{d(\lambda \cdot \widetilde{c}_v)}{dt}\right) &= P_{\lambda \cdot \widetilde{c}_v}\left(d\mu_\lambda\left(\frac{d\tilde{c}_v}{dt}\right)\right) = (\mu_\lambda^* P_{\lambda \cdot \widetilde{c}_v})\left(\frac{d\tilde{c}_v}{dt}\right) = P_{\widetilde{c}_v}\left(\frac{d\tilde{c}_v}{dt}\right) \\ &= \tilde{c}_v^* P\left(\frac{d}{dt}\right) \\ &= 0. \end{aligned}$$

Therefore $\lambda \cdot \widetilde{c}_v(t)$ is the horizontal lift of $c = c(t)$ through the point $\lambda \cdot v \in E_{c(0)}$. The uniqueness of the solution of (2.49) shows that $\lambda \cdot \widetilde{c}_v = \widetilde{c}_{\lambda \cdot v}(t)$, and so we obtain $\widetilde{c}_{\lambda \cdot v}(1) = \lambda \cdot \widetilde{c}_v(1) = \lambda \cdot \tau_c(v)$, namely

$$\tau_c(\lambda \cdot v) = \lambda \cdot \tau_c(v) \tag{2.51}$$

for any $v \in E_{c(0)}$ and $\lambda \in \mathbb{C}^*$. The derivative $d\tau_{c,0}$ of τ_c at $v = 0$ is given by

$$d\tau_{c,0}(v) = \lim_{t \to 0} \frac{\tau_c(0 + t \cdot v) - \tau_c(0)}{t} = \lim_{t \to 0} \frac{\tau_c(t \cdot v)}{t} = \tau_c(v).$$

Denoting τ_c by $\tau_c = (\tau_c^1, \cdots, \tau_c^r)$, τ_c is given by

$$\tau_c(v) = \begin{pmatrix} \dfrac{\partial \tau_c^1}{\partial \zeta^1}(0) & \cdots & \dfrac{\partial \tau_c^1}{\partial \zeta^r}(0) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial \tau_c^r}{\partial \zeta^r}(0) & \cdots & \dfrac{\partial \tau_c^r}{\partial \zeta^r}(0) \end{pmatrix} \begin{pmatrix} \zeta^1 \\ \\ \vdots \\ \\ \zeta^r \end{pmatrix} = \left(\sum a_i^1 \zeta^i, \cdots, \sum a_i^r \zeta^i\right), \quad a_i^h := \frac{\partial \tau_c^h}{\partial \zeta^i}(0).$$

Therefore τ_c is a linear map. Since $\tau_{c^{-1}} = (\tau_c)^{-1}$, the map $\tau_c : E_{c(0)} \to E_{c(1)}$ defined by (2.50) is a linear isomorphism between the fibers.

Definition 2.13. The $\mathbb{C}$-linear isomorphism τ_c is called the *parallel displacement* of v along c with respect to H.

Let X be a tangent vector at $z \in M$, and let $c = c(t)$ be the integral curve of X through the point $z = c(0)$. Then, for the parallel displacement τ_c, we define $\nabla_X : \mathcal{A}(E) \to \mathcal{A}(E)$ by

$$\nabla_X v = \left.\frac{d}{dt}\right|_{t=0} \left[\tau_c^{-1}\left(v(c(t))\right)\right] = \lim_{t\to 0} \frac{\tau_c^{-1}\left(v(c(t))\right) - v(z)}{t} \quad (z = c(0)). \tag{2.52}$$

Since the parallel displacement τ_c is a linear isomorphism, the map ∇_X is a linear morphism. Furthermore

$$\begin{aligned}
\nabla_X(f \cdot v) &= \lim_{t\to 0} \frac{\tau_c^{-1}\left(f(c(t))v(c(t))\right) - f(z)v(z)}{t} \\
&= \lim_{t\to 0} \frac{f(c(t))\tau_c^{-1}\left(v(c(t))\right) - f(z)v(z)}{t} \\
&= \lim_{t\to 0} \frac{f(c(t))\left[\tau_c^{-1}\left(v(c(t))\right) - v(z)\right]}{t} + \lim_{t\to 0} \frac{\left[f(c(t)) - f(z)\right]v(z)}{t} \\
&= \lim_{t\to 0} f(c(t)) \frac{\tau_c^{-1}\left(v(c(t))\right) - v(z)}{t} + \lim_{t\to 0} \frac{f(c(t)) - f(z)}{t} v(z) \\
&= f(z)\nabla_X v + X(f)v(z)
\end{aligned}$$

shows that ∇ satisfies the Leibnitz rule

$$\nabla_X(f \cdot v) = f \cdot \nabla_X v + X(f) \cdot v. \tag{2.53}$$

Consequently any Ehresmann connection H determines a covariant derivative ∇ in E.

Conversely we shall show that any covariant derivative ∇ on E determines an Ehresmann connection on T_E. Then the natural action μ of $\mathbb{C}^*$ on E induces a holomorphic vector field $\mathcal{E}$ on T_E defined by

$$\mathcal{E}(v) = \sum \zeta^i \left(\frac{\partial}{\partial \zeta^i}\right)_v \cong (v, v) \tag{2.54}$$

for all $v = (z, \zeta) \in E$.

Definition 2.14. The vector field $\mathcal{E}$ on E defined by (2.54) is called the *tautological section* of V or *radial vector field* on E.

The tautological section $\mathcal{E}$ is invariant by the action μ of $\mathbb{C}^*$ on E, i.e.,

$$d\mu_\lambda(\mathcal{E}) = \mathcal{E}. \tag{2.55}$$

Indeed, $d\mu_\lambda \left(\frac{\partial}{\partial \zeta^i}\right)_\zeta = \lambda \left(\frac{\partial}{\partial \zeta^i}\right)_{\lambda\cdot\zeta}$ implies

$$\begin{aligned} d\mu_\lambda(\mathcal{E}) &= d\mu_\lambda \left(\sum \zeta^i \left(\frac{\partial}{\partial \zeta^i}\right)_{(z,\zeta)}\right) \\ &= \sum \zeta^i d\mu_\lambda \left(\frac{\partial}{\partial \zeta^i}\right)_{(z,\zeta)} \\ &= \sum \zeta^i \lambda \left(\frac{\partial}{\partial \zeta^i}\right)_{(z,\lambda\cdot\zeta)} \\ &= \sum \lambda\zeta^i \left(\frac{\partial}{\partial \zeta^i}\right)_{(z,\lambda\cdot\zeta)} \\ &= \mathcal{E}(z, \lambda\cdot\zeta). \end{aligned}$$

Hence $\mathcal{E}$ is the fundamental vector field on E with respect to the action of $\mathbb{C}^*$.

Let $\nabla : \mathcal{A}(E) \to \mathcal{A}^1(E)$ be a connection on E such that $\nabla^{0,1} = \overline{\partial}$. Then ∇ induces a natural Ehresmann connection H. Indeed the induced connection $\pi^*\nabla := \widetilde{\nabla} : \mathcal{A}(V) \to \mathcal{A}^1(V)$ defines a covariant derivative on V, since $V \cong \widetilde{E}$. For local expression of $\widetilde{\nabla}$, we denote by $\{\widetilde{dz^1}, \cdots, \widetilde{dz^m}\}$ the dual of vector fields $\left\{\widetilde{\frac{\partial}{\partial z^1}}, \cdots, \widetilde{\frac{\partial}{\partial z^m}}\right\}$ on T_E defined by

$$\widetilde{\frac{\partial}{\partial z^\alpha}} := \pi^*\left(\frac{\partial}{\partial z^\alpha}\right).$$

Then we can easily prove that the V-valued $(1,0)$-form P defined by

$$P = \widetilde{\nabla}\mathcal{E} = \sum \left(\frac{\partial}{\partial \zeta^i}\right)_v \otimes \left(d\zeta^i + \sum \zeta^j \Gamma^i_{j\alpha}(z)\widetilde{dz^\alpha}\right) \cong (v, \nabla v) \in E \oplus \mathcal{A}^1(E) \tag{2.56}$$

satisfies $P(Z) = Z$ for every $Z \in \mathcal{A}(V)$, i.e., $H = \ker(P)$ is an Ehresmann connection.

Further P satisfies $P_{f\cdot v} = (f\cdot v, \nabla(f\cdot v)) = (f\cdot v, df \otimes v + f\cdot \nabla v)$ for any $v \in \mathcal{A}(E)$ and $f \in C^\infty(M)$. Thus any Ehresmann connection H is determined by a section (v, σ) of the bundle $E \oplus \mathcal{A}^1(E)$ satisfying

$$f \cdot (v, \sigma) = (f \cdot v, df \otimes v + f \cdot \sigma)$$

for any $v \in \mathcal{A}(E)$, $\sigma \in \mathcal{A}^1(E)$ and $f \in C^\infty(M)$.

As shown in the above, any connection ∇ of $(1,0)$-type on E defines an Ehresmann connection H in the sense of Definition 2.11. In particular, the Hermitian connection ∇ on a Hermitian bundle (E, h) defines a natural Ehresmann connection H.

Let E be a holomorphic vector bundle with a Hermitian metric $h = \sum h_{i\bar{j}}(z) e^i \otimes e^j$. The Hermitian connection ∇ of (E, h) is given by (2.15).

Proposition 2.15. *Let $c : (0,1) \to M$ be a smooth curve. Then the parallel displacement τ_c along c is linear isomorphism and preserves the metric h, i.e., for all u, $v \in E_{c(0)}$*

$$h\left(\tau_c(u), \tau_c(v)\right) = h(u, v). \tag{2.57}$$

Let $z_0 \in M$ be a fixed point, and C_0 be the set of all closed curve based on z_0. We put $\Psi_0 = \{\tau_c \mid c \in C_0\}$. Since the displacement $\tau_c : E_{z_0} \to E_{z_0}$ is a linear isomorphism, relative to the basis $\{s_1(z_0), \cdots, s_r(z_0)\}$ of E_{z_0}, any element of Ψ_0 is represent by an element (τ^i_j) of $GL(r, \mathbb{C})$. Then, by (2.57) we have $\sum h_{r\bar{s}} \tau^r_i \overline{\tau^s_j} = h_{i\bar{j}}$. Hence Ψ_0 is a subgroup of unitary group $U(r)$.

Definition 2.15. The group Φ_0 is called the *holonomy group* of (E, h) with reference point $z_0 \in M$.

Let (E, h) be a Hermitian bundle, and $P : T_E \to V$ the connection on E defined by (2.56):

$$P = \sum \frac{\partial}{\partial \zeta^i} \otimes P^i,$$

where we put $P^i := \nabla \zeta^i = d\zeta^i + \sum \omega^i_j \zeta^j$. Then we define $T \in \mathcal{A}^2(V)$ by

$$T = \widetilde{\nabla} P. \tag{2.58}$$

By definition, we have

$$T(X, Y) = \widetilde{\nabla}_X P(Y) - \widetilde{\nabla}_Y P(X) - P([X, Y]).$$

From (2.56), we have $T = \widetilde{\nabla}^2 \mathcal{E} = (\pi^* R)(\mathcal{E})$. In local coordinate, the form T is given by

$$T = \sum \frac{\partial}{\partial \zeta^i} \otimes \left(\sum \Omega^i_j \zeta^j \right) = \sum \frac{\partial}{\partial \zeta^i} \otimes \left(\sum R^i_{j\alpha\bar{\beta}}(z) \zeta^j \widetilde{dz^\alpha} \wedge \widetilde{d\bar{z}^\beta} \right).$$

We call T the *torsion form* of $(V_E, \widetilde{\nabla})$.

In the case of $E = T_M$, i.e., E is the holomorphic tangent bundle of a complex manifold M, then the bundle $\widetilde{T_M}$ is naturally identified with V. Therefore we may consider the differential $d\pi$ as another morphism $d\pi : T_E \longrightarrow V$ with the form

$$d\pi = \sum \frac{\partial}{\partial \zeta^\alpha} \otimes \widetilde{dz^\alpha}.$$

Then we define another torsion $\widetilde{T} \in \mathcal{A}^2(V)$ by

$$\widetilde{T} = \widetilde{\nabla}(d\pi), \tag{2.59}$$

that is,

$$\widetilde{T}(X, Y) = \widetilde{\nabla}_X d\pi(Y) - \widetilde{\nabla}_Y d\pi(X) - d\pi[X, Y]$$

for all $X, Y \in \mathcal{A}(T_E)$. Then we have

$$\widetilde{T} = \sum \widetilde{\nabla} \frac{\partial}{\partial \zeta^\beta} \wedge dz^\beta = \sum \frac{\partial}{\partial \zeta^\alpha} \otimes \omega^\alpha_\beta \wedge dz^\beta = \sum \frac{\partial}{\partial \zeta^\alpha} \otimes \left(\sum \Gamma^\alpha_{\beta\gamma}(z) \widetilde{dz^\beta} \wedge \widetilde{dz^\gamma} \right).$$

Therefore $\widetilde{T} \equiv 0$ if and only if (2.27) is satisfied, i.e., (M, h) is a Kähler manifold.

Proposition 2.16. *A Hermitian manifold (M, h) is Kähler if and only if its torsion form $\widetilde{T}$ vanishes identically.*

Chapter 3

Finsler metrics and connections

In this chapter, we will focus on the geometry of complex Finsler vector bundles. As an application of the geometry of Kähler fibrations, we shall study the geometry of the vertical sub-bundle V with the Hermitian metric g defined by a Rizza metric F. We shall introduce the notion of Rizza-negativity of complex Finsler metrics. The fundamental tool in this chapter is a partial connection on V ([Ha-Ai], [Ai6], [Ai8]).

3.1 Complex Finsler metrics

3.1.1 Complete circular domains and Minkowski functionals

We shall recall the notion of (complex) Minkowski space (cf. [Th], [Pa-Wo]). Let $\mathbb{V}$ be a complex vector space of $\dim_{\mathbb{C}} \mathbb{V} = n$.

Definition 3.1. We call a function $\|\cdot\| : \mathbb{V} \to \mathbb{R}$ a *Minkowski norm* on $\mathbb{V}$ if it satisfies

(1) $\|\zeta\| \geq 0$, and $\|\zeta\| = 0$ if and only if $\zeta = 0$,

(2) $\|\lambda\zeta\| = |\lambda| \cdot \|\zeta\|$ for all $\lambda \in \mathbb{C}$ and $\zeta \in \mathbb{V}$,

(3) $\|\zeta\|$ is C^∞ on $\mathbb{V} \setminus \{0\}$, and is continuous on $\mathbb{V}$.

A complex vector space $\mathbb{V}$ is said to be a *complex Minkowski space* if a Minkowski norm $\|\cdot\| : \mathbb{V} \to \mathbb{R}$ is defined on $\mathbb{V}$. The unit ball $\mathcal{D} = \{\zeta \in \mathbb{C}^n \mid \|\zeta\| < 1\}$ is called the *indicatrix* of the given Minkowski norm.

We shall fix a basis $\{s_1, \cdots, s_n\}$ of $\mathbb{V}$ and identify $\mathbb{V}$ with $\mathbb{C}^n$ with coordinate system $\left(\zeta^1, \cdots, \zeta^n\right)$. If we set $f(\zeta) = \|\zeta\|^2$, then f satisfies the following conditions:

(f 1) $f(\zeta) \geq 0$, and $f(\zeta) = 0$ if and only if $\zeta = 0$,

(f 2) $f(\lambda\zeta) = |\lambda|^2 \cdot f(\zeta)$ for all $\lambda \in \mathbb{C}$ and $\zeta \in \mathbb{V}$,

(f 3) f is C^∞ on $\mathbb{V} \setminus \{0\}$, and is continuous on $\mathbb{V}$..

The function f is called a *complex Finsler metric* on $\mathbb{V} \cong \mathbb{C}^n$. A complex Finsler metric f is said to be *strongly pseudo-convex* if f is strongly pluri-subharmonic outside of the origin, i.e., the *Levi form*

$$L_f(Z, Z) = \sum \frac{\partial^2 f}{\partial \zeta^i \partial \bar{\zeta}^j} Z^i \bar{Z}^j$$

is positive-definite for all $Z = \left(Z^1, \cdots, Z^n\right)$. Hence the complex Hessian $(f_{i\bar{j}})$ defined by

$$f_{i\bar{j}} := \frac{\partial^2 f}{\partial \zeta^i \partial \bar{\zeta}^j} \tag{3.1}$$

is positive-definite.

Definition 3.2. ([Pa-Wo]) A domain $\mathcal{D}$ in $\mathbb{C}^n$ satisfying the following conditions is called a *complete circular domain.*

(1) If $\zeta \in \mathcal{D}$ and $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$, then $\lambda\zeta = (\lambda\zeta^1, \cdots, \lambda\zeta^n) \in \mathcal{D}$

(2) If $\zeta \in \bar{\mathcal{D}}$ and $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, then $\lambda\zeta \in \mathcal{D}$.

In the sequel we usually treat complete circular domains with smooth boundaries. For a bounded complete circular domain $\mathcal{D}$, its *Minkowski functional* $m_{\mathcal{D}}$ is defined by

$$m_{\mathcal{D}}(\zeta) := \inf \left\{ \frac{1}{t} \;\mid t\zeta \notin \mathcal{D}, t > 0 \right\}, \ \zeta \in \mathbb{C}^n. \tag{3.2}$$

Moreover, if we set

$$f_{\mathcal{D}} = m_{\mathcal{D}}^2. \tag{3.3}$$

Then it is trivial that $f_{\mathcal{D}}$ satisfies $f_{\mathcal{D}}(\lambda\zeta) = |\lambda|^2 f_{\mathcal{D}}(\zeta)$ for all $\zeta \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$, i.e., $f_{\mathcal{D}}$ is a complex Finsler metric on $\mathbb{C}^n$. Moreover it is also trivial that $\zeta \in \mathcal{D}$ if and only if $f_{\mathcal{D}}(\zeta) < 1$, i.e., the domain $\mathcal{D}$ is the *indicatrix* of the corresponding $f_{\mathcal{D}}$. If $\mathcal{D}$ is strongly pseudo-convex, then

$$\left((f_{\mathcal{D}})_{i\bar{j}}\right) > 0, \ \ \left((\log f_{\mathcal{D}})_{i\bar{j}}\right) \geq 0. \tag{3.4}$$

Hence there exists a one-to-one correspondence between the set of all complete circular and strongly pseudo-convex domains with smooth boundaries and the set of all strongly pseudo-convex Finsler metrics.

Proposition 3.1. ([Pa-Wo]) *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two complete circular domains in $\mathbb{C}^n$ with smooth boundaries. Then, $\mathcal{D}_1$ is biholomorphic to $\mathcal{D}_2$ if and only if the Finsler metric $f_{\mathcal{D}_1}$ of $\mathcal{D}_1$ is related to $f_{\mathcal{D}_2}$ of $\mathcal{D}_2$ by $f_{\mathcal{D}_1} = f_{\mathcal{D}_2} \circ A$ for some $A \in GL(n, \mathbb{C})$.*

By this proposition, the following characterization of Hermitian inner product is proved:

Proposition 3.2. ([Pa-Wo]) *Let $\mathcal{D}$ be a complete circular domain in $\mathbb{C}^n$. The following statements are equivalent:*

(1) *$\mathcal{D}$ is biholomorphic to the unit ball $B = \left\{\zeta \in \mathbb{C}^n \mid \sum |\zeta^i|^2 < 1\right\}$,*

(2) *the associated Finsler metric $f_{\mathcal{D}}$ is of the form*

$$f_{\mathcal{D}}(\zeta) = \sum_{j=1}^{n} \left| \sum_{k=1}^{n} A_k^j \zeta^k \right|^2$$

for some $A = \left(A_k^j\right) \in GL(n, \mathbb{C})$,

(3) *$f_{\mathcal{D}}$ is smooth at the origin.*

Example 3.1. ([Ai3]) Let $\mathcal{D}$ be a domain in $\mathbb{C}^n$ defined by

$$\mathcal{D} := \left\{ \zeta \in \mathbb{C}^n \mid \sum_{i=1}^{n} \left|\zeta^i\right|^2 + |\zeta^n|^4 < 1 \right\}.$$

$\mathcal{D}$ is complete circular and strongly pseudo-convex domain. We shall construct the complex Finsler metric $f_{\mathcal{D}}$ whose indicatrix is the given $\mathcal{D}$. We suppose that $\lambda\zeta \in \partial\mathcal{D}$ for some $\lambda \in \mathbb{R}$. Then we have $\lambda^2 \cdot \|\zeta\|^2 + \lambda^4 \cdot |\zeta^n|^4 = 1$, where we put $\|\zeta\|^2 = \sum_{i=1}^{n} \left|\zeta^i\right|^2$. Setting $\lambda^2 = 1/m_{\mathcal{D}}^2 = 1/f_{\mathcal{D}}$, we have

$$\frac{\|\zeta\|^2}{f_{\mathcal{D}}} + \frac{|\zeta^n|^4}{f_{\mathcal{D}}^2} = 1.$$

Therefore the function $f_{\mathcal{D}}$ defined by

$$f_{\mathcal{D}}(\zeta) = \frac{1}{2} \left\{ \|\zeta\|^2 + \sqrt{\|\zeta\|^4 + 4\,|\zeta^n|^4} \right\}$$

is a complex Finsler metric on $\mathbb{C}^n$ whose indicatrix is the given $\mathcal{D}$. We note that this Finsler metric $f_{\mathcal{D}}$ is invariant by $U(n-1) \times U(1)$. □

3.1.2 Complex Finsler metrics on $\mathbb{C}^{n+1}$ and Kähler metrics on $\mathbb{P}^n$

In Example 1.9, it is shown that the natural Hermitian metric on $\mathbb{C}^{n+1}$ induces a standard Kähler metric Π_{FS} on the projective space $\mathbb{P}^n$. Conversely it is natural to ask whether any Kähler metric on $\mathbb{P}^n$ is induced from a Hermitian metric on $\mathbb{C}^{n+1}$. This is not true in general. In fact, any Kähler metric on $\mathbb{P}^n$ is induced from a strongly pseudo-convex Finsler metric on $\mathbb{C}^{n+1}$.

For a complex Finsler metric f on $\mathbb{C}^{n+1}$, we shall correspond a real closed $(1,1)$-form

$$\Pi = \sqrt{-1}\,\partial\bar{\partial}\log f = \sqrt{-1}\sum_{i,j=0}^{n}\frac{\partial^2 \log f}{\partial\zeta^i\partial\bar{\zeta}^j}d\zeta^i\wedge d\bar{\zeta}^j.$$

Π is invariant by the action μ of $\mathbb{C}^*$. Indeed, since $\lambda^*\log f(\zeta) = \log f(\lambda\cdot\zeta)$ for any $\lambda\in\mathbb{C}^*$, we have

$$\begin{aligned}
\mu_\lambda^*\Pi &= \sqrt{-1}\,\partial\bar{\partial}\log f(\lambda\cdot\zeta)\\
&= \sqrt{-1}\,\partial\bar{\partial}\log\left(|\lambda|^2\cdot f(\zeta)\right)\\
&= \sqrt{-1}\,\partial\bar{\partial}\left(\log|\lambda|^2 + \log f(\zeta)\right)\\
&= \sqrt{-1}\,\partial\bar{\partial}\log f(\zeta)\\
&= \Pi.
\end{aligned}$$

Therefore there exists a closed real $(1,1)$-form $\Pi_{\mathbb{P}^n}$ on the complex projective space $\mathbb{P}^n$ such that $\rho^*\Pi_{\mathbb{P}^n} = \Pi$, where $\rho : \widehat{\mathbb{C}}^{n+1}\to\mathbb{P}^n$ is the natural projection.

We shall provide a more geometrical description of the Kähler metric $\Pi_{\mathbb{P}^n}$ by the same way as in Example 1.9. The holomorphic tangent bundle $T_{\mathbb{P}^n}$ is locally spanned by the vector fields $\{d\rho\left(\partial/\partial\zeta^i\right)\}$ with the relation (1.16), and $\ker(d\rho)$ is spanned by the tautological section $\mathcal{E}$ on $\widehat{\mathbb{C}}^{n+1}$ defined in Example 1.9. Any strongly pseudo-convex Finsler metric f defines a Hermitian metric h on $\widehat{\mathbb{C}}^{n+1}$ by $h = \sum f_{i\bar{j}}d\zeta^i\otimes d\bar{\zeta}^j$, where $f_{i\bar{j}}$ are defined by (3.1). Then we also define a Hermitian metric δ on $\widehat{\mathbb{C}}^{n+1}$ by

$$\delta(Y,Z) = \frac{1}{f(\zeta)}h(Y,Z) \tag{3.5}$$

for all $Y,Z\in T_\zeta\widehat{\mathbb{C}}^{n+1}$. Here we notice that the norm $\|\mathcal{E}\|$ of $\mathcal{E}$ is defined by $\|\mathcal{E}\|^2 = f(\zeta)$. The homogeneity condition (f 2) implies that the Hermitian metric δ is invariant by the natural action μ of $\mathbb{C}^*$, i.e., $\mathcal{L}_{\mathcal{E}}\delta\equiv 0$. Therefore there exists a Hermitian metric $\mathfrak{g}$ on $\mathbb{P}^n$ such that $\delta = \rho^*\mathfrak{g}$.

Let $\mathcal{E}_\zeta^\perp \subset T_\zeta \widehat{\mathbb{C}}^{n+1}$ denote the δ-orthogonal complement to the line bundle spanned by $\mathcal{E}$, and $p : T_\zeta \widehat{\mathbb{C}}^{n+1} \to \mathcal{E}_\zeta^\perp$ the orthogonal projection, i.e., $p(Y) = Y - \delta(Y, \mathcal{E})\mathcal{E} := Y^\perp$. Then $\mathcal{E}_\zeta^\perp$ is naturally identified with $T_{[\zeta]}\mathbb{P}^n$, and a Hermitian metric $\mathfrak{g}$ on $T_{[\zeta]}\mathbb{P}$ is defined by

$$\begin{aligned}(\rho^*\mathfrak{g})(Y^\perp, Z^\perp) &= \delta\left(Y^\perp, Z^\perp\right)\\ &= \frac{1}{f(\zeta)}\left[h(Y,Z) - \frac{1}{f(\zeta)}h(Y,\mathcal{E})h(\mathcal{E},Z)\right]\\ &= \sum \frac{\partial^2 \log f}{\partial\zeta^i \partial\bar{\zeta}^j} Y^i \overline{Z^j}\end{aligned}$$

for all $Y, Z \in T_\zeta \widehat{\mathbb{C}}^{n+1}$. The fundamental form $\mathit{\Pi}_{\mathbb{P}^n}$ of this metric is given by

$$\mathit{\Pi}_{\mathbb{P}^n} = \sqrt{-1}\partial\bar{\partial}\log f. \tag{3.6}$$

For the non-homogeneous coordinate $(\zeta^1, \cdots, \zeta^n)$ on $U_j = \{[\zeta] \in \mathbb{P}^n \mid \zeta^i \neq 0\}$, the local function

$$g_j([\zeta]) := \log f(\zeta) - \log|\zeta^j|^2 = \log\left[\frac{1}{|\zeta^j|^2} f(\zeta)\right]$$

satisfies $\sqrt{-1}\partial\bar{\partial} g_i = \sqrt{-1}\partial\bar{\partial} g_j = \sqrt{-1}\partial\bar{\partial}\log f$ on $U_i \cap U_j$. Hence the real $(1,1)$-form

$$\mathit{\Pi}_{\mathbb{P}^n} = \sqrt{-1}\partial\bar{\partial} g_i \tag{3.7}$$

defines the Kähler metric on $\mathbb{P}^n$ with Kähler potentials $\{g_j\}$. Especially, if the function f is given by $f(\zeta) = \sum \zeta^i \bar{\zeta}^i$ (i.e., $f(\zeta)$ is the fundamental function of the flat metric $\sum d\zeta^i \otimes d\bar{\zeta}^i$ on $\mathbb{C}^{n+1}$), the induced Kähler metric on $\mathbb{P}^n$ is the Fubini-Study metric $\mathit{\Pi}_{FS}$.

We shall show that the converse of this fact is also true.

Proposition 3.3. *A Kähler metric $\mathit{\Pi}_{\mathbb{P}^n}$ on the projective space $\mathbb{P}^n$ defines a strongly pseudo-convex Finsler metric on $\mathbb{C}^{n+1}$ uniquely up to a positive constant multiple.*

For the proof of this proposition, we use the following example.

Example 3.2. Let $\mathcal{H}$ be the sheaf of germs of pluri-harmonic functions on M. By the $\partial\bar{\partial}$-Poincaré lemma, for any pluri-harmonic function K_U on $U \subset M$, there exists a holomorphic function φ_U satisfying $K_U = (\varphi_U + \overline{\varphi_U})/2$. Moreover, if $Re(\varphi_U) := (\varphi_U + \overline{\varphi_U})/2 = 0$, then $\varphi_U = \theta_U \times \sqrt{-1}$ for some real-valued constant function θ_U on U. Hence we have the following exact sequence of sheaves:

$$0 \longrightarrow \mathbb{R} \overset{\times\sqrt{-1}}{\longrightarrow} \mathcal{O} \overset{Re}{\longrightarrow} \mathcal{H} \longrightarrow 0, \tag{3.8}$$

Therefore, we have the long exact sequence of cohomology groups in the case of $M = \mathbb{P}^n$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \check{H}^0(\mathbb{P}^n, \mathbb{R}) & \longrightarrow & \check{H}^0(\mathbb{P}^n, \mathcal{O}) & \longrightarrow & \check{H}^0(\mathbb{P}^n, \mathcal{H}) & \longrightarrow & \check{H}^1(\mathbb{P}^n, \mathbb{R}) & \longrightarrow \\ & & \| & & \| & & & & \| & \\ & & \mathbb{R} & & \mathbb{C} & & & & 0 & \end{array}$$

This implies $\check{H}^0(\mathbb{P}^n, \mathcal{H}) = \mathbb{R}$. Hence any pluri-harmonic function on $\mathbb{P}^n$ is constant.

Proof of Proposition 3.3. We express $\mathit{\Pi}_{\mathbb{P}^n}$ locally as $\mathit{\Pi}_{\mathbb{P}^n} = \sqrt{-1}\partial\bar{\partial}\mathfrak{g}_j$ on U_j for a C^∞-function $\mathfrak{g}_j$ on U_j. Since $\mathfrak{g}_j - \mathfrak{g}_i$ is pluri-harmonic, Remark 1.2 implies that there exists a one-cocycle $K_{ij} \in Z^1(U_i \cap U_j, \mathcal{O}_{\mathbb{P}^n})$ satisfying $\mathfrak{g}_j - \mathfrak{g}_i = K_{ij} + \overline{K_{ij}}$ on $U_i \cap U_j \neq \phi$. Then $\{K_{ij}\}$ is a one-cocycle on $\mathbb{P}^n$, and since $H^1(\mathbb{P}^n, \mathcal{O}) = 0$, we may put $K_{ij} = (K_j - \log\zeta^j) - (K_i - \log\zeta^i)$ for a zero-cochain $\{K_j\}$ on $\mathbb{P}^n$. Hence we have

$$\mathfrak{g}_j - (K_j + \overline{K_j}) + \log|\zeta^j|^2 = \mathfrak{g}_i - (K_i + \overline{K_i}) + \log|\zeta^i|^2$$

If we put

$$f_j([\zeta]) = \exp\{\mathfrak{g}_j - (K_j + \overline{K_j})\}$$

on U_j, we have $|\zeta^j|^2 f_j([\zeta]) = |\zeta^i|^2 f_i([\zeta])$. Thus we have a function $f(\zeta) = |\zeta^j|^2 f_j([\zeta])$ on $\mathbb{C}^{r+1}$. It is clear that f satisfies the conditions (f 1), (f 2) and (f 3). Moreover, because of $\sqrt{-1}\partial\bar{\partial}\log f = \sqrt{-1}\partial\bar{\partial}\log f_j = \sqrt{-1}\partial\bar{\partial}\mathfrak{g}_j > 0$ and

$$\sqrt{-1}\partial\bar{\partial}f = \sqrt{-1}f\left(\partial\bar{\partial}\log f + \partial\log f \wedge \bar{\partial}\log f\right),$$

the function f defines a strongly pseudo-convex Finsler metric on $\mathbb{C}^{n+1}$.

We suppose that we get another Finsler metric $\tilde{f}$ from another Kähler potential $\{\tilde{\mathfrak{g}}_j\}$. Then, since $\sqrt{-1}\partial\bar{\partial}\tilde{\mathfrak{g}}_j = \sqrt{-1}\partial\bar{\partial}\mathfrak{g}_j$, the function $\log\tilde{f} - \log f$ is pluri-harmonic function on $\mathbb{P}^n$. Hence it is a constant c. Consequently we have $\tilde{f} = e^c f$.

Q.E.D.

3.2 Complex Finsler bundles

3.2.1 Complex Finsler metric on vector bundles

Let E be a holomorphic vector bundle over a complex manifold. If $\mathrm{rank}(E) = 1$, then any Finsler metric on E is reducible to a Hermitian metric, and so we assume $\mathrm{rank}(E) \geq 2$ in the sequel.

Definition 3.3. A *complex Finsler metric* on E is a smooth assignment to each fiber $E_z = \pi^{-1}(z)$ of a Minkowski norm $\|\cdot\|_z$. We call $(E, \|\cdot\|)$ a *complex Finsler vector bundle.*

We shall fix a local holomorphic frame fields $e_U = (e_1, \cdots, e_n)$ of E on $U \subset M$ and the local coordinate system $(z^1, \cdots, z^n, \zeta^i, \cdots, \zeta^r) \in U \times \mathbb{C}^r$ on $\pi^{-1}(U)$ defined in §2.4. We define a function $F : E \to \mathbb{R}$ by $F(z, \zeta) = \|v\|^2$. Then F satisfies the following conditions:

(F1) $F(z, \zeta) \geq 0$ and $F(z, \zeta) = 0$ if and only if $v = (z, \zeta) = 0$,

(F2) $F(z, \lambda\zeta) = |\lambda|^2 F(z, \zeta)$ for all $\lambda \in \mathbb{C}$,

(F3) F is smooth on $E^0 := E \setminus \{0\}$.

Conversely, if a function $F : E \to \mathbb{R}$ satisfying these condition is given on E, then it defines a unique complex Minkowski norm $\|\cdot\|$ on E. Thus, in the sequel, we always identify a complex Minkowski norm $\|\cdot\|$ with F and we shall call F a *complex Finsler metric* in E.

Example 3.3. Let h be an arbitrary Hermitian metric on E. With respect to an open cover $\{\mathcal{U}, (e_U)\}$, we put $h_{i\bar{j}} = h(e_i, e_j)$. The function $F_h : E \to \mathbb{R}$ defined in the proof of Theorem 2.4 is a complex Finsler metric in E. We remark here that this function F_h is smooth on E. Conversely, it is easily shown that if a complex Finsler metric F is smooth on E, then F coincides with the function F_h defined by a Hermitian metric h. □

Proposition 3.4. ([Ko1]) *Any complex Finsler metric on E is identified with a Hermitian metric on the tautological line bundle $\mathbb{L}(E)$.*

Proof. We shall fix an open covering $\mathcal{U}$ of M, and we define an open covering $\{U_j\}$ of $\mathbb{P}(E)$ by $U_j = \phi^{-1}(U) \cap \{\zeta^j \neq 0\}$ for each $U \in \mathcal{U}$. Then the transition functions $\{l_{(ij)}\}$ of $\mathbb{L}(E)$ relative to $\{U_j\}$ are given by (cf. (1.20)):

$$l_{(ij)}([v]) = \frac{\zeta^i}{\zeta^j}.$$

For any complex Finsler metric F on E, we define a positive function F_j on U_j by

$$F_j([v]) = \frac{1}{|\zeta^j|^2} F(z, \zeta).$$

Then, it is easily verified that

$$F_j([v]) = |l_{(ij)}|^2 F_i([v]). \tag{3.9}$$

Thus the family $\{F_j\}$ defines a metric on $\mathbb{L}(E)$.

Conversely, since any Hermitian metric on $\mathbb{L}(E)$ is defined by the family $\{F_j\}$ of positive functions satisfying (3.9), we can define a complex Finsler metric F on E by $F(z,\zeta) = |\zeta^j|^2 F_j([v])$ (see Remark 2.4).

Q.E.D.

Definition 3.4. A complex Finsler metric F is called a *Rizza metric* if F is strongly pseudo-convex on each fiber E_z.

Example 3.4. Let $\mathcal{D}$ be a strongly convex domain in $\mathbb{C}^{n+1}$ with smooth boundary. The *Kobayashi metric* $F_{\mathcal{D}}$ is defined by

$$F_{\mathcal{D}}(z,\zeta) := \inf_{\varphi}\left\{\frac{1}{R}\right\},$$

where the infimum is taken all holomorphic maps $\varphi : \Delta(R) \to \mathcal{D}$ satisfying $\varphi(0) = z$ and $d\varphi(d/dt)_0 = \zeta$ for all $(z,\zeta) \in T_{\mathcal{D}}$. By the early work due to Lempert[Le], the function $F = F_{\mathcal{D}}^2$ defines a Rizza metric on $T_{\mathcal{D}}$. □

For every point $z \in M$, we define a function $F_z : E_z \to \mathbb{R}$ by $F_z(\zeta) = F(z,\zeta)$. We set

$$g_{i\bar{j}}(z,\zeta) = \frac{\partial^2 F_z}{\partial\zeta^i \partial\bar{\zeta}^j}. \tag{3.10}$$

Since F_z is a strongly pseudo-convex on $E_z \cong \mathbb{C}^r$, the Hermitian matrix $(g_{i\bar{j}})$ is positive-definite. Then each fiber $E_z \cong \mathbb{C}^r$ admits a Kähler metric g_z defined by

$$g_z(Y,Z) = \sum g_{i\bar{j}}(z,\zeta) Y^i \overline{Z^j} \tag{3.11}$$

for all $Y, Z \in T_\zeta E_z$. Hence $E \to M$ is a smooth family of Kähler manifolds $\{E_z, \sqrt{-1}\partial\bar{\partial}F_z\}$ parameterized by $z \in M$.

3.2.2 Construction of Rizza metrics

We take an open covering $\{U_{(j)}\}$ of $\mathbb{P}(E)$ defined in the proof of Proposition 3.4. The collection $\{U_{z,(j)} := \mathbb{P}_z \cap U_{(j)}\}$ defines an open covering of $\mathbb{P}_z$. If a Rizza metric F is given on E, then F gives a Kähler metric on the projective space $\mathbb{P}(E_z) := \mathbb{P}_z$. Indeed, the local function

$$\mathfrak{G}_{(j)}([v]) = \log\left[\frac{1}{|\zeta^j|^2} F_z(\zeta)\right]$$

defined on $U_{z,(j)}$ gives a Kähler form $\mathit{\Pi}_z$ on $\mathbb{P}_z$ by $\mathit{\Pi}_z = \sqrt{-1}\partial\bar{\partial}\mathfrak{G}_{(j)}$, and the real $(1,1)$-form

$$\mathit{\Pi}_{\mathbb{P}(E)} = \sqrt{-1}\partial\bar{\partial}\log F \tag{3.12}$$

is a *pseudo Kähler metric* on $\mathbb{P}(E)$ in the sense that it is positive definite only in vertical directions.

Conversely we suppose that $\mathbb{P}(E) \to M$ is a smooth family of Kähler manifolds $\{\mathbb{P}_z, \mathit{\Pi}_{\mathbb{P}_z}\}_{z\in M}$. Denoted by $\mathit{\Pi}_{\mathbb{P}_z} = \sqrt{-1}\partial\bar{\partial}\mathfrak{G}_{(j)}$ for a smooth function $\mathfrak{G}_{(j)}$ in $U_{z,(j)}$,

$$\mathfrak{G}_{(j)} - \mathfrak{G}_{(i)} = k_{(ij)} + \overline{k_{(ij)}} \tag{3.13}$$

for some $k_{(ij)} \in Z^1(U_{z,(i)} \cap U_{z,(j)}, \mathcal{O}_{\mathbb{P}_z})$ since $\mathfrak{G}_{(j)} - \mathfrak{G}_{(i)}$ is pluri-harmonic on $U_{z,(i)} \cap U_{z,(j)} \neq \emptyset$ (cf. Remark 1.2). Then $H^1(\mathbb{P}_z, \mathcal{O}_{\mathbb{P}_z}) = 0$ assures that we can take $k_{(j)} \in C^0(U_{z,(j)}, \mathcal{O}_{\mathbb{P}_z})$ satisfying

$$k_{(ij)} = (k_{(j)} - \log\zeta^j) - (k_{(i)} - \log\zeta^i),$$

where $\{k_{(i)}\}$ are smooth in $z \in U$. Then (3.13) implies

$$\mathfrak{G}_{(j)} - (k_{(j)} + \overline{k_{(j)}}) + \log|\zeta^j|^2 = \mathfrak{G}_{(i)} - (k_{(i)} + \overline{k_{(i)}}) + \log|\zeta^i|^2.$$

Putting $F_{(j)}(z,[\zeta]) = \exp\left[\mathfrak{G}_{(j)} - (k_{(j)} + \overline{k_{(j)}})\right]$, we have

$$|\zeta^j|^2 F_{(j)}(z,[\zeta]) = |\zeta^i|^2 F_{(i)}(z,[\zeta])$$

on $U_{z,(i)} \cap U_{z,(j)}$. Since $F_{(i)}$ depends on $z \in M$ smoothly, if we define a smooth function $F : E^0 \to \mathbb{R}$ by

$$F(z,\zeta) = |\zeta^j|^2 F_{(j)}(z,[\zeta]), \tag{3.14}$$

then F satisfies (F3) for any $(z,\zeta) \in E^0$ and $\lambda \in \mathbb{C}^*$. Hence we can extend F continuously on the whole of E by setting $F(z,0) = 0$, i.e., F defined by (3.14) is complex Finsler metric on E.

Denoted by $F_z := F|_{E_z^0}$ the restriction of F to the fiber $E_z^0 := E_z\backslash\{0\}$,

$$\sqrt{-1}\partial\bar{\partial}\log F_z = \sqrt{-1}\partial\bar{\partial}\left(\log F_{(j)}|_{E_z^0}\right) = \sqrt{-1}\partial\bar{\partial}\mathfrak{G}_{(j)} \tag{3.15}$$

from the construction of F. Further

$$\partial\bar{\partial}\log F_z = \frac{1}{F_z}\left(\partial\bar{\partial}F_z - \frac{1}{F_z}\partial F_z \wedge \bar{\partial}F_z\right) \tag{3.16}$$

show that the matrix $(g_{i\bar{j}})$ defined by (3.10) is positive-definite. Thus any pseudo Kähler metric $\Pi_{\mathbb{P}(E)}$ on $\mathbb{P}(E)$ determines a Rizza metric F on E.

Proposition 3.5. *Let E be a holomorphic vector bundle over a compact complex manifold M. If $\mathbb{P}(E) \to M$ is a family of Kähler manifolds parametrized by M, then E admits a Rizza metric.*

The Kähler potentials $\{\mathfrak{G}_{(j)}\}$ of $\Pi_{\mathbb{P}_z}$ are not uniquely determined. Let $\widetilde{F}$ be a Finsler metric obtained from another Kähler potentials $\{\widetilde{\mathfrak{G}}_{(j)}\}$. It follows from $\sqrt{-1}\partial\bar{\partial}\mathfrak{G}_{(j)} = \sqrt{-1}\partial\bar{\partial}\widetilde{\mathfrak{G}}_{(j)}$ that $\log \widetilde{F}_z - \log F_z$ is pluri-harmonic on $\mathbb{P}_z$, and thus it is a constant σ_z in $\mathbb{P}_z$. Consequently we have $\widetilde{F}_z = e^{\sigma_z} F_z$. The corresponding Finsler metrics F and $\widetilde{F}$ satisfy the relation $\widetilde{F} = e^{\sigma(z)} F$ for a smooth function $\sigma(z)$ on M. Hence the Rizza metric obtained from $\Pi_{\mathbb{P}(E)}$ is unique up to the conformal factor $e^{\sigma(z)}$ on M.

Corollary 3.1. *Any pseudo Kähler metric $\Pi_{\mathbb{P}(E)}$ on $\mathbb{P}(E)$ determines the conformal class of a Rizza metric F on E.*

3.2.3 Partial connection D on (V, g)

A horizontal sub-bundle H of the holomorphic tangent bundle T_E over E is called a *complex non-linear connection* if

(N1) H is invariant by the natural action μ of $\mathbb{C}^*$ on E,

(N2) H is smooth on E^0 and continuous on E.

If H is smooth on E, then H is an Ehresmann connection on E in the sense of Definition 2.11.

Let F be a Rizza metric on E derived from the given pseudo Kähler metric $\Pi_{\mathbb{P}(E)}$ on $\mathbb{P}(E)$, and g the Hermitian matrix defined by (3.10). The tangent space $T_\zeta E_z$ is naturally identified with the fiber $V_{(z,\zeta)}$ of the vertical sub-bundle V over $(z, \zeta) \in E$. Hence g defines a Hermitian metric on V by

$$g\left(\frac{\partial}{\partial \zeta^i}, \frac{\partial}{\partial \zeta^j}\right) := g_{i\bar{j}}(z, \zeta). \tag{3.17}$$

If a complex non-linear connection H is given on E, then we can define a *partial connection* $D : \mathcal{A}(V) \to \mathcal{A}(H^* \otimes V)$ of $(1,0)$-type on V. Denoted by

$$P = \sum \frac{\partial}{\partial \zeta^i} \otimes \left(d\zeta^i + \sum N^i_\alpha dz^\alpha\right)$$

the projection $P : T_E \to V$ with $\ker(P) = H$, a *partial connection* D on V is defined by

$$D_{X^H} Z = P\left(\mathcal{L}_{X^H} Z\right), \tag{3.18}$$

where $\mathcal{L}_{X^H}$ denotes the Lie derivative by X^H. We shall determine a non-linear connection H so that D satisfies

$$D_{X^H} g = 0 \tag{3.19}$$

for all $X \in \mathcal{A}(T_M)$. If we write the horizontal lifts $X_1, \cdots, X_m$ with respect to H of local frame fields $\dfrac{\partial}{\partial z^1}, \cdots, \dfrac{\partial}{\partial z^m}$ as

$$X_\alpha = \frac{\partial}{\partial z^\alpha} - \sum N_\alpha^l \frac{\partial}{\partial \zeta^l},$$

then (3.19) can be written as

$$\frac{\partial}{\partial \zeta^i}\left(\frac{\partial^2 F}{\partial z^\alpha \partial \bar{\zeta}^j} - \sum N_\alpha^l g_{l\bar{j}}\right) = 0.$$

Hence we define H by

$$N_\alpha^l = \sum g^{\bar{j}l} \frac{\partial^2 F}{\partial z^\alpha \partial \bar{\zeta}^j} = \sum g^{\bar{j}l} \frac{\partial g_{k\bar{l}}}{\partial z^\alpha} \zeta^k, \tag{3.20}$$

where $g^{\bar{j}l}$ denote the components of the inverse of $\left(g_{i\bar{j}}\right)$.

Proposition 3.6. ([Ai1], [Ai6]) *Let F be a Rizza metric on a holomorphic vector bundle E. Then there exists a complex non-linear connection H on E so that the partial connection D associated with H satisfies* (3.19).

From the homogeneity condition (F3), the local functions N_α^l satisfy

$$N_\alpha^l(z, \lambda \cdot \zeta) = \lambda N_\alpha^l(z, \zeta) \tag{3.21}$$

for any $\lambda \in \mathbb{C}^*$. This homogeneity is equivalent to the assumption (N1), and thus we obtain

Lemma 3.1. *The local basis $\{X_1, \cdots, X_m\}$ of H is invariant by the action μ of $\mathbb{C}^*$.*

PROOF. Indeed we obtain

$$\begin{aligned}
d\mu_\lambda(X_\alpha) &= d\mu_\lambda\left(\left(\frac{\partial}{\partial z^\alpha}\right)_{(z,\zeta)} - \sum N^l_\alpha(z,\zeta)\left(\frac{\partial}{\partial \zeta^l}\right)_{(z,\zeta)}\right) \\
&= d\mu_\lambda\left(\frac{\partial}{\partial z^\alpha}\right)_{(z,\zeta)} - \sum N^l_\alpha(z,\zeta)d\mu_\lambda\left(\frac{\partial}{\partial \zeta^l}\right)_{(z,\zeta)} \\
&= \left(\frac{\partial}{\partial z^\alpha}\right)_{(z,\lambda\cdot\zeta)} - \sum N^l_\alpha(z,\zeta)\lambda\left(\frac{\partial}{\partial \zeta^l}\right)_{(z,\lambda\cdot\zeta)} \\
&= \left(\frac{\partial}{\partial z^\alpha}\right)_{(z,\lambda\cdot\zeta)} - \sum N^l_\alpha(z,\lambda\cdot\zeta)\left(\frac{\partial}{\partial \zeta^l}\right)_{(z,\lambda\cdot\zeta)} \\
&= X_\alpha(z,\lambda\cdot\zeta).
\end{aligned}$$

Q.E.D.

The partial connection D defined by (3.18) is of $(1,0)$-type:

$$D_\alpha\frac{\partial}{\partial\zeta^i} := D_{X_\alpha}\frac{\partial}{\partial\zeta^i} = \sum \varGamma^l_{i\alpha}\frac{\partial}{\partial\zeta^l}, \tag{3.22}$$

where $\varGamma^l_{i\alpha} = \partial_i N^l_\alpha$. From (3.19) the coefficients $\varGamma^l_{i\alpha}$ also expressed as

$$\varGamma^l_{i\alpha} = \sum g^{l\bar{m}}X_\alpha(g_{i\bar{m}}) = \sum g^{l\bar{m}}\left(\frac{\partial g_{i\bar{m}}}{\partial z^\alpha} - \sum N^k_\alpha\frac{\partial g_{i\bar{m}}}{\partial\zeta^k}\right). \tag{3.23}$$

The action μ of $\mathbb{C}^*$ on E induces the tautological section $\mathcal{E}$ of V defined by (2.54). Then we obtain

Proposition 3.7. *The partial connection D on (V,g) satisfies*

$$D_\alpha\mathcal{E} = 0 \tag{3.24}$$

and the Rizza metric F is constant along H, i.e.,

$$X_\alpha F = 0. \tag{3.25}$$

3.3 Negative vector bundles and Rizza-negativity

A characterization of negative holomorphic vector bundles is given by Kobayashi[Ko1]. In this section, we shall discuss the negativity (or ampleness) of holomorphic vector bundles by using complex Finsler geometry, and we present a proof of Kobayashi's theorem.

A holomorphic vector bundle E over a compact complex manifold M is negative if its tautological line bundle $\mathbb{L}(E)$ is negative. From Remark 2.1 and Proposition 3.4, the first Chern class $c_1(\mathbb{L}(E))$ is given by $\left[\frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial\log F\right]$ for a complex Finsler metric F on E. Thus $\mathbb{L}(E)$ is negative if and only if

$$\sqrt{-1}\bar{\partial}\partial\log F < 0 \tag{3.26}$$

or equivalently $\Pi_{\mathbb{P}(E)} = \sqrt{-1}\partial\bar{\partial}\log F$ defines a Kähler metric on the total space $\mathbb{P}(E)$. Kobayashi's characterization is obtained by analyzing the positivity of the form $\Pi_{\mathbb{P}(E)}$.

The curvature form Ω^i_j of D is defined by

$$D^2\frac{\partial}{\partial\zeta^j} = \sum\frac{\partial}{\partial\zeta^i}\otimes\Omega^i_j,$$

and we write $\Omega^i_j = \sum K^i_{j\alpha\bar{\beta}}dz^\alpha\wedge d\bar{z}^\beta$ so that

$$K_{i\bar{j}\alpha\bar{\beta}} := \sum g_{l\bar{j}}K^l_{i\alpha\bar{\beta}} = -X_{\bar{\beta}}X_\alpha g_{i\bar{j}} + \sum g_{k\bar{l}}\Gamma^k_{i\alpha}\overline{\Gamma^l_{j\beta}}. \tag{3.27}$$

3.3.1 A curvature formula and Kobayashi's theorem

We will use the following proposition for the proof of Kobayashi's theorem,

Proposition 3.8. *Suppose that a holomorphic vector bundle E admits a Rizza metric F. Then the curvature $\bar{\partial}\partial\log F$ of the Hermitian metric on $\mathbb{L}(E)$ is given by*

$$\bar{\partial}\partial\log F = \frac{1}{F}\sum\Psi_{\alpha\bar{\beta}}dz^\alpha\wedge d\bar{z}^\beta - \sum\partial_i\partial_{\bar{j}}(\log F)P^i\wedge\overline{P^j}, \tag{3.28}$$

where $\partial_i := \partial/\partial\zeta^i$, $\partial_{\bar{j}} := \partial/\partial\bar{\zeta}^j$, and we put

$$\Psi_{\alpha\bar{\beta}} := \sum K_{i\bar{j}\alpha\bar{\beta}}\zeta^i\overline{\zeta^j}$$

with $P^i := d\zeta^i + \sum N^i_\alpha dz^\alpha$.

For the proof of the formula (3.28), we first show some useful formulae.

Lemma 3.2. *Let N^i_α be the coefficients of the non-linear connection H defined by* (3.20)*. Then*

$$\sum F_i A^i_{\alpha\bar{j}} = 0, \qquad \sum F_{\bar{i}}\overline{A^i_{\alpha\bar{j}}} = 0, \tag{3.29}$$

where $A^i_{\alpha\bar{j}} := \partial_{\bar{j}}N^i_\alpha$, $F_i := \partial_i F$ and $F_{\bar{i}} := \partial_{\bar{i}}F$.

PROOF. It is sufficient to prove the first identity. From (3.20), we have

$$\begin{aligned}
\sum F_i A^i_{\alpha\bar{k}} &= \sum F_i \frac{\partial N^i_\alpha}{\partial \bar{\zeta}^k} \\
&= \sum g_{i\bar{m}} \overline{\zeta^m} \frac{\partial}{\partial \bar{\zeta}^k} \left(\sum g^{i\bar{l}} \frac{\partial^2 F}{\partial z^\alpha \partial \bar{\zeta}^l} \right) \\
&= \sum g_{i\bar{m}} \overline{\zeta^m} \left(\sum \frac{\partial g^{i\bar{l}}}{\partial \bar{\zeta}^k} \frac{\partial^2 F}{\partial z^\alpha \partial \bar{\zeta}^l} + \sum g^{i\bar{l}} \frac{\partial^3 F}{\partial z^\alpha \partial \bar{\zeta}^k \partial \bar{\zeta}^l} \right) \\
&= \sum \overline{\zeta^m} \left(\sum g_{i\bar{m}} \frac{\partial g^{i\bar{l}}}{\partial \bar{\zeta}^k} \frac{\partial^2 F}{\partial z^\alpha \partial \bar{\zeta}^l} + \frac{\partial^3 F}{\partial z^\alpha \partial \bar{\zeta}^k \partial \bar{\zeta}^m} \right) \\
&= \sum \overline{\zeta^m} \left(-\sum \frac{\partial g_{i\bar{m}}}{\partial \bar{\zeta}^k} g^{i\bar{l}} \frac{\partial^2 F}{\partial z^\alpha \partial \bar{\zeta}^l} + \frac{\partial^3 F}{\partial z^\alpha \partial \bar{\zeta}^k \partial \bar{\zeta}^m} \right) \\
&= \sum \overline{\zeta^m} \left(-\sum \frac{\partial g_{\bar{k}\bar{m}}}{\partial \zeta^i} g^{i\bar{l}} \frac{\partial^2 F}{\partial z^\alpha \partial \bar{\zeta}^l} + \frac{\partial g_{\bar{k}\bar{m}}}{\partial z^\alpha} \right) \\
&= 0,
\end{aligned}$$

since $\sum \overline{\zeta^m} g_{\bar{k}\bar{m}} = 0$ (cf. [Ko3]).

Q.E.D.

Lemma 3.3.

$$X_\alpha F_{\bar{j}} = 0, \qquad X_{\bar{\alpha}} F_j = 0. \tag{3.30}$$

PROOF. Indeed, from (3.25) and (3.29), we have

$$\begin{aligned}
X_\alpha F_{\bar{j}} &= \frac{\partial}{\partial z^\alpha} \left(\frac{\partial F}{\partial \bar{\zeta}^j} \right) - \sum N^l_\alpha \frac{\partial}{\partial \zeta^l} \left(\frac{\partial F}{\partial \bar{\zeta}^j} \right) \\
&= \frac{\partial}{\partial \bar{\zeta}^j} \left(\frac{\partial F}{\partial z^\alpha} \right) - \sum N^l_\alpha \frac{\partial}{\partial \bar{\zeta}^j} \left(\frac{\partial F}{\partial \zeta^l} \right) \\
&= \frac{\partial}{\partial \bar{\zeta}^j} \left(\frac{\partial F}{\partial z^\alpha} - \sum N^l_\alpha \frac{\partial F}{\partial \zeta^l} \right) + \sum F_l A^l_{\alpha\bar{j}} \\
&= 0,
\end{aligned}$$

and $X_{\bar{\alpha}} F_j = \overline{X_\alpha F_{\bar{j}}} = 0$.

Q.E.D.

PROOF of Proposition (3.28). From (3.25) we have

$$\frac{\partial F}{\partial z^\alpha} = \sum N^l_\alpha \frac{\partial F}{\partial \zeta^l}, \qquad \frac{\partial F}{\partial \bar{z}^\alpha} = \sum \overline{N^l_\alpha} \frac{\partial F}{\partial \bar{\zeta}^l}. \tag{3.31}$$

Then (3.30) implies

$$\bar{\partial}\partial \log F = -\frac{1}{F^2}\bar{\partial}^V F \wedge \partial^V F + \frac{1}{F}\bar{\partial}(\partial^V F) = \frac{1}{F^2}\sum F_i F_{\bar{j}} P^i \wedge \overline{P^j} + \frac{1}{F}\bar{\partial}\left(\sum F_i P^i\right)$$

and

$$\begin{aligned}
&\bar{\partial}\left(\sum F_i P^i\right) \\
&= \sum \bar{\partial} F_i \wedge P^i + \sum F_i \bar{\partial} P^i \\
&= \sum \left(\sum X_{\bar{\alpha}} F_i \overline{dz^\alpha} + \sum F_{i\bar{j}} \overline{P^j}\right) \wedge P^i + \sum F_i \left(\sum X_{\bar{\beta}} N^i_\alpha \overline{dz^\beta} \wedge dz^\alpha + \sum A^i_{j\alpha} \overline{P^j} \wedge dz^\alpha\right) \\
&= \sum F_i X_{\bar{\beta}} N^i_\alpha \, \overline{dz^\beta} \wedge dz^\alpha + \sum g_{i\bar{j}} \overline{P^j} \wedge P^i,
\end{aligned}$$

where ∂^V denotes the partial derivative in the vertical direction. Therefore we have

$$\begin{aligned}
\bar{\partial}\partial \log F &= -\sum \left(\frac{g_{i\bar{j}}}{F} - \frac{F_i F_{\bar{j}}}{F^2}\right) P^i \wedge \overline{P^j} + \frac{1}{F}\sum F_i X_{\bar{\beta}} N^i_\alpha \, \overline{dz^\beta} \wedge dz^\alpha \\
&= -\sum (\log F)_{i\bar{j}} P^i \wedge \overline{P^j} + \frac{1}{F}\sum F_i X_{\bar{\beta}} N^i_\alpha \, \overline{dz^\beta} \wedge dz^\alpha.
\end{aligned}$$

Here we note that

$$\begin{aligned}
\sum F_i X_{\bar{\beta}} N^i_\alpha &= \sum X_{\bar{\beta}}(F_i N^i_\alpha) - \sum (X_{\bar{\beta}} F_i) N^i_\alpha \\
&= \sum X_{\bar{\beta}}\left(\frac{\partial F}{\partial z^\alpha}\right) \\
&= \frac{\partial^2 F}{\partial z^\alpha \partial \bar{z}^\beta} - \sum g^{\bar{m}l} \frac{\partial F_l}{\partial \bar{z}^\beta} \frac{\partial F_{\bar{m}}}{\partial z^\alpha} \\
&= \sum \left(X_{\bar{\beta}} X_\alpha g_{i\bar{j}} - \sum g^{\bar{m}l} X_\alpha g_{i\bar{m}} X_{\bar{\beta}} g_{l\bar{j}}\right) \zeta^i \overline{\zeta^j} \\
&= \sum \left(X_{\bar{\beta}} X_\alpha g_{i\bar{j}} - \sum g_{k\bar{l}} \Gamma^k_{i\alpha} \overline{\Gamma^l_{j\beta}}\right) \zeta^i \overline{\zeta^j} \\
&= -\sum K_{\alpha\bar{\beta}i\bar{j}} \zeta^i \overline{\zeta^j}
\end{aligned}$$

Therefore

$$\begin{aligned}
\bar{\partial}\partial \log F &= -\sum (\log F)_{i\bar{j}} P^i \wedge \overline{P^j} + \frac{1}{F}\left(-\sum K_{\alpha\bar{\beta}i\bar{j}} \zeta^i \overline{\zeta^j}\right) \overline{dz^\beta} \wedge dz^\alpha \\
&= \frac{1}{F}\sum \left(\sum K_{\alpha\bar{\beta}i\bar{j}} \zeta^i \overline{\zeta^j}\right) dz^\alpha \wedge \overline{dz^\beta} - \sum (\log F)_{i\bar{j}} P^i \wedge \overline{P^j}.
\end{aligned}$$

i.e., we complete the proof of the curvature formula (3.28).

Q.E.D.

As shown in Theorem 2.4, if E is Griffiths-negative, then E is negative. The converse is not true in general.

Theorem 3.1. ([Ko1]) *A holomorphic vector bundle E over a compact complex manifold M is negative if and only if E admits a Rizza metric F with negative Ψ.*

Proof. Since the second term of (3.28) is negative definite in the vertical direction in $\mathbb{P}(E)$, (3.26) is satisfied if and only if Ψ is negative.

Q.E.D.

Definition 3.5. ([Ha-Ai]) A holomorphic vector bundle E is said to be *Rizza-negative* if E admits a Rizza metric F of negative curvature, i.e.,

$$K(Z \otimes X^H) := \sum K_{i\bar{j}\alpha\bar{\beta}} Z^i X^\alpha \overline{Z^j X^\beta} < 0$$

at any point $(z, \zeta) \in E^0$ for any non-zero $Z \in V_{(z,\zeta)}$ and $X^H \in H_{(z,\zeta)}$.

From the curvature formula (3.28) and Theorem 3.1, we obtain

Theorem 3.2. ([Ha-Ai]) *If E is Rizza-negative, then E is negative.*

Proof. Since F is a Rizza structure, the identity $g(\mathcal{E}, \mathcal{E}) = F(z, \zeta)$ and Schwarz inequality assure the negativity of the second term of the expression of $\bar{\partial}\partial \log F$ in each $T_{[\zeta]}\mathbb{P}_z$. Further, if (E, F) is Rizza-negative, then $\sum \Psi_{\alpha\bar{\beta}} X^\alpha \overline{X}^\beta = K(\mathcal{E} \otimes X^H)$ implies (3.26), i.e., E is negative.

Q.E.D.

As a special case, we consider the case where the Rizza metric F is derived from a Hermitian metric h, i.e., $F = F_h(z, \zeta) = \sum h_{i\bar{j}}(z)\zeta^i \overline{\zeta^j}$ for the components $h_{i\bar{j}}$ of h. Then $\Psi_{\alpha\bar{\beta}}$ is given by $\Psi_{\alpha\bar{\beta}} = \sum R_{i\bar{j}\alpha\bar{\beta}}(z)\zeta^i\overline{\zeta^j}$ for the curvature tensor $R_{i\bar{j}\alpha\bar{\beta}}$ of (E, h). Therefore, if (E, h) is Griffiths-negative, then $\sqrt{-1}\bar{\partial}\partial \log F_h < 0$ which gives another proof of Theorem 2.4.

We shall state another characterization of negative vector bundles due to [Ca-Wo]. We denote by $\odot^m E$ the symmetric tensor product of E. Then we have Grothendieck's identification:

$$H^p\left(\mathbb{P}(E), \mathbb{H}\right) \cong H^p(M, \odot^m E^*) \tag{3.32}$$

for all $p \geq 0$ and $m \geq 0$. Let $\gamma : H^p\left(\mathbb{P}(E), \mathbb{H}\right) \longrightarrow H^p(M, \odot^m E^*)$ be the isomorphism. The bases $\{\sigma_0, \cdots, \sigma_N\}$ of $H^0\left(\mathbb{P}(E), \mathbb{H}\right)$ is identified with a bases $\{\omega_0, \cdots, \omega_N\}$ of

$H^0(M, \odot^m E^*)$ by setting $\gamma^*\omega_b = \sigma_b$. Then a Hermitian metric $h^{\otimes m}$ on $\odot^m E$ is defined by

$$h^{\otimes m}(A, B) = \sum_{b=0}^{N} \omega_b(A)\bar{\omega}_b(\bar{B}) \tag{3.33}$$

for all $A, B \in \mathcal{A}(\odot^m E)$. Then it induces a Finsler metric F on E by setting

$$F(v) = \left[h^{\otimes m}(\otimes^m v, \otimes^m v)\right]^{1/2m} = \sqrt[2m]{h^{\otimes m}(\otimes^m v, \otimes^m v)} \tag{3.34}$$

In [Ca-Wo], it is proved that the metric $h^{\otimes m}$ on $\odot^m E$ has negative curvature, and thus the Finsler metric F defined by (3.34) has negative curvature Ψ. From the discussion above, we have

Theorem 3.3. ([Ca-Wo]) *Let $\pi : E \to M$ be a holomorphic vector bundle of* $\mathrm{rank}(E) \geq 2$ *over a compact complex manifold M. The following statements are equivalent.*

(1) *E^* is ample,*

(2) *E admits a Rizza metric with negative Ψ,*

(3) *there exists a sufficiently large $m \in \mathbb{Z}$ and a Hermitian metric $h^{\otimes m}$ on the symmetric product $\odot^m E$ with negative curvature, namely, $\odot^m E$ is Griffiths-negative.*

3.3.2 A construction of Rizza metrics on negative vector bundles

For a negative vector bundle E over M, we shall construct a Rizza metric F with negative Ψ (cf. [Ai6] and [Ha-Ai]). From Definition 2.9 the line bundle $\mathbb{L}(E)$ is negative, and so $\mathbb{L}(E)^*$ is ample. Hence there exists a sufficiently large $m \in \mathbb{Z}$ such that $L := \mathbb{L}(E)^{*\otimes m}$ is very ample. Then Theorem 2.3 shows that we can take a bases $\{\sigma_0, \cdots, \sigma_N\}$ of $H^0(\mathbb{P}(E), L)$ such that

$$\mathbb{P}(E) \ni [v] \longrightarrow (\sigma_0([v]) : \cdots : \sigma_N([v])) \in \mathbb{P}^N$$

defines a holomorphic embedding $\varphi_f : \mathbb{P}(E) \to \mathbb{P}^N$. Then Lemma 2.2 shows that $L \cong \varphi_f^*\mathbb{H}$ for the hyperplane bundle $\mathbb{H}$ over $\mathbb{P}^N$.

Since $\mathbb{P}^N$ admits the Fubini-Study metric, the first Chern form of $\mathbb{H}$ is given by

$$\frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial \log\left(\frac{\sum_{b=0}^{N}|T^b|^2}{|T^a|^2}\right)^{-1}$$

on $V_a = \left\{[T^1 : \cdots : T^N] \in \mathbb{P}^N \,\middle|\, T^a \neq 0\right\}$. On $U_j := p^{-1}(U) \cap \{\zeta^j \neq 0\} \subset \mathbb{P}(E)$, we put $\sigma_b = \{\sigma_{j,b}\}$, $(b = 0 \cdots, N)$, where $\sigma_{j,b}$ are holomorphic functions on U_j. Then a canonical

Hermitian metric $h_{\varphi_f^*\mathbb{H}}$ of $\varphi_f^*\mathbb{H}$ is defined by

$$h_{\varphi_f^*\mathbb{H},a}([v]) = \left(\frac{\sum_{b=0}^N |\sigma_{j,b}([v])|^2}{|\sigma_{j,a}([v])|^2}\right)^{-1}$$

on $\varphi_f^{-1}(V_a) \cap U_j$. Since $\mathbb{H}$ is ample and φ_f is holomorphic embedding, we have

$$\frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial \log h_{\varphi_f^*\mathbb{H},a}([v]) > 0. \tag{3.35}$$

The corresponding Hermitian metric h_L on L is given by the functions

$$h_{L,j}([v]) = \left(\sum_{b=0}^N |\sigma_{j,b}([v])|^2\right)^{-1}$$

on each U_j. Since $L = \mathbb{L}(E)^{*m}$, the corresponding Hermitian metric on $\mathbb{L}(E)$ is given by the functions

$$h_{\mathbb{L}(E),j}([v]) = \sqrt[m]{\sum_{b=0}^N |\sigma_{j,b}([v])|^2}$$

on U_j. Then, since $\tau(v) = \zeta^j t_j([v]) \cong \left([v], \zeta^j\right)$ on U_j, we shall define a complex Finsler metric F on E by

$$F(v) := \left|\zeta^j\right|^2 h_{\mathbb{L}(E),j} = \sqrt[m]{\sum_{b=0}^N |\sigma_{j,b}([v])|\zeta^j|^m|^2}. \tag{3.36}$$

This definition is independent on the choice of the neighborhood U_j, since $\{h_{\mathbb{L}(E),j}\}$ satisfies $|\zeta^j|^2 h_{\mathbb{L}(E),j} = |\zeta^i|^2 h_{\mathbb{L}(E),i}$ on $U_i \cap U_j \neq \emptyset$. From (3.35), the function F defined by

$$F(v) = \left[\sum_{b=0}^N \sigma_b([v]) \otimes \bar{\sigma}_b([v])\right]^{1/2m} = \sqrt[m]{\sum_{b=0}^N |\sigma_b([v])|^2} \tag{3.37}$$

is a Rizza metric satisfying (3.26).

Proposition 3.9. ([Ai6], [Ha-Ai]) *Let E be a negative vector bundle over a compact complex manifold M. For the holomorphic embedding $\varphi_f : \mathbb{P}(E) \to \mathbb{P}^N$, the function F defined by* (3.37) *is a Rizza metric on E with negative Ψ.*

Chapter 4

Averaged metrics and connections

In this chapter, we shall be concerned with averaged Hermitian metrics and connections on holomorphic vector bundles. In the first section, we consider a smooth family of compact Kähler manifolds and a proposition due to Schumacher[Sc3] which gives a basic idea for this research will be quickly reviewed. In the third section, we shall introduce the notions of averaged Hermitian metrics and averaged connections analogously to the real Finsler geometry (see [Ma-Ra-Tr-Ze] and [To-Et]), and in the last section, we show that Rizza-negetivity implies Griffiths-negativity of holomorphic vector bundles.

4.1 Family of Kähler manifolds

Let $\mathscr{X}$ and M be complex manifolds, and let $\phi : \mathscr{X} \to M$ be a holomorphic submersion. Denoted by $T_{\mathscr{X}}$ and T_M the holomorphic tangent bundles over $\mathscr{X}$ and M respectively, we obtain a short exact sequence of holomorphic vector bundles:

$$\mathbb{O} \longrightarrow \mathscr{V} \xrightarrow{\iota} T_{\mathscr{X}} \xrightarrow{\widetilde{d\phi}} \widetilde{T_M} \longrightarrow \mathbb{O}, \tag{4.1}$$

where the vertical sub-bundle $\mathscr{V}$ is defined by $\mathscr{V} := \ker\{\widetilde{d\phi} : T_{\mathscr{X}} \to \widetilde{T_M}\}$ and $\widetilde{d\phi} = (\phi, d\phi)$ for the derivative $d\phi$ of ϕ. The fiber $\mathscr{V}_{(z,w)}$ over (z, w) can be identified with the tangent space $T_w\mathscr{X}_z$, i.e., $\mathscr{V}_{(z,w)} = T_w\mathscr{X}_z$, where $\mathscr{X}_z := \phi^{-1}(z)$. Then any smooth complement of $\mathscr{V}$ defines a smooth horizontal vector sub-bundle $\mathscr{H} \subset T_{\mathscr{X}}$. Such a bundle $\mathscr{H}$ is defined by $\mathscr{H} = \ker(\mathscr{P})$ for a smooth morphism $\mathscr{P} : T_{\mathscr{X}} \to \mathscr{V}$ satisfying $\mathscr{P} \circ \iota = id$.

We denote by $(z, w) = (z^1, \cdots, z^m, w^1, \cdots, w^r)$ a local complex coordinate for $\mathscr{X}$, where $z = (z^1, \cdots, z^m)$ and $w = (w^1, \cdots, w^r)$ denote the ones for M and $\mathscr{X}_z$ respectively. Since $\mathscr{P}$ may be considered as $\mathscr{V}$-valued $(1, 0)$-form on $\mathscr{X}$ then $\mathscr{P}$ can be written as

$\mathscr{P} = \sum \frac{\partial}{\partial w^c} \otimes (dw^c + \sum \mathcal{N}^c_\alpha dz^\alpha)$ for some local functions $\mathcal{N}^c_\alpha$. Then the horizontal lifts $\mathcal{X}_\alpha$ with respect to $\mathscr{H}$ of local frame fields $\{\partial/\partial z^\alpha\}$ are given by

$$\mathcal{X}_\alpha = \frac{\partial}{\partial z^\alpha} - \sum \mathcal{N}^c_\alpha \frac{\partial}{\partial w^c}.$$

For any tangent vector X on M and its horizontal lift $X^{\mathscr{H}}$ with respect to $\mathscr{H}$, we set

$$\mathscr{D}_{X^{\mathscr{H}}} v := \mathscr{P}\left(\mathcal{L}_{X^{\mathscr{H}}} v\right), \tag{4.2}$$

where $\mathcal{L}_{X^{\mathscr{H}}}$ is the Lie derivative by $X^{\mathscr{H}}$. Then $\mathscr{D}_{X^{\mathscr{H}}} v$ is linear in $X^{\mathscr{H}}$ and satisfies the Leibniz rule $\mathscr{D}_{X^{\mathscr{H}}}(fv) = X^{\mathscr{H}}(f)v + f\mathscr{D}_{X^{\mathscr{H}}} v$, i.e., $\mathscr{D}$ is a *partial connection* on $\mathscr{V}$.

We suppose that $\mathscr{X}$ admits a pseudo Kähler metric $\mathit{\Pi}_{\mathscr{X}}$, i.e., $\mathit{\Pi}_{\mathscr{X}}$ is a smooth family of Kähler forms $\mathit{\Pi}_z$ on the fibers $\mathscr{X}_z$ of $\mathscr{X}$. Thus we consider $\phi : \mathscr{X} \to M$ as a family of compact Kähler manifolds $\{\mathscr{X}_z, \mathit{\Pi}_z\}$ smoothly parametrized by $z \in M$.

Since the bundle $\mathscr{V}$ may be thought as the bundle of vectors that tangent to the fibers of ϕ, the relative Kähler forms $\{\mathit{\Pi}_z\}$ define a Hermitian metric $\mathfrak{g}$ in $\mathscr{V}$ by

$$\mathfrak{g}\left(\frac{\partial}{\partial w^a}, \frac{\partial}{\partial w^b}\right) := \mathfrak{g}_{a\bar{b}}, \quad \mathit{\Pi}_z = \frac{\sqrt{-1}}{2} \sum \mathfrak{g}_{a\bar{b}} dw^a \wedge d\bar{w}^b.$$

Then we assume that $\mathscr{D}$ satisfies

$$\mathscr{D}\mathfrak{g} = 0. \tag{4.3}$$

Since $\mathscr{D}$ is defined by the Lie derivative in the horizontal direction $\mathscr{H}$, this assumption means that the the metric $\mathfrak{g}$ is preserved by the parallel displacement relative to $\mathscr{H}$, i.e., $(\mathcal{L}_{X^{\mathscr{H}}}\mathfrak{g})|_{\mathscr{X}_z} = 0$.

We express $\mathfrak{g}_{a\bar{b}}$ as $\mathfrak{g}_{a\bar{b}} = \partial_a \partial_{\bar{b}} \mathfrak{G}$ for a local smooth function $\mathfrak{G}$, where $\partial_a = \partial/\partial w^a$ and $\partial_{\bar{b}} = \partial/\partial \bar{w}^b$. Then the assumption (4.3) can be written as

$$\frac{\partial}{\partial w^a}\left(\frac{\partial^2 \mathfrak{G}}{\partial z^\alpha \partial \bar{w}^b} - \sum \mathfrak{g}_{c\bar{b}} \mathcal{N}^c_\alpha\right) = 0.$$

Hence, if we define $\mathscr{H}$ by

$$\mathcal{N}^c_\alpha := \sum \mathfrak{g}^{\bar{b}c} \frac{\partial^2 \mathfrak{G}}{\partial z^\alpha \partial \bar{w}^b} \tag{4.4}$$

for the the components $\mathfrak{g}^{\bar{b}c}$ of the inverse of $(\mathfrak{g}_{c\bar{b}})$, the partial connection $\mathscr{D}$ satisfies the assumption (4.3). For a smooth (r,r)-form η on $\mathscr{X}$,

$$\frac{\partial}{\partial z^\alpha}\int_{\mathscr{X}_z}\eta=\int_{\mathscr{X}_z}\mathcal{L}_{\mathcal{X}_\alpha}\eta \quad \text{and} \quad \frac{\partial}{\partial \bar{z}^\alpha}\int_{\mathscr{X}_z}\eta=\int_{\mathscr{X}_z}\mathcal{L}_{\mathcal{X}_{\overline{\alpha}}}\eta, \tag{4.5}$$

where $\mathcal{X}_{\overline{\alpha}}:=\overline{\mathcal{X}_\alpha}$ denotes the conjugate of $\mathcal{X}_\alpha$.

Proposition 4.1. [Sc3] *Let $\phi:\mathscr{X}\to M$ be a family of compact Kähler manifolds $\{\mathscr{X}_z,\varPi_z\}$ parametrized by $z\in M$, and let $\mathscr{H}$ be the horizontal sub-bundle of $T_{\mathscr{X}}$ defined by* (4.4)*. Then we have*

$$(\mathcal{L}_{\mathcal{X}_\alpha}\varPi_z)|_{\mathscr{X}_z}=0 \quad \text{and} \quad (\mathcal{L}_{\mathcal{X}_\alpha}dv)|_{\mathscr{X}_z}=0, \tag{4.6}$$

where $dv=(\sqrt{-1})^r\det(\mathfrak{g}_{a\bar{b}})dw^1\wedge\overline{dw^1}\wedge\cdots\wedge dw^r\wedge\overline{dw^r}=\varPi_z^r/r!$ is the relative volume form in $\mathscr{V}$ induced from $\varPi_z$. Further we obtain

$$X\left(\int_{\mathscr{X}_z}dv\right)=\int_{\mathscr{X}_z}\mathcal{L}_{X^{\mathscr{H}}}dv=0 \tag{4.7}$$

for any vector filed X in M. Thus the volume of each fiber $(\mathscr{X}_z,\varPi_z)$ is constant.

Remark 4.1. Let $\varOmega_M$ denote the holomorphic cotangent bundle, and let $\widetilde{dz^1},\cdots,\widetilde{dz^m}$ denote the local frame field of the pull-back $\widetilde{\varOmega_M}=\phi^*\varOmega_M$. For the zero-cochain $\mathscr{N}=\sum\mathscr{N}_\alpha^c(\partial/\partial w^c)\otimes\widetilde{dz^\alpha}$ of $\varGamma(T_{\mathscr{X}}\otimes\widetilde{\varOmega_M})$,

$$[\overline{\partial}\mathscr{N}]:=\left[\sum\overline{\partial}\mathscr{N}_\alpha^c\otimes\frac{\partial}{\partial w^c}\otimes\widetilde{dz^\alpha}\right]\in H^{0,1}_{\overline{\partial}}(\mathscr{X},\mathscr{V}\otimes\widetilde{\varOmega_M})$$

is an obstruction for the existence of holomorphic splitting of (4.1), i.e., *extension class*, and

$$\left[\overline{\partial}\mathscr{N}\left(\frac{\partial}{\partial z^\alpha}\right)\right]:=\left[\sum\frac{\partial\mathscr{N}_\alpha^c}{\partial\bar{w}^b}\frac{\partial}{\partial w^c}\otimes d\bar{w}^b\right]\in H^{0,1}_{\overline{\partial}}(\mathscr{X}_z,T_{\mathscr{X}_z})$$

is an obstruction for the infinitesimal triviality of (4.1), i.e., *Kodaira-spencer* class. □

4.2 Partial connections on $\mathscr{V}$

In this section, we shall be concerned with the special case of the previous section, i.e., we treat the case of $\mathscr{X}=\mathbb{P}(E)$ for a holomorphic vector bundle E over a compact complex manifold M. Following to [Sc3], we will apply Lie derivation only to relative tensor fields, i.e., to smooth families of tensor fields on the fibers $\mathbb{P}(E_z):=\mathbb{P}_z$.

We suppose that E admits a Rizza metric F which induces a Hermitian metric $g = \sum g_{i\bar{j}} d\zeta^i \otimes d\bar{\zeta}^j$ on the vertical sub-bundle V defined by (3.17). We denote by $\mathcal{E}^\perp$ the complex distribution of codimension one defined by the complex orthogonal to the tautological section $\mathcal{E}$, i.e., $\mathcal{E}^\perp := \{Z \in V \,|\, g(Z, \mathcal{E}) = 0\}$, and by $Z^\perp = p(Z)$ the orthogonal projection of Z onto $\mathcal{E}^\perp$:

$$Z^\perp := Z - \frac{1}{F} g(Z, \mathcal{E})\mathcal{E}.$$

Then

$$\begin{aligned} d\mu_\lambda \left(\frac{\partial}{\partial \zeta^i}\right)^\perp_{(z,\zeta)} &= d\mu_\lambda \left[\left(\frac{\partial}{\partial \zeta^i}\right)_{(z,\zeta)} - \frac{1}{F(z,\zeta)} g\left(\left(\frac{\partial}{\partial \zeta^i}\right)_{(z,\zeta)}, \mathcal{E}(z,\zeta)\right)\mathcal{E}(z,\zeta)\right] \\ &= d\mu_\lambda \left(\frac{\partial}{\partial \zeta^i}\right)_{(z,\zeta)} - \frac{1}{F(z,\zeta)} g\left(\left(\frac{\partial}{\partial \zeta^i}\right)_{(z,\zeta)}, \mathcal{E}(z,\zeta)\right) d\mu_\lambda(\mathcal{E}(z,\zeta)) \\ &= \lambda \left(\frac{\partial}{\partial \zeta^i}\right)_{(z,\lambda\zeta)} - \frac{1}{F(z,\zeta)} g\left(\left(\frac{\partial}{\partial \zeta^i}\right)_{(z,\zeta)}, \mathcal{E}(z,\zeta)\right)\mathcal{E}(z,\lambda\zeta) \end{aligned}$$

and

$$\begin{aligned} g\left(\left(\frac{\partial}{\partial \zeta^i}\right)_{(z,\zeta)}, \mathcal{E}(z,\zeta)\right) &= \sum g_{i\bar{j}}(z,\zeta)\overline{\zeta^j} \\ &= \sum g_{i\bar{j}}(z,\lambda\zeta)\overline{\lambda\zeta^j} \times \frac{1}{\overline{\lambda}} \\ &= g\left(\left(\frac{\partial}{\partial \zeta^i}\right)_{(z,\lambda\zeta)}, \mathcal{E}(z,\lambda\zeta)\right) \times \frac{1}{\overline{\lambda}} \end{aligned}$$

lead to

$$\begin{aligned} d\mu_\lambda \left(\frac{\partial}{\partial \zeta^i}\right)^\perp_{(z,\zeta)} &= \lambda \left(\frac{\partial}{\partial \zeta^i}\right)_{(z,\lambda\zeta)} - \frac{1}{F(z,\zeta)} g\left(\left(\frac{\partial}{\partial \zeta^i}\right)_{(z,\lambda\zeta)}, \mathcal{E}(z,\lambda\zeta)\right) \times \frac{1}{\overline{\lambda}}\mathcal{E}(z,\lambda\zeta) \\ &= \lambda \left[\left(\frac{\partial}{\partial \zeta^i}\right)_{(z,\lambda\zeta)} - \frac{1}{|\lambda|^2}\frac{1}{F(z,\zeta)} g\left(\left(\frac{\partial}{\partial \zeta^i}\right)_{(z,\lambda\zeta)}, \mathcal{E}(z,\lambda\zeta)\right)\mathcal{E}(z,\lambda\zeta)\right] \\ &= \lambda \left[\left(\frac{\partial}{\partial \zeta^i}\right)_{(z,\lambda\zeta)} - \frac{1}{F(z,\lambda\zeta)} g\left(\left(\frac{\partial}{\partial \zeta^i}\right)_{(z,\lambda\zeta)}, \mathcal{E}(z,\lambda\zeta)\right)\mathcal{E}(z,\lambda\zeta)\right] \\ &= \lambda \left(\frac{\partial}{\partial \zeta^i}\right)^\perp_{(z,\lambda\zeta)}. \end{aligned}$$

Therefore $d\mu_\lambda(Z^\perp) = \lambda Z^\perp$, and thus we have

Lemma 4.1. *The orthogonal projection $Z^\perp$ satisfies*

$$d\mu_\lambda(Z^\perp) = \lambda Z^\perp \tag{4.8}$$

for all $\lambda \in \mathbb{C}^*$.

We define a Hermitian metric δ on V by

$$\delta(Z, W) := \frac{1}{F} g(Z, W)$$

for all $Z, W \in \mathcal{A}(V)$. Let H be the non-linear connection on E determined in Proposition 3.6, and let D be the partial connection on (V, g) determined by (3.18). Then

$$D_\alpha Z^\perp = D_\alpha \left(Z - \frac{1}{F} g(Z, \mathcal{E})\mathcal{E} \right) = D_\alpha Z - \frac{1}{F} g(D_\alpha Z, \mathcal{E})\mathcal{E}$$

implies

$$D_\alpha Z^\perp = (D_\alpha Z)^\perp \tag{4.9}$$

for any $Z \in \mathcal{A}(V)$. Further we have

Proposition 4.2. *The partial connection D is compatible with δ:*

$$D\delta = 0. \tag{4.10}$$

Proof.

$$\begin{aligned}
&(D_\alpha \delta)\left(Z^\perp, W^\perp\right) \\
&= X_\alpha \left(\delta(Z^\perp, W^\perp) \right) - \delta\left(D_\alpha Z^\perp, W^\perp \right) - \delta\left(Z^\perp, D_\alpha W^\perp \right) \\
&= X_\alpha \left(\frac{1}{F} \Big[g(Z, W) - \frac{1}{F} g(Z, \mathcal{E}) g(\mathcal{E}, W) \Big] \right) \\
&- \frac{1}{F} \Big[g(D_\alpha Z, W) - \frac{1}{F} g(D_\alpha Z, \mathcal{E}) g(\mathcal{E}, W) \Big] - \frac{1}{F} \Big[g(Z, D_\alpha W) - \frac{1}{F} g(Z, \mathcal{E}) g(\mathcal{E}, D_\alpha W) \Big] \\
&= \frac{1}{F} \Big[g(D_\alpha Z, W) + g(Z, D_\alpha W) - \frac{1}{F} g(D_\alpha Z, \mathcal{E}) g(\mathcal{E}, W) - \frac{1}{F} g(Z, \mathcal{E}) g(\mathcal{E}, D_\alpha W) \Big] \\
&- \frac{1}{F} \Big[g(D_\alpha Z, W) - \frac{1}{F} g(D_\alpha Z, \mathcal{E}) g(\mathcal{E}, W) \Big] - \frac{1}{F} \Big[g(Z, D_\alpha W) - \frac{1}{F} g(Z, \mathcal{E}) g(\mathcal{E}, D_\alpha W) \Big] \\
&= 0
\end{aligned}$$

Q.E.D.

Let $\mathscr{V}$ be the vertical sub-bundle of $T_{\mathbb{P}(E)}$. The fiber $\mathscr{V}_{(z,[\zeta])}$ of $\mathscr{V}$ over $(z, [\zeta]) \in \mathbb{P}(E)$

is naturally identified with $\mathcal{E}^{\perp}_{(z,\zeta)}$. Since δ is invariant by the action μ of $\mathbb{C}^*$, there exists a Hermitian metric $\mathfrak{g}$ on $\mathscr{V}$ such that $\delta = \rho^*\mathfrak{g}$:

$$(\rho^*\mathfrak{g})(Z^{\perp}, W^{\perp}) = \delta(Z^{\perp}, W^{\perp}) = \frac{1}{F}\left[g(Z,W) - \frac{1}{F}g(Z,\mathcal{E})g(\mathcal{E},W)\right]. \tag{4.11}$$

for any $Z, W \in \mathcal{A}(V)$.

Let $X_1, \cdots, X_m$ be the horizontal lifts of local frame fields $\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^m}$ with respect to H. From Lemma 3.1 the basis $\{X_1, \cdots, X_m\}$ is invariant by the action μ of $\mathbb{C}^*$. Further, since the projection $\rho : E^0 \to \mathbb{P}(E)$ commutes with the action μ, we have

$$d\rho\left((X_\alpha(z, \lambda\cdot\zeta)\right) = d\rho\left(d\mu_\lambda(X_\alpha(z,\zeta)\right) = d(\rho\circ\mu_\lambda)(X_\alpha(z,\zeta)) = d\rho\left((X_\alpha(z,\zeta)\right).$$

Thus $\mathcal{X}_\alpha(z,[\zeta]) = d\rho(X_\alpha)(z,[\zeta])$ makes sense, and hence we shall define a horizontal subbundle $\mathscr{H}$ of $T_{\mathbb{P}(E)}$ by $\mathscr{H} = d\rho(H)$. Then $\{\mathcal{X}_1, \cdots, \mathcal{X}_m\}$ defined by

$$\mathcal{X}_\alpha(z,[\zeta]) := d\rho_{(z,\zeta)}(X_\alpha) \tag{4.12}$$

spans the horizontal sub-space $\mathscr{H}_{(z,[\zeta])}$ at $(z,[\zeta]) \in \mathbb{P}(E)$. We also define a partial connection $\mathscr{D}$ on $\mathscr{V}$ by the Lie derivative with respect to $\mathscr{H}$ similarly to the previous section (cf. (4.2)). We set $\widetilde{Z} = d\rho(Z^{\perp})$ for any $Z \in \mathcal{A}(V)$. Then $\mathscr{P}\circ d\rho = d\rho\circ P$ implies

$$\mathscr{D}_\alpha\widetilde{Z} = \mathscr{P}([\mathcal{X}_\alpha, \widetilde{Z}]) = \mathscr{P}\left(\left[d\rho(X_\alpha), d\rho(Z^{\perp})\right]\right) = d\rho\left(P\left(\left[X_\alpha, Z^{\perp}\right]\right)\right),$$

where $\mathscr{D}_\alpha := \mathscr{D}_{\mathcal{X}_\alpha}$. Hence

$$\mathscr{D}_\alpha\widetilde{Z} = d\rho(D_\alpha Z^{\perp}). \tag{4.13}$$

Then Proposition 4.2 and

$$\begin{aligned}
(\mathscr{D}_\alpha\mathfrak{g})(\widetilde{Z}, \widetilde{W}) &= \mathcal{X}_\alpha\left(\mathfrak{g}(\widetilde{Z}, \widetilde{W})\right) - \mathfrak{g}\left(\mathscr{D}_\alpha\widetilde{Z}, \widetilde{W}\right) - \mathfrak{g}\left(\widetilde{Z}, \mathscr{D}_\alpha\widetilde{W}\right) \\
&= d\rho(X_\alpha)\left(\mathfrak{g}(\widetilde{Z}, \widetilde{W})\right) - \mathfrak{g}\left(d\rho(D_\alpha Z^{\perp}), d\rho(W^{\perp})\right) - \mathfrak{g}\left(d\rho(Z^{\perp}), d\rho(D_\alpha W^{\perp})\right) \\
&= d\rho(X_\alpha)\left(\mathfrak{g}(\widetilde{Z}, \widetilde{W})\right) - \delta\left(D_\alpha Z^{\perp}, W^{\perp}\right) - \delta\left(Z^{\perp}, D_\alpha W^{\perp}\right) \\
&= X_\alpha\left((\rho^*\mathfrak{g})(Z^{\perp}, W^{\perp})\right) - \delta\left(D_\alpha Z^{\perp}, W^{\perp}\right) - \delta\left(Z^{\perp}, D_\alpha W^{\perp}\right) \\
&= X_\alpha\left(\delta(Z^{\perp}, W^{\perp})\right) - \delta\left(D_\alpha Z^{\perp}, W^{\perp}\right) - \delta\left(Z^{\perp}, D_\alpha W^{\perp}\right) \\
&= (D_\alpha\delta)\left(Z^{\perp}, W^{\perp}\right) \\
&= 0
\end{aligned}$$

lead to

Proposition 4.3. ([Ha-Ai]) *The partial connection $\mathscr{D}$ on $\mathscr{V}$ is compatible with $\mathfrak{g}$:*

$$\mathscr{D}\mathfrak{g} = 0. \tag{4.14}$$

The parallel displacement with respect to $\mathscr{H}$ is an isometry with respect to the metric $\mathfrak{g}$ on $\mathscr{V}$. Then the volume form dv defined by $dv = \Pi_z^{r-1}/(r-1)!$ for the restriction $\Pi_z := \Pi_{\mathbb{P}(E)}|_{\mathbb{P}_z}$ is preserved by the parallel displacement, i.e., $[\mathcal{L}_{\mathcal{X}_\alpha} dv]|_{\mathbb{P}_z} = 0$. Hence we have

$$\frac{\partial}{\partial z^\alpha}\int_{\mathbb{P}_z} dv = \int_{\mathbb{P}_z} \mathcal{L}_{\mathcal{X}_\alpha} dv = 0. \tag{4.15}$$

Hence we obtain

Proposition 4.4. ([Ha-Ai]) *The volume of each fiber $\{\mathbb{P}_z, \mathfrak{g}_z\}$ is constant.*

Remark 4.2. Yan[Ya] proved this proposition by direct methods in the case where E is the holomorphic tangent bundle T_M over a complex manifold M. □

From Proposition 4.4, if a Rizza metric F is given in E, then we can apply Proposition 4.1 to the family of Kähler manifolds $\phi : \mathbb{P}(E) \to M$ with the pseudo Kähler form $\Pi_{\mathbb{P}(E)}$.

4.3 Averaged metrics and connections

In this section, we shall show that we can define a Hermitian metric on E from the metric $\mathfrak{g}$ on $\mathscr{V}$ by taking an L^2 inner product.

We define $Z_u \in \mathcal{A}(\mathcal{E}^\perp)$ by $Z_u := \left(u^V\right)^\perp$ for any $u \in \mathcal{A}(E)$, where $u^V = \sum u^i(z)\dfrac{\partial}{\partial \zeta^i} \in \mathcal{A}(V)$ denotes the vertical lift of u. Further we introduce a Hermitian structure h on E by the L^2-inner product

$$h(u, v) := \int_{\mathbb{P}_z} \mathfrak{g}(\widetilde{u}, \widetilde{v})dv, \tag{4.16}$$

for any $u, v \in \mathcal{A}(E)$, where $\widetilde{u} = d\rho(Z_u)$ and $\widetilde{v} = d\rho(Z_v)$.

Definition 4.1. The Hermitian metric h on E defined by (4.16) is called the *averaged metric* of $\mathfrak{g}$.

Further we define $\nabla : \mathcal{A}(T_M) \times \mathcal{A}(E) \ni (X, u) \longmapsto \nabla_X u \in \mathcal{A}(E)$ by

$$h(\nabla_X u, v) := \int_{\mathbb{P}_z} \mathfrak{g}(\mathscr{D}_{X^{\mathscr{H}}}\widetilde{u}, \widetilde{v})dv, \tag{4.17}$$

where $X^{\mathscr{H}}$ denotes the horizontal lift of X to $\mathbb{P}(E)$ with respect to $\mathscr{H}$. Since $\mathscr{D}$ is of $(1,0)$-type, ∇ is also of $(1,0)$-type. Further it is obvious that $\nabla_X u$ is linear in X and the Leibniz rule is checked as follows:

$$\begin{aligned} h(\nabla_X(fu), v) &= \int_{\mathbb{P}_z} \{\mathfrak{g}(X(f)\widetilde{u} + f\mathscr{D}_{X^{\mathscr{H}}}\widetilde{u}, \widetilde{v})\}\, dv \\ &= h\left(X(f)u + f\nabla_X u, v\right) \end{aligned}$$

for any $f \in C^\infty(M)$ and $u, v \in \mathcal{A}(E)$.

Theorem 4.1. ([Ha-Ai]) *The connection ∇ is the Hermitian connection on (E,h).*

Proof. Since $\mathscr{D}$ is of $(1,0)$-type, the connection ∇ is also of $(1,0)$-type. Hence it is enough to show that ∇ is compatible with h. From (4.17) and Proposition 4.3, we obtain

$$\begin{aligned} (\nabla_X h)(u,v) &= X(h(u,v)) - h(\nabla_X u, v) - h(u, \nabla_X v) \\ &= X\left(\int_{\mathbb{P}_z} \mathfrak{g}(\widetilde{u},\widetilde{v})dv\right) - \int_{\mathbb{P}_z} \mathfrak{g}(\mathscr{D}_{X^{\mathscr{H}}}\widetilde{u},\widetilde{v})dv - \int_{\mathbb{P}_z} \mathfrak{g}(\widetilde{u},\mathscr{D}_{X^{\mathscr{H}}}\widetilde{v})dv \\ &= \int_{\mathbb{P}_z} \left\{X^{\mathscr{H}}(\mathfrak{g}(\widetilde{u},\widetilde{v})) - \mathfrak{g}(\mathscr{D}_{X^{\mathscr{H}}}\widetilde{u},\widetilde{v}) - \mathfrak{g}(\widetilde{u},\mathscr{D}_{X^{\mathscr{H}}}\widetilde{v})\right\} dv \\ &= \int_{\mathbb{P}_z} (\mathscr{D}_{X^{\mathscr{H}}}\mathfrak{g})(\widetilde{u},\widetilde{v})dv \\ &= 0. \end{aligned}$$

Hence ∇ is compatible with h.

Q.E.D.

Definition 4.2. The Hermitian connection ∇ defined by (4.18) is called the *averaged connection* of $\mathscr{D}$.

4.4 Rizza-negativity and Griffiths-negativity

Let h be the Hermitian metric on E defined by (4.16), and let $h_{i\bar{j}} = h(s_i, s_j)$ be the components of h with respect to a local holomorphic frame filed $s = \{s_1, \cdots, s_r\}$ of E. We write the curvature form Ω^i_j of ∇ as $\Omega^i_j = \sum R^i_{j\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$ so that

$$R_{i\bar{j}\alpha\bar{\beta}} := \sum h_{l\bar{j}} R^l_{i\alpha\bar{\beta}} = -\partial_{\bar{\beta}}\partial_\alpha h_{i\bar{j}} + \sum h^{l\bar{m}} \partial_\alpha h_{i\bar{m}} \partial_{\bar{\beta}} h_{l\bar{j}}, \tag{4.18}$$

where $\partial_\alpha := \partial/\partial z^\alpha$ and $\partial_{\bar\beta} := \partial/\partial \bar z^\beta$. On the other hand, (4.14) and (4.15) imply

$$\partial_\alpha h_{i\bar j} = \int_{\mathbb{P}_z} \mathcal{X}_\alpha(\mathfrak{g}(\widetilde{s}_i, \widetilde{s}_j))dv = \int_{\mathbb{P}_z} \mathfrak{g}(\mathscr{D}_\alpha \widetilde{s}_i, \widetilde{s}_j)dv$$

and

$$\partial_{\bar\beta}\partial_\alpha h_{i\bar j} = \int_{\mathbb{P}_z} \mathcal{X}_{\overline{\beta}}(\mathfrak{g}(\mathscr{D}_\alpha \widetilde{s}_i, \widetilde{s}_j))dv = \int_{\mathbb{P}_z} \Big\{\mathfrak{g}(\mathscr{D}_{\overline{\beta}}\mathscr{D}_\alpha \widetilde{s}_i, \widetilde{s}_j) + \mathfrak{g}(\mathscr{D}_\alpha \widetilde{s}_i, \mathscr{D}_\beta \widetilde{s}_j)\Big\}dv,$$

where $\mathscr{D}_{\overline{\beta}} := \mathscr{D}_{\mathcal{X}_{\overline{\beta}}}$. Hence (4.18) implies

$$-R_{i\bar j\alpha\bar\beta} + \sum h^{l\overline{m}}\partial_\alpha h_{i\overline{m}}\partial_{\bar\beta} h_{l\bar j} = \int_{\mathbb{P}_z} \Big\{\mathfrak{g}(\mathscr{D}_{\overline{\beta}}\mathscr{D}_\alpha \widetilde{s}_i, \widetilde{s}_j) + \mathfrak{g}(\mathscr{D}_\alpha \widetilde{s}_i, \mathscr{D}_\beta \widetilde{s}_j)\Big\}dv.$$

Setting $\mathfrak{g}(\mathscr{D}_{\overline{\beta}}\mathscr{D}_\alpha \widetilde{s}_i, \widetilde{s}_j) := -\mathscr{R}_{i\bar j\alpha\bar\beta}$, we get

$$R_{i\bar j\alpha\bar\beta} - \sum h^{l\overline{m}}\partial_\alpha h_{i\overline{m}}\partial_{\bar\beta} h_{l\bar j} = \int_{\mathbb{P}_z} \Big\{\mathscr{R}_{i\bar j\alpha\bar\beta} - \mathfrak{g}(\mathscr{D}_\alpha \widetilde{s}_i, \mathscr{D}_\beta \widetilde{s}_j)\Big\}dv. \tag{4.19}$$

Let $z_0 \in M$ be an arbitrary point, and let $s = \{s_1, \cdots, s_r\}$ be a local frame field in (E, h) such that s is normal at $z_0 \in M$, i.e., $h_{i\bar j}(z_0) = \delta_{ij}$ and $\partial_\alpha h_{i\bar j}(z_0) = 0$. Then (4.19) implies

$$R_{i\bar j\alpha\bar\beta}(z_0) = \left[\int_{\mathbb{P}_z} \Big\{\mathscr{R}_{i\bar j\alpha\bar\beta} - \mathfrak{g}(\mathscr{D}_\alpha \widetilde{s}_i, \mathscr{D}_\beta \widetilde{s}_j)\Big\}dv\right]_{z=z_0},$$

and thus

$$\sum R_{i\bar j\alpha\bar\beta}(z_0)u^i X^\alpha \overline{u^j X^\beta} = \left[\int_{\mathbb{P}_z} \Big\{\sum \mathscr{R}_{i\bar j\alpha\bar\beta}u^i X^\alpha \overline{u^j X^\beta} - \Big\|\sum(\mathscr{D}_\alpha \widetilde{s}_i)u^i X^\alpha\Big\|^2\Big\}dv\right]_{z=z_0}$$

for all $(u^i) \in \mathbb{C}^r$ and $(X^\alpha) \in \mathbb{C}^m$, where we put

$$\Big\|\sum(\mathscr{D}_\alpha \widetilde{s}_i)u^i X^\alpha\Big\|^2 = \mathfrak{g}\left(\sum(\mathscr{D}_\alpha \widetilde{s}_i)u^i X^\alpha, \sum(\mathscr{D}_\beta \widetilde{s}_j)u^j X^\beta\right).$$

This implies

$$\sum R_{i\bar j\alpha\bar\beta}(z_0)u^i X^\alpha \overline{u^j X^\beta} \le \left[\int_{\mathbb{P}_z} \Big\{\sum \mathscr{R}_{i\bar j\alpha\bar\beta}u^i X^\alpha \overline{u^j X^\beta}\Big\}dv\right]_{z=z_0} \tag{4.20}$$

We will use the notation $\mathscr{R}(\widetilde{u} \otimes X^{\mathscr{H}})$ for the Hermitian form

$$\mathscr{R}(\widetilde{u} \otimes X^{\mathscr{H}}) := \sum \mathscr{R}_{i\bar j\alpha\bar\beta}u^i X^\alpha \overline{u^j X^\beta}.$$

From (4.20) we obtain

Theorem 4.2. ([Ha-Ai]) *Let $\mathscr{D}$ be the partial connection on $(\mathscr{V}, \mathfrak{g})$ determined by the horizontal sub-bundle $\mathscr{H}$, and let ∇ be the averaged connection of $\mathscr{D}$. Then the curvatures R of ∇ and $\mathscr{R}$ of $\mathscr{D}$ satisfy*

$$R(u \otimes X) \leq \int_{\mathbb{P}_z} \mathscr{R}(\widetilde{u} \otimes X^{\mathscr{H}}) dv \tag{4.21}$$

for all $u \in E_z$ and $X \in T_z M$ at every point $z \in M$.

Definition 4.3. We say that the partial connection $\mathscr{D}$ has *negative curvature* (resp. *semi-negative curvature*) if $\mathscr{R}(\widetilde{u} \otimes X^{\mathscr{H}}) < 0$ (resp. $\mathscr{R}(\widetilde{u} \otimes X^{\mathscr{H}}) \leq 0$) for all non-zero $u \in E_z$ and non-zero $X \in T_z M$ at any point $z \in M$.

Hence (4.21) implies

Theorem 4.3. ([Ha-Ai]) *If the partial connection $\mathscr{D}$ on $(\mathscr{V}, \mathfrak{g})$ has negative curvature (resp. semi-negative curvature), then E is Griffiths-negative (resp. Griffiths-semi-negative).*

By the definition of $\mathscr{D}$ and (4.13), we have

$$\mathscr{D}_{\bar{\beta}} \mathscr{D}_{\alpha} \widetilde{s}_i = \mathscr{P}\left(\left[d\rho(X_{\bar{\beta}}), d\rho\left(D_{\alpha}\left(\frac{\partial}{\partial \zeta^i}\right)^{\perp}\right)\right]\right) = d\rho\left(D_{\bar{\beta}} D_{\alpha}\left(\frac{\partial}{\partial \zeta^i}\right)^{\perp}\right)$$

and

$$\begin{aligned}
D_{\bar{\beta}} D_{\alpha}\left(\frac{\partial}{\partial \zeta^i}\right)^{\perp} &= D_{\bar{\beta}} D_{\alpha}\left(\frac{\partial}{\partial \zeta^i} - \frac{1}{F} g\left(\frac{\partial}{\partial \zeta^i}, \mathcal{E}\right) \mathcal{E}\right) \\
&= D_{\bar{\beta}} D_{\alpha} \frac{\partial}{\partial \zeta^i} - \frac{1}{F} g\left(D_{\bar{\beta}} D_{\alpha} \frac{\partial}{\partial \zeta^i}, \mathcal{E}\right) \mathcal{E} \\
&= -\sum K^l_{i\alpha\bar{\beta}}\left(\frac{\partial}{\partial \zeta^l} - \frac{1}{F} g\left(\frac{\partial}{\partial \zeta^l}, \mathcal{E}\right) \mathcal{E}\right) \\
&= -\sum K^l_{i\alpha\bar{\beta}}\left(\frac{\partial}{\partial \zeta^l}\right)^{\perp}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathfrak{g}\left(\mathscr{D}_{\bar{\beta}} \mathscr{D}_{\alpha} \widetilde{s}_i, \widetilde{s}_j\right) &= \mathfrak{g}\left(d\rho\left(D_{\bar{\beta}} D_{\alpha}\left(\frac{\partial}{\partial \zeta^i}\right)^{\perp}\right), d\rho\left(\frac{\partial}{\partial \zeta^j}\right)^{\perp}\right) \\
&= -\frac{1}{F} g\left(\sum K^l_{i\alpha\bar{\beta}}\left(\frac{\partial}{\partial \zeta^l}\right)^{\perp}, \left(\frac{\partial}{\partial \zeta^j}\right)^{\perp}\right)
\end{aligned}$$

implies

Proposition 4.5. *Let D be the Finsler connection on (V, g), and let $\mathscr{D}$ be the partial connection on $(\mathscr{V}, \mathfrak{g})$ as above. Then the curvatures K of D and $\mathscr{R}$ of $\mathscr{D}$ satisfy the relation*

$$\mathscr{R}(\widetilde{u} \otimes X^{\mathscr{H}}) = \frac{1}{F} K(Z_u \otimes X^H) \tag{4.22}$$

at every point $(z, \zeta) \in E^0$ for any $u \in E_z$ and $X \in T_z M$.

Therefore, if F has negative curvature, then (4.22) implies

$$R(u \otimes X) \leq \int_{\mathbb{P}_z} \mathscr{R}(\widetilde{u} \otimes X^{\mathscr{H}}) dv < 0,$$

and we have

Theorem 4.4. ([Ha-Ai]) *Let E be a holomorphic vector bundle over a complex manifold M. If E is Rizza-negative, then E is Griffiths-negative.*

Therefore Theorem 2.4 concludes the following relation:

$$E \text{ is Rizza-negative.} \Longrightarrow E \text{ is Griffiths-negative.} \Longrightarrow E \text{ is negative.}$$

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