The N-Queens Problem: An Experimental Study

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Abstract

The N-Queens Problem is an old difficult problem. In this paper we investigate the queen problem from three viewpoints. These viewpoints are widely known, but relations with the queen problem have been not been studied. We have made extensive computer simulation to produce new data about the queen problem. Based on these data we propose three conjectures.

Keyword : Rectagular chess board, Cayley distance, Random sequential place-ment

1 Introduction

The N-Queens problem is as follows: Place n queens on $n \times n$ chess board so as no queen attacks others. The problem as well as many various variants have been extensively studied (see [1],[2]). However, as research progresses deeper, we become to recognize clearer about difficulty of the problem (see [3]).

For example, consider the numbers of solutions for the original N-queens problem. Let us denote these numbers by q_n . At present we know them only for $n \leqq 26$, which are shown below:

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n	q_n	n	q_n
1	1	14	365,596
2	0	15	2,279,184
3	0	16	14,772,512
4	2	17	95,815,104
5	10	18	666,090,624
6	4	19	4,968,057,848
7	40	20	39,029,188,884
8	92	21	314,666,222,712
9	352	22	2,691,008,701,644
10	724	23	24,233,937,684,440
11	2,680	24	227,514,171,973,736
12	14,200	25	2,207,893,435,808,352
13	73,712	26	22,317,699,616,364,044

Until now any explicit formulas which represent q_n have not been discoverd. On contrary we have some strong evidence that there could not exist such formulas.

It is obvious that an inequality $q_n < n!$ holds. From the above numerical table, it is plausible that a "reverse" inequality $q_n > \sqrt{n!}$ holds. However we can not prove this weak conjecture.

The value q_{26} is obtained by using one-year computation with very fast machine [1]. Thus we can not expect to know, for example, q_{100}, in near future.

In this paper we study several problems related to the original n Queens problem. Our method is not theoretical, but experimental by using computer simulation.

2 Queens problem on rectangular chess board

Consider $n \times k$ (n rows and k colums) rectangular chess board. Of course assume that $k \leqq n$. On this rectangular chess board we place queens so as any queens do not attack other ones. Denote the numbers of solutions by $q_{n,k}$. Then, the numbers of solutions of the original N-Queens problem is equal to $q_{n,n}$. Our research plan is as follows: first to obtain explicit formulas of $q_{n,k}$ for "small" k, next to extrapolate these formulas up to $k = n$.

To obtain data for $q_{n,k}$, we make the following simple program (Our programming language is Haskell).

```
import Data.List

find_spaces :: Int -> Int -> [Int] -> [Int]
find_spaces n k js = [1..n] \\ ys
  where ys = concat [[j-(k-i),j,j+(k-i)] | (i,j) <- zip [1..(k-1)] js ]
```

[1] http://www.nqueens.de/sub/WorldRecord.en.html

```
queen :: Int -> Int -> [[Int]]
queen n k
  | k == 1    = [[j] | j <- [1..n] ]
  | otherwise = [ js ++ [q] | js <- queen n (k-1), q <- find_spaces n k js ]
```

A computation of small amount gives the next data.

Table 1

1	2	3	4	5	6	7	8	9	10	11	12
4	6	4	2								
5	12	14	12	10							
6	20	36	46	40	4						
7	30	76	140	164	94	40					
8	42	140	344	568	550	312	92				
9	56	234	732	1614	2292	2038	1066	352			
10	72	364	1400	3916	7552	9632	7828	4040	724		
11	90	536	2468	8492	21362	37248	44148	34774	15116	2680	
12	110	756	4080	16852	52856	120104	195270	222720	160964	68264	14200

From these data we have the following conjecture.

Conjecture 1

Fix k. Then there exists a large $N(k)$ such that for all $n \geqq N(k)$, $q_{n,k}$ can be expressed as a polynomial of variable n with degree k :

$$q_{n,k} = n^k - \frac{3}{2}\,k(k-1)\,n^{k-1} + \cdots$$

For example we have

$$\begin{aligned} q_{n,2} &= n^2 - 3n + 2 \\ q_{n,3} &= n^3 - 9n^2 + 3n - 36 \\ q_{n,4} &= n^4 - 18n^3 + 139n^2 - 534n + 840 \\ q_{n,5} &= n^5 - 30n^4 + 407n^3 - 3098n^2 + 13104n - 24332 \end{aligned}$$

We make some attempt to get approximate formulas of $q_{n,k}$. One candidate is

$$f_1 \times f_2 \times f_3 \times \cdots \times f_k.$$

Here we define $f_1 = n$ and

$$f_j = (n - j + 1)\left(1 - \frac{2}{n} + \frac{j-2}{n^2}\right)^{j-1}$$

For $n = 8$ we compare approximate values obtained by the formula with exact values in the previous table. (The question mark shows that difference between exact values ans aproximate ones are very large.))

$k =$	1	2	3	4	5	6	7	8
exact	8	42	140	344	568	550	312	92
approximate	8	42.0	147.7	352.2	?	?	?	?

3 Distibutions of the Cayley distance

3.1 Definition of the Cayley distance

Each solution of the N-Queens problem can be represented by a permutation of $1, 2, \cdots, n$. In fact, suppose that n queens are placed at $(i_1, 1), (i_2, 2), \cdots, (i_n, n)$, where notation (i, j) means that a queen is placed at the cell of i-th row and j-th columns. Then we represent this configuration of n queens by a permutation $[i_1, i_2, \cdots, i_n]$.

Let $\mathcal{P}$ be the set of all permutations of $1, 2, \cdots, n$. The set $\mathcal{P}$ consists of $n!$ elements. Into $\mathcal{P}$ we introduce the well-known Cayley distance.

Consider two arbitrary elements $\rho, \sigma \in \mathcal{P}$. By applying appropriate transpositions some times to ρ, we can transform it to σ. That is, we have $\rho\tau_1\tau_2\cdots\tau_k = \sigma$, where $\tau_1, \tau_2, \cdots, \tau_k$ are appropriate transpositions. The Cayley distance is defined as the minimum of the number of such k.

We call each element of $\mathcal{P}$ "vertex". When the Cayley distance between two vertices is equal to 1, we combine them by an "edge". Then a set $\mathcal{P}$ becomes a "graph". In this section we call it a Cayley graph.

For example, consider permutations of $1, 2, 3$. Then a Cayley graph cnsistes of vertices

$$v_1 = [1, 2, 3],\ v_2 = [2, 1, 3],\ v_3 = [3, 2, 1],\ v_4 = [2, 3, 1],\ v_5 = [3, 1, 2],\ v_6 = [1, 3, 2]$$

and edges

$$\{(1, 2), (1, 3), (1, 6), (2, 4), (2, 5), (3, 4), (3, 5), (4, 6), (5, 6)\},$$

where notation (i, j) denotes an edge that combines v_i and v_j. When we decompose the set of vertices into two parts as $V_1 \cup V_2$ where $V_1 = \{v_1, v_4, v_5\}, V_2 = \{v_2, v_3, v_6\}$, we see that every edge combines a vertex in V_1 and a vertex V_2. Thus the Cayley graph is a bipartite graph.

Let X, Y be mutually independent random permutations, and consider the Caley distance between them, which we denote by $d(X, Y)$. We will study the probability distribution of $D_{\mathcal{P}} := d(X, Y)$. For this purpose it is sufficient to investigate the probability distribution of $d(X, I)$, where I is the identity $[1, 2, \cdots, n]$. The following table shows numerators of the probabilities, while denominators are $n! - 1$. Thus, for example, when $n = 11$,

$$P(d(X, I) = 7) = \frac{f_7}{11! - 1} = \frac{8409500}{39916799}.$$

Table 2-1

n	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
3	3	2								
4	6	11	6							
5	10	35	50	24						
6	15	85	225	274	120					
7	21	175	735	1624	1764	720				
8	28	322	1960	6769	13132	13068	5040			
9	36	546	4536	22449	67284	118124	109584	40320		
10	45	870	9450	63273	269325	723680	1172700	1026576	362880	
11	55	1320	18150	157773	902055	3416930	8409500	12753576	10628640	3628800

From Table 2-1 we can immediately derive the mean and variance, $E(d(X,I)), Var(d(X,I))$.

Table 2-2

n	mean	variance
3	$\frac{7}{5} = 1.4$	$\frac{6}{25} = 0.24$
4	2	$\frac{12}{23} \approx 0.522$
5	$\frac{326}{119} \approx 2.74$	$\frac{10820}{14161} \approx 0.764$
6	$\frac{2556}{719} \approx 3.55$	$\frac{487180}{516961} \approx 0.942$
7	$\frac{22212}{5039} \approx 4.41$	$\frac{27357316}{25391521} \approx 1.077$
8	$\frac{212976}{40319} \approx 5.28$	$\frac{1934120216}{1625621761} \approx 1.190$
9	$\frac{2239344}{362879} \approx 6.17$	$\frac{169750080216}{131681168641} \approx 1.289$
10	$\frac{25659360}{3628799} \approx 7.07$	$\frac{18161387317008}{13168182182401} \approx 1.379$
11	$\frac{318540960}{39916799} \approx 7.98$	$\frac{2329229719011888}{1593350842406401} \approx 1.462$

3.2 Distibutions for solutions of *N*-queens problem

Fix n. Let $\mathcal{Q}$ be the set of all permutations that represent solutions of N-queens problem. Consider a random permutation X such that $P(X = x) = 1/q_n$ (q_n denotes the number of elements of $\mathcal{Q}$). Then we say X is a uniform random permutation on $\mathcal{Q}$. Now let X, Y be mutually independent uniform random permutations on $\mathcal{Q}$ and study probability distributions of the Caley distances $D_{\mathcal{Q}} := d(X, Y)$.

The Cayley distance between two permutations can be easily computed using the following program.

```
transposition :: [a] -> Int -> Int -> [a]
transposition xs i j = xs1 ++ [x2] ++ xs2 ++ [x1] ++ xs3
  where
    (xs1, xs') = splitAt i xs
    x1 = head xs'
    xs'' = tail xs'
    (xs2, xs''') = splitAt (j - i - 1) xs''
    x2 = head xs'''
    xs3 = tail xs'''

distance :: [Int] -> [Int] -> Int
distance xs ys
  | xs == [] || ys == []  = 0
  | head xs == head ys    = dist (tail xs) (tail ys)
  | otherwise = 1 + distance xs' ys
    where
      x = head xs
      y = head ys
      j = index y xs
      xs' = transposition xs 0 j
```

A short comoputation using the program gives the following values for f_j s, where they are efined by

$$P(d(X,Y)=j)=\frac{f_j}{q_n}$$

.

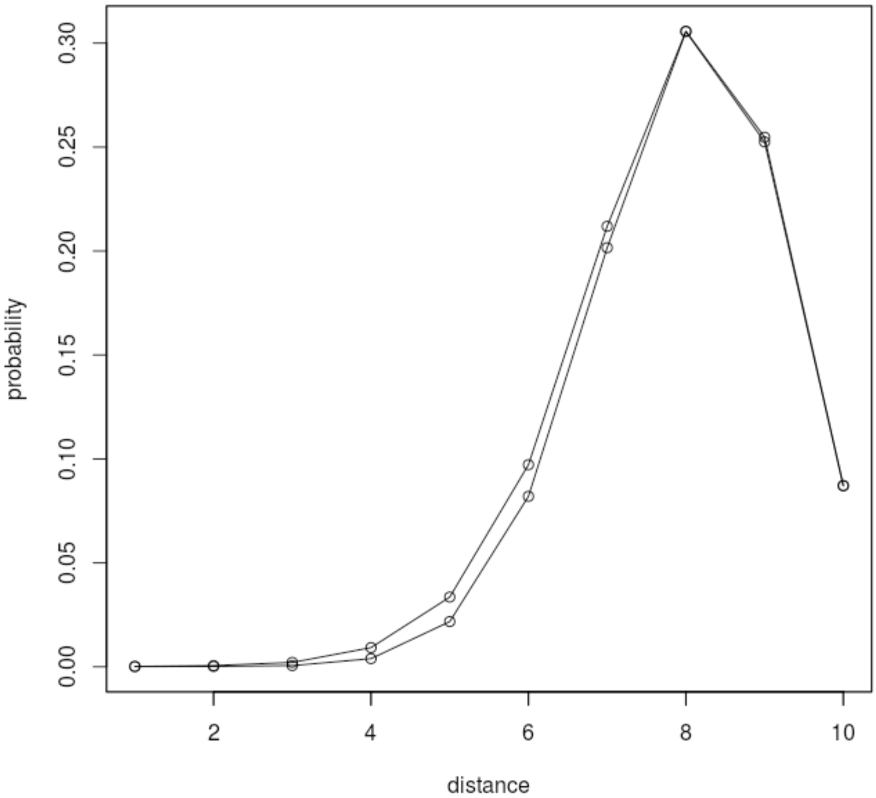
Table 3-1

n	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
4	0	1								
5	0	25	0	20						
6	0	0	2	2	2					
7	4	20	172	216	224	144				
8	8	96	328	854	1296	1124	480			
9	48	408	1376	5248	11736	19624	16792	6544		
10	112	648	1608	7768	22778	53456	80552	68810	25994	
11	552	2048	7616	33148	120408	348976	760648	1097204	906376	312884

Table 3-2

n	mean	variance
4	2	0
5	$\frac{26}{9} \approx 2.89$	$\frac{80}{81} \approx 0.988$
6	4	$\frac{2}{3}$
7	$\frac{284}{65} \approx 4.37$	$\frac{15952}{12675} \approx 1.259$
8	$\frac{10592}{2093} \approx 5.06$	$\frac{6474264}{4380649} \approx 1.478$
9	$\frac{23269}{3861} \approx 6.03$	$\frac{23109059}{14907321} \approx 1.550$
10	$\frac{303370}{43621} \approx 6.95$	$\frac{9437150086}{5708374923} \approx 1.653$
11	$\frac{2353698}{299155} \approx 7.87$	$\frac{460471383308}{268481142075} \approx 1.715$

Using data in Table 2-1 and Table 3-1, we draw histograms of $D_{\mathcal{Q}}$ and $D_{\mathcal{P}}$ for $n = 11$.



The graph shows a fact that difference between two histogramse is unexpectedly small, which suggests us the following conjecture.

Conjecture 2

As n tends to infinity, the difference between the probability distribution of $D_{\mathcal{Q}}$ and that of $D_{\mathcal{P}}$ will vanish. In articular $\lim_{n\to\infty}(E(D_{\mathcal{Q}}) - E(D_{\mathcal{P}})) = 0$.

4 Random sequential placements of queens

When queens are placed at random sequentially, what kind of configurations of queens do we obtain? We formulate the problem more precisely.

- Suppose that some numbers of queens have been already placed in a chess board. We say that a cell is "space" if we can place another queen on this cell.
- Suppose that there are s spaces. Then we select a space at random. In other words the cell is selected with probability $1/s$.

- We place a queen on this cell. Queens which are once placed will be never moved.
- we repeat this trial
- until the time when there are no spaces. Then trial (random placement) ends.

For example, we compute probability that two queens will be placed on 3×3 bord as follows:

```
. . B
A . .
. . .
```

- In case that we first place A then B.
  1. At first the number of spaces is $s = 9$. Thus the probability that place A is 1/9.
  2. The number of spaces that queen A can not attack is $s = 2$. Thus the probability that place B is 1/2.
  3. Accotdingly the probability that first place A then B is $1/9 \times 1/2 = 1/18$.
- A similar calculation shows the probability that first place B then B is also $1/9 \times 1/2 = 1/18$.
- Therefore the probability that the above configuration is realized is $1/18 + 1/18 = 1/9$.

This simple example reveals the important fact. In random placement it is possible that a configuration made of less than n queens.

Suppose that the number X of queens will be placed in random sequential placement. In case $n = 4$ we easily see the probabilities $P(X = r)$.

r	$P(X = r)$
3	29/36
4	7/36

In case $n = 8$, we obtain, with aid of computer program, the following probabilities.

r	$P(X=r)$
5	$\dfrac{271248312897453337616 9}{12252778766867358720000 0} \approx 0.022$
6	$\dfrac{12307334646765143630269663}{29406669040481660928000000} \approx 0.419$
7	$\dfrac{43648342202447408238617731}{88220007121444982784000000} \approx 0.495$
8	$\dfrac{1424168281460119905932 9}{22055001780361245696000 0} \approx 0.065$

Moreover, in case $n = 8$, we have

$$E(X) = \frac{582408743329936600159239251}{88220007121444982784000000} \approx 6.60.$$

From these data we get naturally the next conjecture.

Conjecture 3

$$\lim_{n\to\infty} P(X = n) = 0$$

References

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