Structure of Heron Triangles : An Experimental Study

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Abstract

Triangles with integral sides and integeral area are studied. These triangles are classified into two types. One type are only juxtaposition of two right triangles. Another type are more complex. By computer experiment we infer a plaucible fact that vertices of these triangles can be placed on the integral lattice.

1 Introduction

In this short paper we study triangle such that lengths of its three sides and its area are integers. We call these triangles by **Heron triangles**, whereas the same word sometimes denotes triangle such that lengths of sides and area are rational, not necessarily integral, numbers (see Chapter 5 of [1]).

Let a, b, c stand for three sides. Let g be the g.c.d. of a, b, c. If $g = 1$, we say that a triangle is **primitive**. Obviously any triangle is similar to a primitive triangle. In the below we do not distinguish similar triangles. In other word we study the equivalent class that consist of similar triangles and can be idetified with a primitive one.

Blichfeldt [1] shows that three sides of Heron triangles can be parametrized as

$$\begin{cases} a = (pt + qu)(pu - qt)(r^2 + s^2), \\ b = (ps + qr)(pr - qs)(t^2 + u^2), \\ c = (ru + st)(rt - su)(p^2 + q^2) \end{cases}$$

where r, s, t, u, p, q are integers.

Note that three sides a, b, c computed by (1) correspond to an equivalent class of similar triangles. Thus, if a, b, c are sides of a primitive triangle, we have

$$\begin{cases} ga = (pt + qu)(pu - qt)(r^2 + s^2), \\ gb = (ps + qr)(pr - qs)(t^2 + u^2), \\ gc = (ru + st)(rt - su)(p^2 + q^2) \end{cases} \tag{1}$$

where g is a positive integer.

In this paper we shall propose an algorithm that determine parameters r, s, t, u, p, q when three sides a, b, c are given. Thus we solve a kind of inverse problem to one studied by Blichfeldt.

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2 Algorithm

The following formula gives the area Δ of a primitive triangle with sides a, b, c

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = (a+b+c)/2$.

Then, using (1), we can derive

$$g^2\Delta = (ru+st)(rt-su)(pt+qu)(pu-qt)(ps+qr)(pr-qs) \tag{2}$$

Furthermore, observing (1) and (2), we see

$$g \cdot \frac{abc}{\Delta} = \frac{(ga)(gb)(gc)}{g^2\Delta} = (r^2+s^2)(t^2+u^2)(p^2+q^2).$$

Thus an integral multiple of the quantity abc/Δ is a product of three numbers, each being written as sum of two squares.

3 Experiment

In this section we shall observe the process of determing parameters r, s, t, u, p, q, we prepare a large list of primitive Heron triangles. To generate these triangle exhaustively, we introduce temporary variables $x = s-a, y = s-b, z = s-c$. Then $x+y+z = s$, and triangle inequalities $a < b+c, b < c+a, c < a+b$ are equivalent $x > 0, y > 0, z > 0$.

A program using these temporary variables shows that there are two types of Heron triangles;

- one type, say "type1", is a triangle which is made of two right triangles with integral sides.
- another type, say "type2", is a triangle which is not of type.

To explain type 1 triangle in other words, there is at least one side (base) to which the corresponding height is integral. Thus a type 1 triangle can be formed as juxtaposition of two right triangles. We admit "one" right triangle as a speial kind of type 1 triangles. To speak roughly, type1 triangles are composed of right triangles as if a molecules are composed of atoms.

Thus type 1 triangles seem to be easily imaginable, where as type 2 not. In the below we stydy structure of type 2 triangles experimentally.

3.1 Heron triangles of type 2

By an exhaustive search of Heron triangles with area less than 500, we find the number 78 of type 1 and 16 of type 2. The following table shows these triangles of type 2 as well as parametera r, s, t, u, p, q in this order.

$[a, b, c]$	area	r, s, t, u, p, q
$[5, 29, 30]$	72	$1, 2, 2, 5, 1, 2$
$[5, 51, 52]$	126	$2, 1, 4, 1, 5, 1$
$[10, 35, 39]$	168	$1, 2, 1, 2, 2, 3$
$[13, 40, 45]$	252	$3, 2, 3, 1, 2, 1$
$[15, 34, 35]$	252	$1, 2, 1, 4, 1, 2$
$[17, 65, 80]$	288	$4, 1, 4, 7, 1, 2$
$[15, 52, 61]$	336	$2, 1, 3, 2, 6, 5$
$[17, 40, 41]$	336	$4, 1, 2, 1, 5, 4$
$[34, 55, 87]$	396	$4, 1, 1, 2, 2, 5$
$[13, 109, 120]$	396	$3, 2, 3, 10, 1, 3$
$[25, 34, 39]$	420	$1, 2, 1, 4, 2, 3$
$[26, 51, 73]$	420	$3, 2, 1, 4, 3, 8$
$[26, 73, 97]$	420	$3, 2, 3, 8, 4, 9$
$[41, 50, 89]$	420	$5, 4, 1, 2, 5, 8$
$[39, 58, 95]$	456	$3, 2, 2, 5, 1, 2$
$[17, 55, 60]$	462	$4, 1, 2, 1, 3, 1$

From these data we infer that vertices of any ype triangle can be place on the itegrable lattice. For example, if three vertices have coordinates $\mathrm{A} = (0, 0), \mathrm{B} = (48, 36), \mathrm{C} = (33, 44)$ then the triangle ABC becomes $[17, 55, 60]$. Unfortunately we have yet no proof for this fact.

References

[1] Dickson, *History of the Theory of Numbers*, Vol.2, Chelsea Publishing Company.

[2] H.F.Blichfeldt , "On Triangles with Rational Sides and Having Rational Areas", *Annals of Mathematics*, Vol.11 (1896-7)