Family Tree of Pythagorean Triplets : A Statistical Approach

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Abstract

A parent-child relationship for Pytagorean triplets are studied. This relationship introduces a directed-graph. First we present a characterization of root nodes of the graph. Next we show every offspring of a root node is an innite subgraph. Then we give a useful program which computes vertices and edges of the graph. Several simulations by using the program suggest that offspring is in fact a tree.

Keyword : Pythagorean triplet, directed-graph, tree, Haskell

1 Introduction

The Pythagorean triplet has various amusing, amazing, astonishing, and astounding properties. In particular, it has a parent-child property.

Berggren (1934) showed that all primitive Pythagorean triples can be generated from the $(3, 4, 5)$ triangle by using the three linear transformations

$$A_1 = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix},\ A_3 = \begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix}.$$

If a primitive triplet $T_0 = (a, b, c)$ is regarded as a column vector, we can make three new primitive triplets $T_1 = A_1T_0, T_2 = A_2T_0, T_3 = A_3T_0$. Then we say that T_1, T_2, T_3 are children of T_0, and vice versa T_0 is a parent of T_1, T_2, T_3. This parent-child relationship has been re-discovered and studied by several authors, for example Hall (1970), Roberts (1977), and Alperin (2005).

In the present we consider a different parent-child relationship. Berggren's relationship is of algebraic nature, whereas our relationship is geometric. We will say that T_0 is a parent and T_1 a child when the hypotenuse of T_0 coincides with a non-hypotenuse of T_1. Our aim is to construct a very long chain of Pythagorean triangles that satisfy such a parent-child relationship.

Definition 1 Let us call a triplet of positive integers $T = (a, b, c)$ a **Pythagorean triplet** if it satisfies $a^2 + b^2 = c^2$. T is a Pythagorean triplet when a, b, c are three sides of a right-angled triangle. In this case

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we always suppose that c is its hypotenuse.

Definition 2 In particular we say that T is **primitive** if the greatest common divisor of a, b, c equals 1). T is primitive when there exists no right-angled triangles which are similar to T and smaller than T.

As is well-known, when T is primitive, its hypotenuse c is odd, and one of its non-hypotenuses is even, and the other is odd. In the below we always suppose that a is odd and b is even.

Definition 3 For two primitive Pythagorean triplets $T = (a, b, c)$ and $T' = (a', b', c')$, suppose that the hypotenuse of T and a non-hypotenuse of T' have the same length, i.e. $c = a'$. Then we say that T is a **parent** if T' and T' is a **child** of T. The most simple example is $T = (3, 4, 5), T' = (5, 12, 13)$.

Definition 4 Consider the set of all primitive Pythagorean triplets and denote it by $\mathcal{P}$. We introduce a directed-graph $\mathcal{G}$ with vertices $\mathcal{P}$. For $T, T' \in \mathcal{P}$, draw an egde from T to T' if and only if T' is a child of T. We call this directed-graph $\mathcal{G}$ the **Pythagorean graph**.

Definition 5 A sequence of vertices $T_0, T_1, \cdots T_n$ is called a **path** from T_0 to T_n when for all $0 \geqq i < n$, two vertices T_i, T_{i+1} are combined by edge.

Definition 6 If $T \in \mathcal{P}$ has no parent, we say that T is a **progenitor**. Obviously, if T is not a progenitor, there is at least one path from a certain progenitor to T.

Definition 7 For any progenitor T_0, define the set $\mathcal{T}(T_0) = \{T \in \mathcal{P} : \text{there is a path from} T_0 \text{to} T\}$. We call the set the **offspring** of T_0 and denote it by $\mathcal{O}(T_0)$.

2 Well-known results

The following famous theorem first appeared in Euclid' element.

Theorem 1 *Suppose that $T = (a, b, c)$ is primitive. Then there exist positive integers u, v such that*

$$a = u^2 - v^2,\ b = 2uv,\ c = u^2 + v^2,$$

where u, v are coprime and have different parity (i.e. one is odd and the other even)

In this article we say that a positive integer n has an expression as a sum of squares if there exist positive integers u, v such that $n = u^2 + v^2$ with u, v being coprime and of different parity.

The next theorem is called Fermat's christmas theorem [1]

Theorem 2 *Let p be a prime such that $p \equiv 1 \pmod 4$. Then p has an expression as a sum of squares. Furthermore, such expressions is unique.*

From Theorem 2 we can derive the following theorem. Consider an odd number n with factorizaion

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} q_1^{f_1} q_2^{f_2} \cdots q_l^{f_l}$$

[1] Fermat wrote this theorem in a letter to Mersenne dated December 25, 1640

where $p_1, p_2, \cdots, p_k$ are primes of form $\equiv 1 \pmod 4$, and $q_1, q_2, \cdots, q_l$ are primes of form $\equiv -1 \pmod 4$.

Theorem 3 *An odd number n has an expression as a sum of squares if and only if all exponents $f_1, f_2, \cdots, f_l$ vanish.*

(Proof) It is easy to prove 'if part' by using Theorem 2 with the aid of a famous algebraic identity

$$(u_1^2 + v_1^2)(u_2^2 + v_2^2) = (u_1u_2 \mp v_1v_2)^2 + (u_1v_2 \pm v_1u_2)^2.$$

To prove 'only if part', suppose that some f_j is positive. To simplify notation we write q, f instead of q_j, f_j. We can write $n = q^f m$ where m is not divisible by q. By assumption there are u, v such that $q^f m = u^2 + v^2$. Then we can see that $q \nmid u, q \nmid v$ because if q can divide either u or v then it can divide both, which contradicts to that u, v are coprime. Hence, as $f > 0$, we have $u^2 \equiv -v^2 \pmod q$.

Now, since $q \equiv -1 \pmod 4$, we can write $q = 4n - 1$, i.e. $(q-1)/2 = 2n - 1$. Accordingly we have

$$u^{q-1} = (u^2)^{2n-1} \equiv (-v^2)^{2n-1} = -(v^2)^{2n-1} = -v^{q-1}.$$

As $q \nmid u$, Fermat's little theorem shows that $u^{q-1} \equiv 1 \pmod q$. Similarly $v^{q-1} \equiv 1 \pmod q$. Thus $1 \equiv -1 \pmod q$, which is a contradiction.

Therefore we conclude that every f_j vanishes. (Q.E.D.)

Example 1

n	the number of expressions	$u^2 + v^2$
$5^2 = 25$	1	$4^2 + 3^2$
$13^2 = 169$	1	$12^2 + 5^2$
$17^2 = 289$	1	$15^2 + 8^2$
$5^3 = 125$	1	$11^2 + 2^2$
$13^3 = 2197$	1	$46^2 + 9^2$
$17^2 = 4913$	1	$52^2 + 47^2$
$5 \times 13 = 65$	2	$7^2 + 4^2, 8^2 + 1^2$
$5 \times 17 = 85$	2	$7^2 + 6^2, 9^2 + 2^2$
$13 \times 17 = 221$	2	$11^2 + 10^2, 14^2 + 5^2$
$5 \times 13 \times 17 = 1105$	4	$24^2 + 23^2, 31^2 + 12^2, 32^2 + 9^2, 33^2 + 4^2$

Consider an odd number $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, and try to express it as a sum of squares in many ways. Then the following result is only a special case of a foonote that Gauss has stated in his famous book *Disquisitiones Arithmeticae*.

Theorem 4 *The number of ways of expresstions is equal to 2^k.*

3 Main results

3.1 Shapes of progenitors

The next theorem is only a simple consequence of Theorem 3 and Theorem 4.

Theorem 5
(1) $T = (a, b, c)$ *is a progenitor if and only if* a *has a prime factor* q *of form* $\equiv -1 \pmod 4$.
(2) If $T = (a, b, c)$ *is not a progenitor, the number of parents of* T *is equal to* 2^k, *where* k *denotes the number of primes of form* $\equiv 1 \pmod 4$ *in the factorization of* a.

Now we study shape of right-angled triangles. In a right-angled triangle $(u^2 - v^2, 2uv, u^2 + v^2)$, let us denote by 2β the angle between sides $u^2 - v^2$ and $2uv$. Obviously $\tan\beta = v/u$. We studu distribution of ratio $r := v/u$ experimentally. The following table shows the number of all "admissible" (u, v) and progenitors for $u \leqq 1000$ ("admissible" means that u, v are coprime and of different parity).

r	all admisibles	progenitors
$\ldots < 0.1$	20275	19006
$0.1 \leqq \ldots < 0.2$	20294	18978
$0.2 \leqq \ldots < 0.3$	20286	18949
$0.3 \leqq \ldots < 0.4$	20292	18960
$0.4 \leqq \ldots < 0.5$	20279	18960
$0.5 \leqq \ldots < 0.6$	20284	18925
$0.6 \leqq \ldots < 0.7$	20285	18939
$0.7 \leqq \ldots < 0.8$	20286	18891
$0.8 \leqq \ldots < 0.9$	20285	18862
$0.9 \leqq \ldots$	20295	18624

From these data we conjecture that

- the majority (approximately 93%) of primitive triplets are progenitors.
- distribution of ratios v/u are uniform for both all admisibles and progenitors. The p-values of chi-squared test of uniformity for both cases are 1.0 and 0.7792 respectively.

3.2 Graph structure

Theorem 6 *Every offspring is an infinite graph.*

(Proof) To show that $\mathcal{O}(T_0)$ is infinite, it suffices to show that any T has a child. Let c be a hyotenuse of T. If we define $u = (c+1)/2, v = (c-1)/2$ and $T' = (u^2 - v^2, 2uv, u^2 + v^2$, then T' is a child of T. (Q.E.D.)

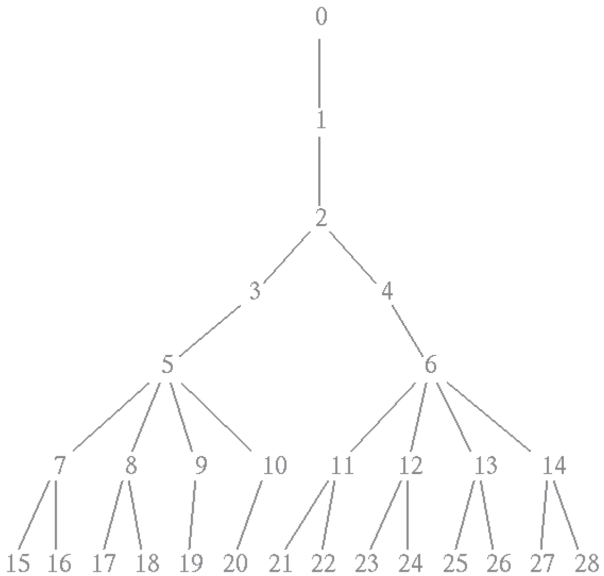
Several examples, one of which is Example 2, suggest that every offspring is really a tree. However the authors can not prove this conjecture.

Conjecture 7 *Every offspring is an infinite tree.*

Example 2 $\mathcal{O}((3,4,5))$

id	(u,v)
0	$(2,1)$
1	$(3,2)$
2	$(7,6)$
3	$(11,6)$
4	$(43,42)$
5	$(79,78)$
6	$(1807,1806)$
7	$(227,198)$
8	$(259,234)$
9	$(371,354)$
10	$(6163,6162)$
11	$(22579,22434)$
12	$(112547,112518)$
13	$(652691,652686)$
14	$(3263443,3263442)$

id	(u,v)
15	$(1127,1086)$
16	$(45367,45366)$
17	$(871,798)$
18	$(60919,60918)$
19	$(131479,131478)$
20	$(37976407,37976406)$
21	$(31831,342)$
22	$(506547799,506547798)$
23	$(32554343,32553954)$
24	$(12663563767,12663563766)$
25	$(1860272071,1860271842)$
26	$(426002278039,426002278038)$
27	$(45708399047,45708398814)$
28	$(10650056950807,10650056950806)$



The vertices and edges of $\mathcal{O}((3,4,5))$ are computed by using the following program.

```
type Tri = (Integer, Integer)  -- (u, v)
type Edge = (Tri, Tri)

divisors_pair :: Integer -> [Tri]
divisors_pair n
  = [(x, q) | x <- [m, (m-1)..1], let (q, r) = n `divMod` x, r == 0, q > x]
  where
    m = floor $ sqrt $ fromIntegral n

edge_forward :: Tri -> [Edge]
edge_forward (u0, v0) = [((u0, v0), (u, v)) | (d1, d2) <- ys,
      let u = (d2 + d1) `div` 2, let v = (d2 - d1) `div` 2, gcd u v == 1]
  where
    c0 = u0 * u0 + v0 * v0
    ys = divisors_pair c0

tree_forward :: Tri -> Int -> [Edge]
tree_forward progenitor niter = concat $ take niter $ iterate f [(progenitor, progenitor)]
  where
    f :: [Edge] -> [Edge]
    f es = concat [edge_forward v1 | (v0, v1) <- es]
```

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