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On $(a,\ b,\ f)$-Metrics

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Abstract

In our recent study we introduced the notion of *generalized Randers metric* $L = \alpha+\beta$ on a differentiable manifold M, where α and β are respectively a Riemannian metric and a singular Riemannian metric on M. Given a covariant vector field b and an almost Hermitian structure f on a Riemannian manifold $(M,\ \alpha)$ we have an interesting example called an $(a,\ b,\ f)$-*metric.* In the present paper we show that a *normal* $(a,\ b,\ f)$-metric gives a non-trivial example of a Kaehlerian Finsler manifold. The conformal theory of $(a,\ b,\ f)$-metrics is also discussed.

Key words: Kaehlerian Finsler manifold, Rizza manifold, Generalized Randers metric, Conformal change.

1 Introduction

Given a Riemannian metric α and a non-vanishing 1-form β on a differentiable manifold M, we have a Finsler metric $L = \alpha + \beta$ on M called a *Randers metric.* In our previous paper [9] we generalized the notion of Randers metric by replacing β by a singular Riemannian metric, and obtained a condition that such a metric L be *locally flat*, that is, the Finsler manifold $(M,\ L)$ be a locally Minkowski space, and further under some assumption we obtained a condition that L be *conformally flat*, that is, $(M,\ L)$ be locally conformal to a locally Minkowski space, as follows.

Definition 1.1 On an m-dimensional differentiable manifold M, let α be a Riemannian metric and β a singular Riemannian metric. A Finsler metric $L = \alpha + \beta$ on M is called a *generalized Randers metric* and then the Finsler manifold $(M,\ L)$ is called a *generalized Randers space.*

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Now, denoting a point of M and a tangent vector at the point by $x = (x^i)$ and $y = (y^i)$ respectively, we put

$$\alpha(x,\, y) = (a_{ij}(x)y^i y^j)^{1/2}, \ \beta(x,\, y) = (b_{ij}(x)y^i y^j)^{1/2}, \tag{1.1}$$

where a_{ij} and b_{ij} are symmetric tensor fields on M and it is assumed that the matrix (a_{ij}) is positive-definite and the matrix (b_{ij}) has the rank r such that $0 < r < m$. With respect to the Levi-Civita connection $\Gamma = (\{{}_j{}^i{}_k\})$ of the associated Riemannian manifold $(M,\, \alpha)$ we denote the covariant differentiation and the curvature tensor field by ∇_k and $R_h{}^i{}_{jk}$ respectively. Then we have

Theorem 1.1 *A generalized Randers space $(M,\ \alpha + \beta)$ is a Berwald space if and only if $\nabla_k b_{ij} = 0$, and $(M,\ \alpha + \beta)$ is a locally Minkowski space if and only if $R_h{}^i{}_{jk} = 0,\ \nabla_k b_{ij} = 0$.*

Putting $(a^{ij}) = (a_{ij})^{-1},\ b^i{}_j = a^{ir} b_{rj},\ b^{ij} = a^{jr} b^i{}_r$, and

$$\mu = a^{ij} b_{ij}, \ \nu = b^{ij} b_{ij}, \tag{1.2}$$

we assume $m\nu - \mu^2 \neq 0$. Then we can put

$$L_j = (m/(m\nu - \mu^2))\{b^{rs}\nabla_r b_{sj} - (\mu/m)\nabla_r b^r{}_j\}, \tag{1.3}$$

$$L_j{}^i{}_k = \{{}_j{}^i{}_k\} + \delta_j{}^i L_k + \delta_k{}^i L_j - a_{jk} L^i, \tag{1.4}$$

where $L^i = a^{ir} L_r$. $\overset{l}{\Gamma} = (L_j{}^i{}_k)$ defines a conformally invariant symmetric linear connection on M. Denoting the curvature tensor field of $\overset{l}{\Gamma}$ by $L_h{}^i{}_{jk}$, we have

Theorem 1.2 *A generalized Randers metric $L = \alpha + \beta$ satisfying $m\nu - \mu^2 \neq 0$ is conformally flat if and only if*

$$L_h{}^i{}_{jk} = 0, \ \nabla_k L_j = \nabla_j L_k, \ \nabla_k b_{ij} = b_{kj} L_i + b_{ki} L_j - a_{ik} b_{jr} L^r - a_{jk} b_{ir} L^r. \tag{1.5}$$

In terms of the conformally invariant linear connection $\overset{l}{\Gamma}$ the condition (1.5) *is expressed as*

$$L_h{}^i{}_{jk} = 0, \ \overset{l}{\nabla}_k L_j = \overset{l}{\nabla}_j L_k, \ \overset{l}{\nabla}_k b_{ij} = -2 L_k b_{ij}, \tag{1.6}$$

where $\overset{l}{\nabla}_k$ denotes the covariant differentiation with respect to $\overset{l}{\Gamma}$.

In the present paper we shall consider the case where $\nabla_k b_{ij} = 0$. From Theorem 1.1 we have

Theorem 1.3 *A generalized Randers space $(M,\ \alpha+\beta)$ satisfying $\nabla_k b_{ij}=0$ is a Berwald space. Then $(M,\ \alpha+\beta)$ is a locally Minkowski space if and only if the associated Riemannian manifold $(M,\ \alpha)$ is locally flat.*

Since $\nabla_k b_{ij}=0$ implies $L_j=0$, $L_j{}^i{}_k=\{{}_j{}^i{}_k\}$ and $L_h{}^i{}_{jk}=R_h{}^i{}_{jk}$, we have from Theorem 1.2

Theorem 1.4 *A generalized Randers metric $L=\alpha+\beta$ satisfying $\nabla_k b_{ij}=0$, $m\nu-\mu^2\neq 0$ is conformally flat if and only if L itself is locally flat, that is, α is locally flat.*

As an interesting example of a generalized Randers metric $L=\alpha+\beta$ satisfying $m\nu-\mu^2\neq 0$ we have an $(a,\ b,\ f)$*-metric*, which was introduced in Ichijyō [4] as an example of an almost Hermitian Finsler metric. A Finaler manifold $(M,\ L)$ with an almost Hermitian Finsler metric L is called a *Rizza manifold*, which is a Finsler manifold corresponding to an almost Hermitian manifold in Riemannian geometry (cf. Ichijyō [6], Rizza [14, 15])..

As an example of an $(a,\ b,\ f)$-metric L satisfying further $\nabla_k b_{ij}=0$, we shall define a *normal* $(a,\ b,\ f)$-metric and show that $(M,\ L)$ is a Kaehlerian Finsler manifold (Theorem 3.3). A Kaehlerian Finsler manifold is a Finsler manifold corresponding to a Kaehler manifold in Riemannian geometry, and there are known some studies (cf. Aikou [1], Dragomir-Ianuş [2], Fukui [3], Ichijyō [4, 5, 6, 7], Kobayashi [10], Royden [16], Rund [17], etc.). Theorem 3.3 seems important in the sense that it gives a non-Riemannian example of a Kaehlerian Finsler manifold.

We shall also discuss the conformal change of an $(a,\ b,\ f)$-metric. Then a condition that an $(a,\ b,\ f)$-metric be locally conformal to a normal one is obtained in terms of a new conformally invariant tensor field $f^i{}_{jk}$ (Theorem 4.1).

2 $(a,\ b,\ f)$-metrics

Let $(M,\ \alpha)$ be a Riemannian manifold of even dimension $m=2n$. Given a non-vanishing covariant vector field $b_i(x)$ and an almost Hermitian structure $f^i{}_j(x)$ on $(M,\ \alpha)$:

(2.1) $$f^i{}_r f^r{}_j=-\delta^i{}_j,\quad a_{rs}f^r{}_i f^s{}_j=a_{ij},$$

where $\alpha(x,\ y)=(a_{ij}(x)y^iy^j)^{1/2}$, we put

(2.2) $$\beta(x,\ y)=\{(b_i(x)y^i)^2+(b_i(x)f^i{}_j(x)y^j)^2\}^{1/2}.$$

Since $\beta(x,\ y)$ has a form $\beta(x,\ y)=(b_{ij}(x)y^iy^j)^{1/2}$, where

(2.3) $$b_{ij}=b_ib_j+b_rb_sf^r{}_if^s{}_j,$$

we have

Proposition 2.1 *For the matrix* (b_{ij}) *we have* $\mathrm{rank}(b_{ij}) = 2$.

Proof Putting $f_i = b_r f^r{}_i$, we have $b_{ij} = b_i b_j + f_i f_j$, so the minor determinants of order 3 vanish. On the other hand, if $\begin{vmatrix} b_{ii} & b_{ij} \\ b_{ji} & b_{jj} \end{vmatrix} = (b_i f_j - b_j f_i)^2 = 0$, then we have $b_{ij} = 0$, which contradicts $b_i \neq 0$. Thus we have $\mathrm{rank}(b_{ij}) = 2$. □

Proposition 2.2 *For the Finsler metric* $L = \alpha + \beta$ *we have* $m\nu - \mu^2 = 4(n-1)b^4$, *where* μ *and* ν *are the scalar fields given by* (1.2) *and* $b = (a^{ij} b_i b_j)^{1/2}$.

Proof Putting $b^i = a^{ir} b_r$, $f^i = a^{ir} f_r$, we have $b^{ij} = b^i b^j + f^i f^j$. Paying attention that the tensor field $a^{ir} f^j{}_r$ is skew-symmetric in i, j (cf. K.Yano [18]), we have $a^{ij} f_i f_j = b^2$, $b_i f^i = b^i f_i = 0$, $f_i f^i = b^2$, so we have $\mu = a^{ij} b_{ij} = 2b^2$, $\nu = b^{ij} b_{ij} = 2b^4$. Since $m = 2n$, we have $m\nu - \mu^2 = 4(n-1)b^4$. □

Hence, if we assume $n \geq 2$, then we have $0 < \mathrm{rank}\,(b_{ij}) < m$ and $m\nu - \mu^2 \neq 0$, so β is a singular Riemannian metric on M, and $L = \alpha + \beta$ defines on M a generalized Randers metric satisfying $m\nu - \mu^2 \neq 0$.

Definition 2.1 On a differentiable manifold M of even dimension $m = 2n$ $(n \geq 2)$, let α be a Riemannian metric and β a singular Riemannian metric given by (2.2). A generalized Randers metric $L = \alpha + \beta$ on M is called an (a, b, f)-*metric,* and then the Finsler manifold (M, L) is called an (a, b, f)-*manifold.*

From Theorem 1.1 and Theorem 1.2 we have

Theorem 2.1 *An* (a, b, f)-*manifold* $(M, \alpha + \beta)$ *is a Berwald space if and only if* $\nabla_k b_{ij} = 0$, *and* $(M, \alpha+\beta)$ *is a locally Minkowski space if and only if* $R_h{}^i{}_{jk} = 0$, $\nabla_k b_{ij} = 0$.

Theorem 2.2 *An* (a, b, f)-*metric* $L = \alpha + \beta$ *satisfies the condition* $m\nu - \mu^2 \neq 0$. L *is conformally flat if and only if* L *satisfies the condition* (1.5) *or* (1.6).

Now, we shall remark that an (a, b, f)-metric on M defines a Rizza manifold. A *Rizza manifold* (M, L, f) is by definition a Finsler manifold (M, L) endowed with an almost complex structure $f^i{}_j(x)$ on M: $f^i{}_r f^r{}_j = -\delta^i{}_j$, satisfying the condition

$$L(x, \phi_\theta y) = L(x, y) \quad (0 \leq \theta \leq 2\pi), \tag{2.4}$$

where $\phi_\theta{}^i{}_j = (\cos\theta)\,\delta^i{}_j + (\sin\theta)\,f^i{}_j$. The condition (2.4) is called the *Rizza condition,* and is also expressed as

$$g_{rs}(x, \phi_\theta y)\,\phi_\theta{}^r{}_i\,\phi_\theta{}^s{}_j = g_{ij}(x, y) \tag{2.5}$$

with respect to the fundamental tensor field $g_{ij} = \dot{\partial}_i \dot{\partial}_j (L^2/2)$, where $\dot{\partial}_i = \partial/\partial y^i$.

The Rizza condition is equivalent to each of the following θ-free conditions (cf. [6]):

$$g_{ij} f^i{}_r y^r y^j = 0, \tag{2.6}$$

$$(g_{ij} - g_{rs} f^r{}_i f^s{}_j) y^j = 0, \tag{2.7}$$

$$g_{ir} f^r{}_j + g_{rj} f^r{}_i + 2C_{ijr} f^r{}_s y^s = 0, \tag{2.8}$$

where $C_{ijk} = \dot{\partial}_k (g_{ij}/2)$.

The condition (2.8) is also expressed as

$$g_{ij} = g_{rs} f^r{}_i f^s{}_j + 2C_{irs} f^r{}_j f^s{}_k y^k. \tag{2.9}$$

Thus in the Riemannian case, where $C_{ijk} = 0$, the Rizza condition $\alpha(x,\ \phi_\theta y) = \alpha(x,\ y)$ means that the almost complex structure $f^i{}_j$ is an almost Hermitian structure: $a_{rs} f^r{}_i f^s{}_j = a_{ij}$. Since for the proof of the equivalence between (2.4) and (2.9) we need not assume that (g_{ij}) is regular, we have also $\beta(x,\ \phi_\theta y) = \beta(x,\ y)$ by showing $b_{rs} f^r{}_i f^s{}_j = b_{ij}$ from (2.3). Thus we have

Theorem 2.3 *An $(a,\ b,\ f)$-manifold is a Rizza manifold.*

Remark 2.1 Theorem 2.3 is also proved by showing (2.4) straight. On the other hand, since β is singular, an $(a,\ b,\ f)$-metric $L = \alpha + \beta$ is not Riemannian, so this metric L gives a non-trivial example of a Rizza manifold.

3 Normal $(a,\ b,\ f)$-metrics

A Rizza manifold $(M,\ L,\ f)$ is called a *Kaehlerian Finsler manifold* if

$$\nabla^*{}_k f^i{}_j = 0 \tag{3.1}$$

is satisfied, where $\nabla^*{}_k$ denotes the h-covariant differentiation with respect to the Cartan connection $C\Gamma$. Since the Nijenhuis tensor field $N^i{}_{jk}$ of $f^i{}_j$ is expressed as

$$N^i{}_{jk} = (\nabla^*{}_r f^i{}_j) f^r{}_k - (\nabla^*{}_r f^i{}_k) f^r{}_j + f^i{}_r (\nabla^*{}_j f^r{}_k - \nabla^*{}_k f^r{}_j), \tag{3.2}$$

if $(M,\ L,\ f)$ is a Kaehlerian Finsler manifold, then $(M,\ f)$ is a complex manifold (cf. Ichijyō [6]).

If L is a Riemannian metric, we have $\nabla^*{}_k f^i{}_j = \nabla_k f^i{}_j$, so $\nabla^*{}_k f^i{}_j = 0$ implies $\nabla_k f^i{}_j = 0$, and a Kaehlerian Finsler manifold is a Kaehler manifold. We shall give an example of a non-Riemannian Kaehlerian Finsler manifold.

Definition 3.1 Let M be a differentiable manifold of even dimension $m = 2n$ ($n \geq 2$). An (a, b, f)-metric $L = \alpha + \beta$ on M is called *normal* if it satisfies

$$\nabla_k b_i = 0, \ \nabla_k f^i{}_j = 0, \tag{3.3}$$

and then the (a, b, f)-manifold (M, L) is called *normal.*

Since (3.3) implies $\nabla_k b_{ij} = 0$, we have an example of a generalized Randers metric satisfying $\nabla_k b_{ij} = 0$, $m\nu - \mu^2 \neq 0$. Thus from Theorem 1.3 and Theorem 1.4 we have

Theorem 3.1 *A normal (a, b, f)-manifold $(M, \ \alpha + \beta)$ is a Berwald space. Then $(M, \ \alpha+\beta)$ is a locally Minkowski space if and only if the associated Riemannian manifold $(M, \ \alpha)$ is locally flat.*

Theorem 3.2 *A normal (a, b, f)-metric $L = \alpha + \beta$ is conformally flat if and only if L itself is locally flat, that is, α is locally flat.*

Now, we shall show that a normal (a, b, f)-manifold (M, L) $(L = \alpha+\beta)$ is a Kaehlerian Finsler manifold. It is noted that a Finsler connection $F\Gamma = (F_j{}^i{}_k, N^i{}_k, V_j{}^i{}_k)$ given by

$$F_j{}^i{}_k = \{{}_j{}^i{}_k\}, \ N^i{}_k = y^j\{{}_j{}^i{}_k\}, \ V_j{}^i{}_k = 0 \tag{3.4}$$

is the Berwald connection $B\Gamma$ of (M, L). In fact, $F\Gamma$ satisfies the system of axioms, which uniquely determines $B\Gamma$, due to Okada [12]:

$$L_{;k} = 0, \ F_j{}^i{}_k = F_k{}^i{}_j, \ N^i{}_k = y^j F_j{}^i{}_k, \ \dot{\partial}_j N^i{}_k = F_j{}^i{}_k, \ V_j{}^i{}_k = 0, \tag{3.5}$$

where ${}_{;k}$ denotes the h-covariant differentiation with respect to $F\Gamma$. In a Berwald space the h-covariant differentiatons with respect to $C\Gamma$ and $B\Gamma$ coincide, and from (3.4) the h-covariant derivative of $f^i{}_j(x)$ with respect to $B\Gamma$ $(= F\Gamma)$ becomes the covariant derivative with respect to the Levi-Civita connection $\Gamma = (\{{}_j{}^i{}_k\})$ of (M, α), so we have $\nabla^*{}_k f^i{}_j = f^i{}_{j;k} = \nabla_k f^i{}_j = 0$. Thus we have

Theorem 3.3 *A normal (a, b, f)-manifold is a Kaehlerian Finsler manifold.*

Remark 3.1 Theorem 3.3 is also proved by showing that a Finsler connection $F\Gamma^* = (F_j{}^i{}_k, N^i{}_k, V_j{}^i{}_k)$ given by

$$F_j{}^i{}_k = \{{}_j{}^i{}_k\}, \ N^i{}_k = y^j\{{}_j{}^i{}_k\}, \ V_j{}^i{}_k = g^{ir}C_{jrk}, \tag{3.6}$$

where $(g^{ij}) = (g_{ij})^{-1}$, is the Cartan connection $C\Gamma$ of (M, L). This is shown by checking that $F\Gamma^*$ satisfies the system of axioms, which uniquely determines $C\Gamma$, due to Matsumoto (cf. [11]):

$$g_{ij|k} = 0, \ \ F_j{}^i{}_k = F_k{}^i{}_j, \ N^i{}_k = y^j F_j{}^i{}_k, \ g_{ij}\,|_k = 0, \ V_j{}^i{}_k = V_k{}^i{}_j, \tag{3.7}$$

where $_{|k}$ and $|_k$ denote the h- and v-covariant differentiations with respect to $F\Gamma^*$, but it is not so trivial that $F\Gamma^*$ satisfies the first axiom.

Since g_{ij} is given by $g_{ij} = \dot\partial_i\dot\partial_j(L^2/2)$, in order to show $g_{ij|k} = 0$ it is sufficient to prove that $_{|k}$ commutes with the partial differentiation $\dot\partial_h$. This follows from the Ricci identitiy, applied to a Finsler tensor field, e.g., $T^i{}_j$,

$$\dot\partial_h(T^i{}_{j;k}) - (\dot\partial_h T^i{}_j)_{;k} = T^r{}_j P_r{}^i{}_{kh} - T^i{}_r P_j{}^r{}_{kh} - T^i{}_{j;r} V_k{}^r{}_h - (\dot\partial_r T^i{}_j) P^r{}_{kh} \tag{3.8}$$

with respect to $F\Gamma$ given by (3.4), where $P^i{}_{kh} = \dot\partial_h N^i{}_k - F_h{}^i{}_k$, $P_j{}^i{}_{kh} = \dot\partial_h F_j{}^i{}_k - V_j{}^i{}_{h;k} + V_j{}^i{}_r P^r{}_{kh}$. In fact, since with respect to $F\Gamma$ the h-covariant differentiation $_{;k}$ coincides with $_{|k}$ and we have $P_j{}^i{}_{kh} = V_j{}^i{}_k = P^i{}_{kh} = 0$, we have $\dot\partial_h(T^i{}_{j|k}) = (\dot\partial_h T^i{}_j)_{|k}$.

Remark 3.2 A normal $(a,\ b,\ f)$-manifold gives a concrete example of a non-Riemannian Kaehlerian Finsler manifold. This was a motive for studying a generalized Randers space. A normal $(a,\ b,\ f)$-manifold is a Berwald space, but it is shown in Ichijyō [7] that a Kaehlerian Finsler manifold is a Landsberg space. So it is an important open problem to find a concrete example of a non-Berwald Kaehlerian Finsler manifold.

On the other hand, H. S. Park [13] generalized the notion of Kaehlerian Finsler manifold by replacing the condition (3.1) by $\nabla^*{}_k f^i{}_j + \nabla^*{}_j f^i{}_k = 0$ and discussed the Rizza manifold which was called a *nearly Kaehlerian Finsler manifold.* For this Finsler manifold an interesting example is also expected.

4 Conformal changes of $(a,\ b,\ f)$-metrics

On a differentiable manifold M of even dimension $m = 2n$ $(n \geq 2)$ we shall consider a conformal change of an $(a,\ b,\ f)$-metric $L = \alpha + \beta$:

$$L(x,\ y) \to \tilde L(x,\ y) = e^{\sigma(x)} L(x,\ y). \tag{4.1}$$

Since α and β are expressed as $\alpha(x,\ y) = (a_{ij}(x)y^iy^j)^{1/2}$ and $\beta(x,\ y) = \{(b_i(x)y^i)^2 + (b_i(x)f^i{}_j(x)y^j)^2\}^{1/2}$ respectively, if we put

$$\tilde a_{ij} = e^{2\sigma} a_{ij},\ \tilde b_i = e^{\sigma} b_i, \tag{4.2}$$

we have a generalized Randers metric $\tilde L = \tilde\alpha + \tilde\beta$ given by $\tilde\alpha(x,\ y) = (\tilde a_{ij}(x)y^iy^j)^{1/2}$, $\tilde\beta(x,\ y) = \{(\tilde b_i(x)y^i)^2 + (\tilde b_i(x)f^i{}_j(x)y^j)^2\}^{1/2}$. Then we have $\tilde a_{rs} f^r{}_i f^s{}_j = \tilde a_{ij}$, so $\tilde L$ is also an $(a,\ b,\ f)$-metric.

We shall find a condition that an $(a,\ b,\ f)$-metric L be locally conformal to a normal one $\tilde L$. Since the Levi-Civita connection $\tilde\Gamma = (\widetilde{\{{}_j{}^i{}_k\}})$ of the Riemannian manifold $(M,\ \tilde\alpha)$ is given by

$$\widetilde{\{{}_j{}^i{}_k\}} = \{{}_j{}^i{}_k\} + \delta_j{}^i\sigma_k + \delta_k{}^i\sigma_j - a_{jk}\sigma^i, \tag{4.3}$$

where $\sigma_j = \partial\sigma/\partial x^j$, $\sigma^i = a^{ir}\sigma_r$, we have

$$\widetilde{\nabla}_k \widetilde{b}_j = e^\sigma(\nabla_k b_j - b_k\sigma_j + b_r\sigma^r a_{jk}), \tag{4.4}$$

where $\widetilde{\nabla}_k$ denotes the covariant differentiation with respect to $\widetilde{\Gamma}$. Eliminating $b_r\sigma^r$ from (4.4) and putting

$$M_j = (1/b^2)\{b^r(\nabla_r b_j) - (\nabla_r b^r)b_j/(m-1)\}, \tag{4.5}$$

we have

$$\sigma_j = M_j - \widetilde{M}_j. \tag{4.6}$$

As is shown in Ichijyō-Hashiguchi [8], substituting (4.6) in (4.3) and putting

$$M_j{}^i{}_k = \{{}_j{}^i{}_k\} + \delta_j{}^i M_k + \delta_k{}^i M_j - a_{jk}M^i, \tag{4.7}$$

where $M^i = a^{ir}M_r$, we have $\widetilde{M}_j{}^i{}_k = M_j{}^i{}_k$. $M_j{}^i{}_k$ defines on M a symmetric linear connection $\overset{m}{\Gamma} = (M_j{}^i{}_k)$, which is invariant by the conformal change (4.2).

In the same way, we have

$$\widetilde{\nabla}_k f^i{}_j = \nabla_k f^i{}_j + \delta_k{}^i\sigma_r f^r{}_j - \sigma^i a_{kr} f^r{}_j - \sigma_j f^i{}_k + \sigma^r f^i{}_r a_{jk}. \tag{4.8}$$

Substituting (4.6) in (4.8) and putting

$$f^i{}_{jk} = \nabla_k f^i{}_j + \delta_k{}^i M_r f^r{}_j - M^i a_{kr} f^r{}_j - M_j f^i{}_k + M^r f^i{}_r a_{jk}, \tag{4.9}$$

we have $\widetilde{f}^i{}_{jk} = f^i{}_{jk}$. $f^i{}_{jk}$ is a tensor field invariant by the conformal change (4.2). It is noted that $M_j{}^i{}_k$ and $f^i{}_{jk}$ are invariant by the conformal change (4.1).

Since $\widetilde{\nabla}_k\widetilde{b}_j = 0$ implies $\widetilde{M}_j = 0$, that is, $M_j = \sigma_j$ is gradient, in the same way as shown in [8], using these conformal invariants $M_j{}^i{}_k$ and $f^i{}_{jk}$, we can obtain a condition that an (a, b, f)-metric L be locally conformal to a normal (a, b, f)-metric $\widetilde{L}$, that is, a condition that there locally exists σ such that $\widetilde{\nabla}_k\widetilde{b}_j$ and $\widetilde{\nabla}_k f^i{}_j$ vanish by a conformal change $L \to \widetilde{L} = e^\sigma L$, as follows.

Theorem 4.1 *By a conformal change* (4.1) *an* (a, b, f)*-metric remains to be an* (a, b, f)*-metric. An* (a, b, f)*-metric is locally conformal to a normal* (a, b, f)*-metric if and only if*

$$\nabla_k M_j = \nabla_j M_k, \ \nabla_k b_j = b_k M_j - b_r M^r a_{jk}, \ f^i{}_{jk} = 0. \tag{4.10}$$

Lastly, it is noted that we can express the condition (4.10) in terms of the linear connection $\overset{m}{\Gamma}$ as follows.

Theorem 4.2 *An* (a, b, f)*-metric is locally conformal to a normal* (a, b, f)*-metric if and only if*

$$\overset{m}{\nabla}_k M_j = \overset{m}{\nabla}_j M_k, \ \overset{m}{\nabla}_k b_j = -M_k b_j, \ \overset{m}{\nabla}_k f^i{}_j = 0, \tag{4.11}$$

where $\overset{m}{\nabla}_k$ *denotes the h-covariant differentiation with respect to* $\overset{m}{\Gamma}$.

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