

## A note on continuity properties of relations

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**Abstract:** This note provides 4 equivalent conditions with the continuity of relations adopted by Brattka and Hertling to investigate computability of relations. As the case of continuous functions, each condition characterises continuous relations by respectively using open sets, neighbourhoods, closures, and closed sets. A notion of sequential continuity is also studied. We show a relevance between continuous relations and sequentially continuous relations which is, again, analogous to the case of functions.

**Keywords:** relations, topological spaces, continuity, sequential continuity

### 1. Introduction

This note provides quite fundamental results on continuity of relations. There are various definitions of continuous relations. This note investigates the continuity adopted by Brattka and Hertling in [2].

Brattka and Hertling have investigated relevance between continuity and computability of relations in the article. Using the notion of continuity, [4] has given a relation algebraic proof of Richardson's theorem [7] which globally characterises a class of nondeterministic cellular automata. We study the continuity from a pure general-topological point of view.

Notion and properties of continuous functions are very fundamental and studied very well. Everyone learns the following facts in math classes.

**Fact 1.** Let  $f: X \rightarrow Y$  be a function. Then the following 5 conditions are equivalent.

1.  $f$  is continuous.
2. For each open set  $V \subseteq Y$  the set  $f^{-1}(V)$  is open in  $X$ .
3. For each point  $x \in X$  the set  $f^{-1}(N(f(x)))$  is a neighbourhood of  $x$  for each neighbourhood  $N(f(x))$  of  $f(x)$ .
4.  $f(\overline{A}) \subseteq \overline{f(A)}$  for each subset  $A \subseteq X$ .
5. For each closed set  $F \subseteq Y$  the set  $f^{-1}(F)$  is closed in  $X$ .

**Fact 2.** Let  $f: X \rightarrow Y$  be a function.

1. If  $f$  is continuous, then it is sequentially continuous.
2. If  $X$  is a first-countable space and  $f$  is sequentially continuous, then it is continuous.

Precisely speaking, this note provides analogous properties of the above facts based on the continuity of relations in [2].

### 2. Images and preimages of relations

This section recalls images and preimages of relations which will be used later.

Let  $R \subseteq X \times Y$  be a relation from a set  $X$  to a set  $Y$ . For  $x \in X$ ,  $A \subseteq X$ ,  $y \in Y$  and  $B \subseteq Y$  we define

$$\begin{aligned} [x]R &= \{y \in Y \mid (x, y) \in R\} & [A]R &= \bigcup_{a \in A} [a]R \\ R[y] &= \{x \in X \mid (x, y) \in R\} & R[B] &= \bigcup_{b \in B} R[b], \end{aligned}$$

and respectively call  $[A]R$  and  $R[B]$  *image* of  $A$  and *preimage* (or, equivalently, *inverse image*) of  $B$  under  $R$ . The domain of  $R$  is denoted by  $\text{dom}(R)$ , that is,  $\text{dom}(R) = R[Y]$ . For a subset  $B \subseteq Y$  *upper preimage*  $R[[B]]$  of  $B$  under  $R$  is defined by

$$R[[B]] = \{x \in X \mid [x]R \subseteq B\}.$$

A relation  $R \subseteq X \times Y$  is called *univalent* (or a *partial function* in other words) if  $(x, y) \in R$  and  $(x, y') \in R$  implies  $y = y'$  for all  $x \in X$  and  $y, y' \in Y$ . If a relation  $R$  satisfies  $\text{dom}(R) = X$ , we call it *total*. If a relation  $R$  is univalent and total, we call it a *function* (or a *map* in other words) and is written by  $R: X \rightarrow Y$ . For a function  $f: X \rightarrow Y$ ,  $f(x)$  and  $f(A)$  denote the value at  $x$  of  $f$  and the image of  $A$  under  $f$ , respectively. Of course,  $\{f(x)\} = [x]f$  and  $f(A) = [A]f$ . Notations  $f^{-1}(y)$  and  $f^{-1}(B)$  for preimages  $f[y]$  and  $f[B]$  of  $y \in Y$  and  $B \subseteq Y$  under a function  $f$  are also used. Note that  $f[B] = f^{-1}(B) = f[[B]]$  for any  $B \subseteq Y$  since

$$x \in f[B] \iff x \in f^{-1}(B) \iff f(x) \in B \iff [x]f \subseteq B \iff x \in f[[B]].$$

The next lemma will be used to prove Proposition 3.7.

**Lemma 2.1.** *Let  $R \subseteq X \times Y$  be a relation. Then the following two inclusions hold for any subsets  $A \subseteq X$  and  $B \subseteq Y$ .*

$$A \subseteq R[[A]R] \quad [R[[B]]]R \subseteq B$$

*Proof.*  $A \subseteq R[[A]R]$  holds by

$$a \in A \implies [a]R \subseteq [A]R \iff a \in R[[A]R].$$

Also  $[R[[B]]]R \subseteq B$  holds by

$$\begin{aligned} y \in [R[[B]]]R &\iff \exists x \in X. x \in R[[B]] \text{ and } y \in [x]R \\ &\iff \exists x \in X. [x]R \subseteq B \text{ and } y \in [x]R \\ &\implies y \in B. \end{aligned}$$

□

The next lemma connects preimages and upper preimages, and will be used to prove Proposition 3.8. For a set  $S$  we denote the complement of  $S$  by  $S^c$ .

**Lemma 2.2.** *Let  $R \subseteq X \times Y$  be a relation. Then the following two equations hold for each subset  $B \subseteq Y$ .*

$$R[B] = (R[[B^c]])^c \quad R[[B]] = (R[B^c])^c$$

*Proof.*  $R[B] = (R[[B^c]])^c$  holds by

$$\begin{aligned} x \in R[B] &\iff \exists y \in Y. y \in B \text{ and } y \in [x]R \\ &\iff \exists y \in Y. y \notin B^c \text{ and } y \in [x]R \\ &\iff [x]R \not\subseteq B^c \\ &\iff x \notin R[[B^c]] \\ &\iff x \in (R[[B^c]])^c \end{aligned}$$

Also  $R[[B]] = (R[B^c])^c$  holds by  $R[[B]] = ((R[[B^c]])^c)^c = (R[B^c])^c$ . □

### 3. Point-wise continuity and its equivalent conditions

Let's recall the definition of continuous relations adopted in [2]. Throughout of the section we assume  $X$  and  $Y$  to be topological spaces.

**Definition 3.1.** Let  $R \subseteq X \times Y$  be a relation.

1. Given a point  $(x, y) \in R$ ,  $R$  is *continuous at*  $(x, y)$  if for each neighbourhood  $N(y)$  of  $y$  there exists a neighbourhood  $N(x)$  of  $x$  such that  $N(y) \cap [x']R \neq \emptyset$  for all  $x'$  in  $N(x) \cap \text{dom}(R)$ .

2.  $R$  is *continuous* if  $R$  is continuous at all points  $(x, y)$  in  $R$ .

For people who know what (point-wise) continuous functions are, this definition might be strange a bit since points in  $R$  are focused on rather than points in  $X$ . Let's think about a definition more like continuous functions.

**Definition 3.2.** Let  $R \subseteq X \times Y$  be a relation.

1. Given a point  $x \in \text{dom}(R)$ ,  $R$  is *continuous at  $x$*  if for each  $y \in [x]R$  and for its each neighbourhood  $N(y)$  there exists a neighbourhood  $N(x)$  of  $x$  such that  $N(y) \cap [x']R \neq \emptyset$  for all  $x'$  in  $N(x) \cap \text{dom}(R)$ .
2.  $R$  is *continuous on  $\text{dom}(R)$*  if  $R$  is continuous at all points  $x$  in  $\text{dom}(R)$ .

A function  $f: X \rightarrow Y$  is continuous at  $x \in X$  iff  $f$  satisfies 1 of Definition 3.2 since  $\text{dom}(f) = X$  and  $y \in [x]R$  is equivalent to  $y = f(x)$  for any  $x \in X$ . Also the following holds since  $(x, y) \in R$  is equivalent to  $x \in \text{dom}(R)$  and  $y \in [x]R$  for any  $(x, y) \in X \times Y$ .

**Proposition 3.3.**  $R$  is *continuous* iff  $R$  is *continuous on  $\text{dom}(R)$* .

Thus the continuity of relations is a generalisation of the (point-wise) continuity of functions. In the rest of this section we provide generalisations of the items 2–5 from Fact 1.

The second one has been provided in [2].

**Proposition 3.4.** A relation  $R \subseteq X \times Y$  is *continuous* iff for each open set  $V \subseteq Y$  the set  $R[V]$  is open in  $\text{dom}(R)$ .

Recalling the definition of neighbourhoods, we have the following property. Its consequence is a generalisation of the third item of Fact 1.

**Proposition 3.5.** Let  $R \subseteq X \times Y$  be a relation such that for each open set  $V \subseteq Y$  the set  $R[V]$  is open in  $\text{dom}(R)$ . Then for each point  $(x, y) \in R$  the set  $R[N(y)]$  is a neighbourhood in  $\text{dom}(R)$  of  $x$  for each neighbourhood  $N(y)$  of  $y$ .

*Proof.* Let  $(x, y) \in R$  and  $N(y)$  be a neighbourhood of  $y$ . Then there exists an open set  $V \subseteq Y$  such that  $y \in V$  and  $V \subseteq N(y)$ . For this open set  $V$ ,  $R[V]$  contains  $x$  since  $(x, y) \in R$  and  $y \in V$ . Also  $R[V]$  is open in  $\text{dom}(R)$ . Moreover  $R[V] \subseteq R[N(y)] \subseteq \text{dom}(R)$ . Therefore  $R[N(y)]$  is a neighbourhood in  $\text{dom}(R)$  of  $x$ .  $\square$

Recalling the definition of closures, we have the following property. Its consequence is a generalisation of the fourth item of Fact 1. The closure of a subset  $S$  of a topological space is denoted by  $\bar{S}$ .

**Proposition 3.6.** Let  $R \subseteq X \times Y$  be a relation such that for each point  $(x, y) \in R$  the set  $R[N(y)]$  is a neighbourhood in  $\text{dom}(R)$  of  $x$  for each neighbourhood  $N(y)$  of  $y$ . Then  $[\bar{A}]R \subseteq \overline{[A]R}$  holds for each subset  $A \subseteq \text{dom}(R)$ .

*Proof.* Let  $A \subseteq \text{dom}(R)$ . If  $y \in [\bar{A}]R$ , then there exists  $x \in X$  such that  $x \in \bar{A}$  and  $(x, y) \in R$ . Hence for each neighbourhood  $N(y)$  of  $y$  the set  $R[N(y)]$  is a neighbourhood in  $\text{dom}(R)$  of  $x$ . Thus there exists a neighbourhood  $N_0(x)$  of  $x$  such that  $R[N(y)] = N_0(x) \cap \text{dom}(R)$  and hence

$$A \cap R[N(y)] = A \cap N_0(x) \cap \text{dom}(R) = A \cap N_0(x) \neq \emptyset$$

hold by  $A \subseteq \text{dom}(R)$  and  $x \in \bar{A}$ . Moreover

$$\begin{aligned} & x_0 \in A \cap R[N(y)] \\ \iff & x_0 \in A \text{ and } \exists y_0 \in Y. y_0 \in N(y) \text{ and } (x_0, y_0) \in R \\ \implies & \exists y_0 \in Y. y_0 \in N(y) \text{ and } y_0 \in [A]R \\ \iff & [A]R \cap N(y) \neq \emptyset. \end{aligned}$$

Therefore  $y \in \overline{[A]R}$ .  $\square$

For preimages of closed sets, we use Lemma 2.1 and 2.2. The consequence of the following proposition is a generalisation of the fifth item of Fact 1.

**Proposition 3.7.** Let  $R \subseteq X \times Y$  be a relation satisfying  $\overline{[A]R} \subseteq \overline{[A]R}$  for each subset  $A \subseteq \text{dom}(R)$ . Then for each closed set  $F \subseteq Y$  the set  $R[F] \cap \text{dom}(R)$  is closed in  $\text{dom}(R)$ .

*Proof.* It is sufficient to show that  $\overline{R[F] \cap \text{dom}(R)} \cap \text{dom}(R) \subseteq R[F] \cap \text{dom}(R)$  for each closed set  $F$ . Assume that  $\overline{[A]R} \subseteq \overline{[A]R}$  for each subset  $A \subseteq \text{dom}(R)$ . Then we have

$$\begin{aligned} \overline{R[F] \cap \text{dom}(R)} \cap \text{dom}(R) &\subseteq \overline{R[(R[F] \cap \text{dom}(R))]R} \cap \text{dom}(R) && \text{by Lemma 2.1} \\ &\subseteq \overline{R[(R[F] \cap \text{dom}(R))]R} \cap \text{dom}(R) && \text{by assumption} \\ &\subseteq \overline{R[(R[F]R)]} \cap \text{dom}(R) \\ &\subseteq \overline{R[F]} \cap \text{dom}(R) && \text{by Lemma 2.1} \\ &= R[F] \cap \text{dom}(R). \end{aligned}$$

□

**Proposition 3.8.** Let  $R \subseteq X \times Y$  be a relation. Then the following two conditions are equivalent:

1. For each closed set  $F \subseteq Y$  the set  $R[F] \cap \text{dom}(R)$  is closed in  $\text{dom}(R)$ .
2. For each open set  $V \subseteq Y$  the set  $R[V]$  is open in  $\text{dom}(R)$ .

*Proof.* 1 implies 2 since

$$\begin{aligned} R[V] &= (R[V^c])^c && \text{by Lemma 2.2} \\ &= (R[V^c] \cap (\text{dom}(R) \cup \text{dom}(R)^c))^c \\ &= ((R[V^c] \cap \text{dom}(R)) \cup (R[V^c] \cap \text{dom}(R)^c))^c \\ &= ((R[V^c] \cap \text{dom}(R)) \cup \text{dom}(R)^c)^c && \text{by } \text{dom}(R)^c \subseteq R[V^c] \\ &= (R[V^c] \cap \text{dom}(R))^c \cap \text{dom}(R) \end{aligned}$$

and  $R[V^c] \cap \text{dom}(R)$  is closed in  $\text{dom}(R)$ . Also 2 implies 1 since

$$R[F] \cap \text{dom}(R) = R[F^c]^c \cap \text{dom}(R)$$

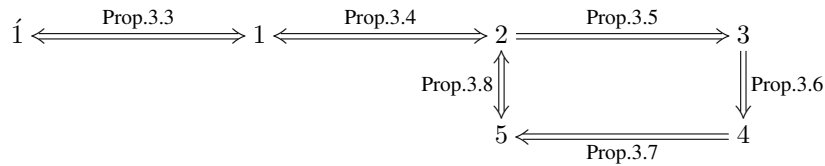
by Lemma 2.2 and  $R[F^c]$  is open in  $\text{dom}(R)$ . □

Summing up the discussion above, we have the following theorem.

**Theorem 3.9.** Let  $R \subseteq X \times Y$  be a relation. Then the following conditions are equivalent:

1.  $R$  is continuous.
- 1'.  $R$  is continuous on  $\text{dom}(R)$ .
2. For each open set  $V \subseteq Y$  the set  $R[V]$  is open in  $\text{dom}(R)$ .
3. For each point  $(x, y) \in R$  the set  $R[N(y)]$  is a neighbourhood in  $\text{dom}(R)$  of  $x$  for each neighbourhood  $N(y)$  of  $y$ .
4.  $\overline{[A]R} \subseteq \overline{[A]R}$  for each subset  $A \subseteq \text{dom}(R)$ .
5. For each closed set  $F \subseteq Y$  the set  $R[F] \cap \text{dom}(R)$  is closed in  $\text{dom}(R)$ .

*Proof.*



□

Now Fact 1 is a corollary of Theorem 3.9.

**Corollary 3.10.** *Let  $f: X \rightarrow Y$  be a function. Then the following conditions are equivalent:*

1.  $f$  is continuous.
2. For each open set  $V \subseteq Y$  the set  $f^{-1}(V)$  is open in  $X$ .
3. For each point  $x \in X$  the set  $f^{-1}(N(f(x)))$  is a neighbourhood of  $x$  for each neighbourhood  $N(f(x))$  of  $f(x)$ .
4.  $f(A) \subseteq \overline{f(A)}$  for each subset  $A \subseteq X$ .
5. For each closed set  $F \subseteq Y$  the set  $f^{-1}(F)$  is closed in  $X$ .

#### 4. Sequential continuity and point-wise continuity

In this section, we study sequential continuity of relations. Again, throughout of the section we assume  $X$  and  $Y$  to be topological spaces.

**Definition 4.1.** Let  $R \subseteq X \times Y$  be a relation.

1. Given a point  $(x, y) \in R$ ,  $R$  is *sequentially continuous at  $(x, y)$*  if for each sequence  $\{x_n\}$  in  $\text{dom}(R)$ , whenever  $\{x_n\}$  converges to  $x$  in  $X$ , then, for each neighbourhood  $N(y)$  of  $y$ , there exists  $M \in \mathbb{N}$  such that  $[x_m]R \cap N(y) \neq \emptyset$  for any  $m \geq M$ .
2.  $R$  is *sequentially continuous* if  $R$  is sequentially continuous at all points  $(x, y)$  in  $R$ .

Analogously to the beginning of the former section, let's think about a definition more like sequentially continuous functions.

**Definition 4.2.** Let  $R \subseteq X \times Y$  be a relation.

1. Given a point  $x \in \text{dom}(R)$ ,  $R$  is *sequentially continuous at  $x$*  if for each  $y \in [x]R$  and for each sequence  $\{x_n\}$  in  $\text{dom}(R)$ , whenever  $\{x_n\}$  converges to  $x$  in  $X$ , then, for each neighbourhood  $N(y)$  of  $y$ , there exists  $M \in \mathbb{N}$  such that  $[x_m]R \cap N(y) \neq \emptyset$  for any  $m \geq M$ .
2.  $R$  is *sequentially continuous on  $\text{dom}(R)$*  if  $R$  is sequentially continuous at all points  $x$  in  $\text{dom}(R)$ .

Note that function  $f: X \rightarrow Y$  is continuous at  $x \in X$  iff  $f$  satisfies 1 of Definition 4.2. Also the following holds similarly to Proposition 3.3.

**Proposition 4.3.**  *$R$  is sequentially continuous iff  $R$  is sequentially continuous on  $\text{dom}(R)$ .*

Thus the sequential continuity of relations is a generalisation of the sequential continuity of functions. Also the continuity of relations implies sequential continuity of them.

**Proposition 4.4.** *If a relation  $R \subseteq X \times Y$  is continuous, then it is sequentially continuous.*

*Proof.* Let  $(x, y) \in R$ ,  $\{x_n\}$  be a sequence in  $\text{dom}(R)$  converging to  $x$  in  $X$ , and  $N(y)$  a neighbourhood of  $y$ . Since  $R$  is continuous, there is a neighbourhood  $N(x)$  of  $x$  such that  $N(y) \cap [x']R \neq \emptyset$  for all  $x' \in N(x) \cap \text{dom}(R)$ . Also, since  $\{x_n\}$  converges to  $x$ , there exists  $M \in \mathbb{N}$  such that  $x_m \in N(x)$  for any  $m \geq M$ . Noting that  $x_n \in \text{dom}(R)$  for any  $n \in \mathbb{N}$ , we have  $[x_m]R \cap N(y) \neq \emptyset$  for  $m \geq M$ .  $\square$

Obviously, the converse of Proposition 4.4 does not holds since it is known that there exists a sequentially continuous function which is not continuous. However, the following holds as the case of functions.

**Proposition 4.5.** *A sequentially continuous relation  $R \subseteq X \times Y$  is continuous if  $X$  is a first-countable space.*

*Proof.* We prove contraposition. Suppose that  $R$  is not continuous. Then, by Theorem 3.9, for some point  $(x_0, y_0) \in R$  there exists a neighbourhood  $N(y_0)$  of  $y_0$  such that  $R[N(y_0)]$  is not a neighbourhood in  $\text{dom}(R)$  of  $x_0$ . Also, since  $X$  is a first-countable space, there exists a countable fundamental neighbourhood system  $\{N_n(x_0)\}$  of  $x_0$  such that  $N_k(x_0) \supseteq N_{k+1}(x_0)$  for each  $k \in \mathbb{N}$ . Now, since  $R[N(y_0)]$  is not a neighbourhood in  $\text{dom}(R)$ ,  $N_k(x_0) \cap \text{dom}(R) \not\subseteq R[N(y_0)]$  for each  $k \in \mathbb{N}$ . Thus it is able to take a sequence  $\{x_n\}$  from  $\text{dom}(R)$  such that  $x_k \in N_k(x_0)$  and  $x_k \notin R[N(y_0)]$  for each  $k \in \mathbb{N}$ . Then, though the sequence  $\{x_n\}$  converges  $x_0$  in  $X$ ,  $[x_k]R \cap N(y_0) = \emptyset$  for each  $k \in \mathbb{N}$ . Therefore  $R$  is not sequentially continuous.  $\square$

Now the 2 items of Fact 2 are corollaries of Proposition 4.4 and 4.5.

**Corollary 4.6.** *Let  $f: X \rightarrow Y$  be a function.*

1. *If  $f$  is continuous, then it is sequentially continuous.*
2. *If  $X$  is a first-countable space and  $f$  is sequentially continuous, then it is continuous.*

## 5. Concluding remarks

In this note we have generalised well-known properties Fact 1 and 2 of continuous functions to ones of continuous relations in the sense of [2]. This is very beginning and there are some space for further study of the continuity.

Pauly and Ziegler have introduced a uniform continuity of relations as an extension of the continuity taken in this note and investigated relevance between the uniform continuity and relative computability of relations in [6]. Using the notion of uniform continuity, [3] has given a global characterisation of nondeterministic cellular automata in general. General-topological fundamentals of the uniform continuity should be prepared to increase convenience.

Talking about continuity of relations, one should mention another notion of continuity studied by Berge [1]. General-topological fundamentals of this notion are well-established and it provides theoretical foundation of mathematical economics [5]. Since the notion of continuity in this note is a generalisation of Berge's continuity, making clear the gap of theories between these two notions could be important.

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